Electromagnetism

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Contents

1 Some reference books

1. D J Griffiths, Introduction to Electrodynamics, Benjamin Cummings, Prentice-Hall of India.
2. R P Feynman, R B Leighton and M Sands, Feynman Lectures on Physics, Vol 2.
3. E M Purcell, Electricity And Magnetism (SI Units) Berkeley Physics Course, Vol. 2, Tata-McGraw Hill.
4. E M Purcell and D J Morin, Electricity and Magnetism (3rd Ed)
5. A Sommerfeld, Electrodynamics: Lectures on Theoretical Physics
6. W Pauli, Electrodynamics, Pauli Lectures on Physics Volume 1, Dover
7. A Zangwill, Modern Electrodynamics, Cambridge Univ Press, 2013.
8. J D Jackson, Classical Electrodynamics, Wiley.
9. M Schwartz, Principles of Electrodynamics, Dover
10. L D Landau and E.M. Lifshitz, The Classical Theory of Fields: Course of Theoretical Physics, Vol. 2, Butterworth Heinemann.
11. A O Barut, Electrodynamics and Classical Theory of Fields and Particles, Dover Publications.

2 Chronology of some developments in electromagnetism

• Thales of Miletus (Greek) c600 BCE: Amber rubbed with fur could attract small light objects.
• W Gilbert (English) 1600: Publication of De Magnete, coined the term electrical. Hypothesised that the amber effect was due to the flow of an electrical fluid.
• C-A de Coulomb (French) 1785: electrostatic force between charged particles
• A Volta (Italian) 1799: electric battery
• H C Oersted (Danish) 1820: electric current deflects a magnetic needle
• J-B Biot and F Savart (French) 1820: Law for magnetic field due to a steady current.

• A-M Ampere (French) 1820-26: force between current carrying wires, circuital law, invented solenoid, electrical telegraph, Memoir on the Mathematical Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience. Maxwell called Ampere the Newton of electricity.

• M Faraday (English) 1831: electromagnetic induction, lines of force, EM fields

• C F Gauss (German) 1835: Gauss' law relating flux of the electric field across a closed surface to the charge enclosed. It turns out a version of this law had already appeared in the work of J-L Lagrange (French) in 1773.

• G Kirchhoff (German) 1845: laws of electrical circuits

• J C Maxwell (Scottish) c1865: displacement current, Maxwell equations, EM waves.

• H Hertz (German) c1886-89: experimentally demonstrated production, transmission and detection of transverse EM (radio) waves validating Maxwell's theory.

• O Heaviside (English) 1884-1902: developed vector calculus, reformulated Maxwell's equations using vector calculus, transmission line theory for electrical telegraph, skin effect, EM fields around a moving charge, magnetic force on moving charged particle, Cerenkov effect

• H A Lorentz (Dutch) 1880-1920: Lorentz force on a charge in an EM field, Lorentz-Fitzgerald contraction, time dilation, covariance of Maxwell's equations in different frames under Lorentz transformations (earlier work by Larmour and full form due to Poincare).

• A Einstein (German) 1905: Special relativity from electrodynamics of moving bodies.

3 Introductory remarks

In the second half of the 19th century, electrodynamics was developed in the intellectual milieu of other continuum mechanical theories such as elasticity and fluid dynamics. By analogy with these, it was supposed that the EM fields existed in a medium dubbed the ether (like the displacement field in an elastic solid). Like sound waves, it was supposed that EM waves and light were propagating waves in ether. It was thought that Maxwell's equations were valid in the frame of the ether. Maxwell, Boltzmann and many others tried to explain the properties of light in terms of those of the ether medium. However, ether had to have very peculiar properties. For example, (a) to allow light to travel very fast, it had to be minimally deformable [i.e., very stiff/rigid, since sound travels faster in a solid than in a gas] but (b) it had to be very rare (not dense) to have evaded detection through its effect on the motion of celestial or terrestrial bodies. With time, models (including so-called ether engines) became very complicated. Moreover, motion relative to the ether was not detected, nor was the expected ether drag on the motion of bodies. The ether concept was eventually abandoned (especially with the work of Einstein) in favor of electric and magnetic fields that could exist in vacuum and in which light could also propagate.

Electromagnetism involves both corpuscular (discrete, particle-like) and field physics. For instance, electric charge is found only in integer multiples of a minimal charge, that of a proton or electron. On the other hand, fields vary (continuously) from point to point. Although the electric and magnetic fields can be described mathematically in isolation, point test charges are used to measure them. Conversely, point charges produce fields of their own. Maxwell's equations (e.g., $\mu_0\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t} = -\mu_0\boldsymbol{j} + \boldsymbol{\nabla}\times\boldsymbol{B}$) govern the dynamics (time evolution) of the electric and magnetic fields in the presence of electric charges and currents. The dynamics of the charges due to the fields is given by the Newton-Lorentz equation of motion $m\ddot{\boldsymbol{r}}(t) = q(\boldsymbol{E}(\boldsymbol{r}(t),t) + \dot{\boldsymbol{r}}(t)\times\boldsymbol{B}(\boldsymbol{r},t))$, which involves the Lorentz force. Although Maxwell's equations are linear equations for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, the Newton-Lorentz equations introduce nonlinearities since the fields generally depend nonlinearly on the particle locations $\boldsymbol{r}$.

• Classical electromagnetic theory has numerous applications: design of Faraday cage and capacitors, electric motors, electric generators, telegraph, transmission of electric currents, wave guides, design of aircraft to evade radar detection, dynamics of charged particles and fields in the plasma of the solar corona or Earth's ionosphere, magnetic confinement fusion in a tokamak, etc.

• Maxwell theory has led to notable theoretical insights including the concept of fields, Lorentz symmetry and the gauge invariance principle (that arose from writing the electric and magnetic fields in terms of scalar and vector potentials). These have had a lasting and continuing impact on the rest of physics (strong interactions, weak interactions, gravity, models for condensed matter, etc.).

4 Vector calculus

4.1 Scalar and vector fields

Scalar fields. At a given instant of time, the pressure $p(\boldsymbol{r})$ in the atmosphere is a real number (scalar) that depends on location $\boldsymbol{r}$. The pressure function is an example of a scalar field defined on a region in 3d Euclidean space. A scalar field assigns a real number to each point $\boldsymbol{r}$. We will suppose that the real number varies sufficiently smoothly as the location changes. This is usually physically justified and allows us to use tools from calculus. Simple examples include (i) $p(\boldsymbol{r}) = \boldsymbol{a}\cdot\boldsymbol{r}$ where $\boldsymbol{a}$ is a fixed vector and (ii) $p(\boldsymbol{r}) = \sigma\boldsymbol{r}\cdot\boldsymbol{r} = \sigma(x^2+y^2+z^2)$ where σ is a real constant. A scalar field on the plane may be visualized via a contour plot: a plot of the level curves (curves along which the field is constant) of the scalar function in the region of interest. For example, the level curves of x^2+y^2 are concentric circles centered at the origin. For a function of three variables, level curves are replaced by level surfaces in $\mathbb{R}^3$. The level surfaces of $\boldsymbol{a}\cdot\boldsymbol{r}$ are planes perpendicular to the fixed vector $\boldsymbol{a}$.

Vector fields. Similarly, we have the concept of a vector field: a smoothly varying vector $\boldsymbol{v}(\boldsymbol{r})$ at each location $\boldsymbol{r}$. The gravitational force felt by a point mass m at various points above the Earth's surface defines a vector field. A vector field may be visualized by drawing arrows pointing along $\boldsymbol{v}(r)$ at each point $\boldsymbol{r}$. The magnitude may be encoded in the lengths of the arrows. The flow velocity at each point of a steadily flowing fluid is an example of a vector field. A vector field on the plane is of the form $\boldsymbol{v} = f(x,y)\hat{x} + g(x,y)\hat{y}$. For instance, $\boldsymbol{v} = \hat{x}$ is a constant vector field of unit length that points in the $\hat{x}$ direction everywhere. A radially outward pointing vector field is $\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{r} = x\hat{x} + y\hat{y}$, its magnitude at $\boldsymbol{r}$ is equal to the distance from the origin.

4.2 Gradient, divergence and curl

Gradient of a scalar field. Given a scalar field $\phi(\boldsymbol{r})$, its gradient is a vector field denoted grad $\phi(\boldsymbol{r})$ or $\boldsymbol{\nabla}\phi(\boldsymbol{r})$ where $\boldsymbol{\nabla}$ is the Greek letter nabla. In Cartesian coordinates,

$$\text{grad}\, \phi(\boldsymbol{r}) = \boldsymbol{\nabla}\phi(\boldsymbol{r}) = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}. \tag{1}$$

For many purposes, we may regard $\boldsymbol{\nabla}$ as the vector first order differential operator $\boldsymbol{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$. Examples: (i) If $\phi(x,y) = x$ then $\boldsymbol{\nabla}\phi = \hat{x}$ is a constant vector field pointing in the x direction at all points of $\mathbb{R}^2$. (ii) If $\phi = \frac{1}{2}(x^2+y^2+z^2)$, then $\boldsymbol{\nabla}\phi = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r}$ is a radially outward pointing vector field on $\mathbb{R}^3$, with magnitude equal to the distance from the origin. The gradient allows us to write the first order Taylor polynomial in a nice way. Recall that for a differentiable function $\phi(x,y,z)$, we have

$$\phi(\boldsymbol{r} + \delta\boldsymbol{r}) \approx \phi(\boldsymbol{r}) + \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z \tag{2}$$

where $\delta\boldsymbol{r} = \delta x\,\hat{x} + \delta y\,\hat{y} + \delta z\,\hat{z}$. We recognize the last three terms as the dot product of $\boldsymbol{\nabla}\phi$ and $\delta\boldsymbol{r}$. Thus $\phi(\boldsymbol{r}+\delta\boldsymbol{r}) \approx \phi(\boldsymbol{r}) + \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi$. This formula holds in any dimension.
• Example. Show that $\boldsymbol{\nabla}(1/r) = -\hat{r}/r^2$.

Interpretation of gradient. At any location $\boldsymbol{r}$, $\boldsymbol{\nabla}\phi$ is a vector that points in the direction of most rapid increase of ϕ. To see why, it is helpful to introduce the level surfaces of ϕ, which are surfaces in $\mathbb{R}^3$ on which ϕ is a constant. For ϕ to change most rapidly, we must move from $\boldsymbol{r}$ along a vector that has no component along the level surface through $\boldsymbol{r}$, i.e., orthogonal to the level surface. We will now argue that at any point $\boldsymbol{r}$, $\boldsymbol{\nabla}\phi(\boldsymbol{r})$ is also orthogonal to the level surface through $\boldsymbol{r}$. Suppose $\boldsymbol{v}$ is a vector at $\boldsymbol{r}$ of small magnitude, then the linear Taylor approximation gives $\phi(\boldsymbol{r}+\boldsymbol{v}) \approx \phi(\boldsymbol{r}) + \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi$. Now, $\boldsymbol{v}$ is tangent to the level surface through $\boldsymbol{r}$ if $\phi(\boldsymbol{r}+\boldsymbol{v}) - \phi(\boldsymbol{r})$ vanishes to first order in $\boldsymbol{v}$. This happens precisely when $\boldsymbol{v}\cdot\boldsymbol{\nabla}\phi = 0$. Thus, $\boldsymbol{\nabla}\phi$ must be perpendicular to the level surface of ϕ and must point either in the direction of most rapid increase or decrease of ϕ. Taking $\boldsymbol{v} = \epsilon\boldsymbol{\nabla}\phi$ for $0 < \epsilon \ll 1$, we find that $\phi(\boldsymbol{r}+\epsilon\boldsymbol{\nabla}\phi) \approx \phi(\boldsymbol{r}) + \epsilon|\boldsymbol{\nabla}\phi|^2 > \phi(\boldsymbol{r})$. Thus we conclude that $\boldsymbol{\nabla}\phi$ must point in the direction of most rapid increase of ϕ.

If ϕ is regarded as a potential function, then its level surfaces are referred to as equipotential surfaces. E.g., (i) For $\phi(x,y) = x$, the level curves are lines parallel to the y axis, and $\boldsymbol{\nabla}\phi = \hat{x}$ points perpendicular to these lines in the direction of most rapid increase in ϕ. (ii) For $\phi = \frac{1}{2}(x^2+y^2+z^2)$, the level surfaces are concentric spheres centered at the origin and $\boldsymbol{\nabla}\phi = \boldsymbol{r}$ is perpendicular to these surfaces.

Divergence of a vector field. The divergence of a vector field $\boldsymbol{v}$, denoted $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ or div $\boldsymbol{v}$ is a scalar field. In Cartesian coordinates, it is defined as $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$. Intuitively, the divergence measures how much a vector field is expanding or contracting at a point. For instance, if $\boldsymbol{v}$ is the velocity field of a flowing fluid, then

$\nabla \cdot \boldsymbol{v}(\boldsymbol{r})$ is the fractional rate of change of volume V of a small parcel of fluid at r

$$\nabla \cdot \boldsymbol{v} = \lim_{V \to 0} \frac{1}{V} \frac{dV}{dt}. \tag{3}$$

A vector field with constant Cartesian components has zero divergence: the arrows representing the vector field are parallel and have the same magnitude everywhere. The vector field $\boldsymbol{v} = x\hat{x} + y\hat{y} + z\hat{z}$ should be expected to have a positive divergence everywhere as it represents a radially expanding flow: in fact, we check that it has divergence 3 everywhere. In other words, the flow is expanding by the same amount everywhere, not just at the origin.

If $\boldsymbol{v} = \nabla\phi$ is a gradient, then its divergence

$$\nabla \cdot \boldsymbol{v} = \nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{4}$$

is a scalar field called the Laplacian of ϕ and denoted $\nabla^2\phi$. The 'Laplace operator' $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is a rotationally invariant second order differential operator.
• For example, it is instructive to find the Laplacian of the scalar function $1/r$. This is the divergence of the radially inward vector field $\boldsymbol{v} = \nabla(1/r) = -\boldsymbol{r}/r^3$. Show that $\nabla \cdot \boldsymbol{v} = 0$ as long as $r \neq 0$.

Curl of a vector field. The curl of a vector field $\boldsymbol{v}$ is the vector field whose Cartesian components are given by

$$\nabla \times \boldsymbol{v} = (\partial_y v_z - \partial_z v_y)\hat{x} + (\partial_z v_x - \partial_x v_z)\hat{y} + (\partial_x v_y - \partial_y v_x)\hat{z}. \tag{5}$$

Here, ∂_x is short-hand for $\frac{\partial}{\partial x}$, etc. The 2^{nd} and 3^{rd} terms are obtained from the 1^{st} by cyclically permuting $x \to y \to z \to x$. In components, we may write $(\nabla \times \boldsymbol{v})_i = \epsilon_{ijk}\partial_i v_j$ where ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol: $\epsilon_{123} = 1$ and it is antisymmetric under interchange of any pair of indices. Being formally a cross product of ∇ and $\boldsymbol{v}$, the formula for the curl should remind us of the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = (yp_z - zp_y)\hat{x} + (zp_x - xp_z)\hat{y} + (xp_y - yp_x)\hat{z}$ of a particle. In fact, the curl is a measure of local rotation in a vector field. If $\boldsymbol{v}$ is the velocity field of a fluid, then $\nabla \times \boldsymbol{v}$ is the vorticity field of the flow and is a local measure of angular velocity or circulation. For example, consider $\boldsymbol{v} = \Omega r\hat{\theta} = \Omega(-y\hat{x} + x\hat{y})$, which is the velocity vector field of a fluid that is uniformly rotating about the $\hat{z}$ axis at angular velocity Ω. Upon calculating its curl we find $\nabla \times \boldsymbol{v} = 2\Omega\hat{z}$. Thus, we may interpret the curl of a vector field as twice the local angular velocity of rotation.

4.2.1 Leibniz product rule for grad, div and curl

• The Leibniz rule for the gradient of a product of scalar functions

$$\nabla(fg) = f\nabla g + (\nabla f)g \tag{6}$$

• Leibniz rule for the divergence of a scalar multiple of a vector field

$$\nabla \cdot (f\boldsymbol{v}) = \nabla f \cdot \boldsymbol{v} + f\nabla \cdot \boldsymbol{v}. \tag{7}$$

• The Leibniz rule for the curl of a scalar multiple of a vector field

$$\nabla \times (f\boldsymbol{v}) = (\nabla f) \times \boldsymbol{v} + f\nabla \times \boldsymbol{v}. \tag{8}$$

4.2.2 Second order derivatives and Helmholtz decomposition

Curl-free and divergence-free vector fields and the Helmholtz decomposition. (i) There is a simple way of constructing a curl-free vector field: take the gradient of a scalar. A gradient vector field has vanishing curl due to the equality of mixed partials:

$$\nabla \times \nabla\phi = (\partial_y\partial_z\phi - \partial_z\partial_y\phi)\hat{x} + (\partial_z\partial_x\phi - \partial_x\partial_z\phi)\hat{y} + (\partial_x\partial_y\phi - \partial_y\partial_x\phi)\hat{z} = 0. \tag{9}$$

Intuitively, a gradient vector field points in the direction of most rapid increase of ϕ and cannot 'circulate'. (ii) Similarly, there is a nice way of constructing a divergence-free vector field: take the curl of any vector field $\boldsymbol{A}$. In other words, the divergence of a curl is identically zero:

$$\nabla\cdot(\nabla\times\boldsymbol{A}) = \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x) = 0, \tag{10}$$

again by the equality of mixed partials. (iii) The Helmholtz decomposition expresses a vector field, quite generally, as a sum of curl-free and divergence-free vector fields: $\boldsymbol{v} = \nabla\phi + \nabla \times \boldsymbol{A}$.

• The **curl of a curl** appears when we write the magnetic field in terms of the vector potential $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in Ampere's law for $\nabla \times \boldsymbol{B}$.

$$\nabla \times (\nabla \times \boldsymbol{A}) = -\nabla^2\boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A}). \tag{11}$$

The curl of a curl is a second order vector differential operator acting on vector fields. There are two other such second order differential operators: the vector Laplacian and the gradient of the divergence. The RHS is a linear combination of these two. We may show this by using the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

4.3 Fundamental theorem of calculus, Stokes', Green's and Gauss' integral theorems

Line integral. Given a vector field $\boldsymbol{v}(\boldsymbol{r}) = (v_x, v_y, v_z)(\boldsymbol{r})$ in three-dimensional space and a parametrized curve $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ for $0 \le t \le 1$, we may define the 'line integral' or 'contour integral' of $\boldsymbol{v}$ along $\boldsymbol{\gamma}$ as the real number

$$\int_{\boldsymbol{\gamma}} \boldsymbol{v}\cdot d\boldsymbol{\gamma} = \int_0^1 \boldsymbol{v}\cdot\frac{d\boldsymbol{\gamma}}{dt}dt = \int_0^1 \left[v_x\frac{dx}{dt} + v_y\frac{dy}{dt} + v_z\frac{dz}{dt}\right]dt. \tag{12}$$

Here, $\dot{\boldsymbol{\gamma}} = \frac{d\boldsymbol{\gamma}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$ is a vector field along the curve $\boldsymbol{\gamma}$ (it is not defined elsewhere in $\mathbb{R}^3$). At each fixed t, it is the tangent vector to the curve at the point $\boldsymbol{\gamma}(t)$ as shown in Fig. 1.

For example, if $\boldsymbol{\gamma}$ is the helix $(\cos t, \sin t, t)$, then $d\boldsymbol{\gamma} = (-\sin t, \cos t, 1)dt$. We may consider $d\boldsymbol{\gamma}$ as the differential of the map $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^3$. The work done by a force field $\boldsymbol{F}(\boldsymbol{r})$ in moving a particle along a curve $\boldsymbol{\gamma}$ is an important example of a line integral: $W_{\boldsymbol{F}}(\boldsymbol{\gamma}) = \int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{\gamma}$.

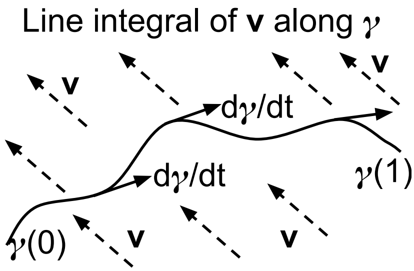

Figure 1: Contour γ along which the line integral of the vector field $\boldsymbol{v}$ is evaluated.

Fundamental theorem of calculus for line integrals. In general, the line integral depends on the values of $\boldsymbol{v}$ all along the curve $\boldsymbol{\gamma}$. However, if $\boldsymbol{v}$ is the gradient of a scalar, $\boldsymbol{v} = \boldsymbol{\nabla}\phi$, then the line integral can be evaluated in terms of the values of ϕ at the endpoints:

$$\begin{aligned} \int_\gamma \boldsymbol{\nabla}\phi \cdot d\boldsymbol{\gamma} &= \int_0^1 \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_0^1 \frac{d\phi(\boldsymbol{r}(t))}{dt} dt = \phi(\boldsymbol{r}(1)) - \phi(\boldsymbol{r}(0)). \end{aligned} \tag{13}$$

Here, we viewed $\phi(x(t), y(t), z(t))$ as a function of t and used the chain rule to differentiate it with respect to t. In particular, if $\boldsymbol{\gamma}$ is a closed curve, then $\boldsymbol{r}(0) = \boldsymbol{r}(1)$ and the line integral of a gradient vanishes $\oint_\gamma \boldsymbol{\nabla}\phi \cdot d\boldsymbol{\gamma} = 0$. Here $\oint$ denotes a line integral around a closed contour.

A vector field that is the gradient of a scalar field is called a gradient vector field. In mechanics, if a force field $\boldsymbol{F}(\boldsymbol{r})$ is the (negative) gradient of a scalar field (or 'potential' $\phi(\boldsymbol{r})$), then it is called a conservative force field. The work done by a conservative force field $-\boldsymbol{\nabla}\phi$ depends only on the initial and final locations of the particle, and not on other details of the path taken. A conservative force field does no work in moving a particle around a closed curve.

We have shown that the integral of a gradient $\boldsymbol{\nabla}\phi$ along a contour $\boldsymbol{\gamma} : [0, 1] \to \mathbb{R}$ is equal to the difference between the values of ϕ at the end and beginning of the contour. The latter difference may itself be viewed as an integral over the boundary of $\boldsymbol{\gamma}$, denoted $\partial\boldsymbol{\gamma}$, which is the set consisting of two points: the final and initial points on the contour. Thus, we rewrite (13) as

$$\int_\gamma \boldsymbol{\nabla}\phi \cdot d\boldsymbol{\gamma} = \int_{\partial\gamma} \phi = \phi(\boldsymbol{\gamma}(1)) - \phi(\boldsymbol{\gamma}(0)). \tag{14}$$

The minus sign for the initial point is to take into account that $\boldsymbol{\gamma}$ is oriented from initial to final point. There are higher dimensional analogs of this integral formula, concerning the surface integral of a curl (Stokes' theorem) and the volume integral of a divergence (Gauss' theorem).

Stokes' theorem. Suppose $\boldsymbol{v}$ is a vector field and S a 2d surface (with area element $d\boldsymbol{S}$) with boundary given by the curve ∂S. Stokes' theorem says that

$$\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{S} = \oint_{\partial S} \boldsymbol{v} \cdot d\boldsymbol{l}. \tag{15}$$

In particular, if $\boldsymbol{v} = \nabla \phi$ is the gradient of a scalar, then $\nabla \times \boldsymbol{v} = 0$ from (9) and Stokes' theorem says that $\oint \boldsymbol{v} \cdot d\boldsymbol{l} = 0$.

Gauss' theorem. On the other hand, suppose Ω is a 3d region bounded by the surface $\partial\Omega$, then Gauss' divergence theorem states that

$$\int_\Omega (\nabla \cdot \boldsymbol{v}) d\boldsymbol{r} = \int_{\partial\Omega} \boldsymbol{v} \cdot d\boldsymbol{S}. \tag{16}$$

Green's theorem. Green's theorem is a planar version of Stokes' theorem. Suppose $\boldsymbol{v} = v_x \hat{x} + v_y \hat{y}$ is a vector field on the x-y plane and let S be a region in the plane bounded by the curve ∂S. Then $\nabla \times \boldsymbol{v}$ has only a z component while $d\boldsymbol{S} = dx\, dy\, \hat{z}$ and $d\boldsymbol{l} = dx\, \hat{x} + dy\, \hat{y}$, so that (15) becomes

$$\int_S (\partial_x v_y - \partial_y v_x) dx\, dy = \oint_{\partial S} (v_x dx + v_y dy). \tag{17}$$

Green's theorem leads to a line integral representation of the area of a region S on the plane. We pick a planar vector field $\boldsymbol{v}$ such that the z-component of its curl is a constant, say $v_x = -y, v_y = x$. Then

$$\text{Area}(S) = \int_S dx dy = \frac{1}{2} \oint_{\partial S} (x\, dy - y\, dx). \tag{18}$$

• **Stokes' theorem in general.** In all these cases, on the left we have the integral of an exact differential (gradient of scalar, curl of a vector, divergence of a scalar) of some quantity ω over some space M. The RHS is an integral of the same quantity ω over the boundary of M. Thus Stokes' theorem may in general be written as

$$\int_M d\omega = \int_{\partial M} \omega. \tag{19}$$

5 Electrostatics

5.1 Electric charge, Coulomb's law for electric force

• **Electric charge.** Macroscopic bodies like rocks and planets are electrically neutral. However, it was found that some bodies can be charged by friction: Thales of Miletus (c 600 BCE) is reported to have observed that amber (fossilized tree resin) rubbed with fur attracts small light objects like hair. Conventionally we say that the amber and fur become negatively and positively charged respectively. Similarly, rubbing a glass rod with silk makes the glass positively charged and silk negatively charged. The microscopic explanation is that electrons have been transferred from the glass to the silk. The phenomenon is the triboelectric effect or 'static electricity'.

• According to the empirically deduced **principle of charge conservation**, electric charge cannot be created or destroyed. Moreover, the total charge of a collection of bodies with charges $q_1, \cdots, q_n$ is the algebraic sum of the individual charges $q_{\text{tot}} = q_1 + \cdots + q_n$.

• A **point charge** is a useful idealization and models a charged body whose linear dimensions are small compared to the scales of interest, such as the separation between bodies.

• **Coulomb's law** for the forces between point charges q_1 and q_2 in free space (vacuum) is deduced from experiment. They are central, proportional to the product of charges, inversely proportional to the square of the separation and form an action-reaction pair:

$$\boldsymbol{F}_{1 \text{ on } 2} = \frac{q_1 q_2 (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{4\pi\epsilon_0 |\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \quad \text{and} \quad \boldsymbol{F}_{2 \text{ on } 1} = -\boldsymbol{F}_{1 \text{ on } 2}. \tag{20}$$

Electric charge (unlike mass) can have either sign. The force between charges of the same sign is repulsive while unlike charges attract. In SI units, the charges are measured in Coulombs. A Coulomb is a rather large unit of charge by microscopic standards: the charge of a proton is about $1.602 \times 10^{-19} C$. Since macroscopic electrical phenomena typically involve very large numbers ($\sim 10^{24}$) of elementary charge carriers, for many practical purposes, the corpuscular character of charge is smoothed out and one speaks of charge varying continuously.

• The proportionality constant $(4\pi\epsilon_0)^{-1} \approx 9 \times 10^9 \; Nm^2/C^2$ while the **permittivity of free space** is $\epsilon_0 \approx 8.85 \times 10^{-12} \; C^2/Nm^2$. It is used to relate the units for electrical charge to the mechanical units for force and distance. This constant can be eliminated by changing the unit of electrical charge (say, to esu), as is done in gaussian (CGS) units. The name permittivity is used for historical reasons. It acquires a physical significance when we ask about the force between charges in a dielectric/polarizable medium rather than in vacuum. The force between charges is reduced by a factor ϵ_r called the relative permittivity, since each of the charges is slightly screened by the polarization of the medium. Thus, roughly, the name (relative) permittivity is meant to convey the extent to which a medium 'permits' polarization and thereby decreases the electrostatic force between charges.

• **Remark on units.** It may be borne in mind that SI units, although in wide use, have conceptual shortcomings. We will see that they assign different dimensions to electric and magnetic fields and thereby partly obscure the Lorentz symmetry that relates electric and magnetic fields under a Lorentz boost.

• Due to the neutrality of macroscopic matter, electric forces between macroscopic bodies are rather weak (comparable to or smaller than Earth's gravity). They are manifested in contact forces such as friction. On the other hand, on atomic scales there is significant charge separation, say between the nucleus and electrons and electrical forces are the dominant ones.

• **Gravity vs electrostatic force.** Compare the magnitudes of the electrostatic force between a pair of protons to the Newtonian gravitational force between them.

• It is empirically found that the electrostatic force obeys the **superposition principle**. The force due to two charges on a third is the vector sum of the individual forces: $\boldsymbol{F}_{1\ \&\ 2\ \text{on}\ 3} = \boldsymbol{F}_{1\ \text{on}\ 3} + \boldsymbol{F}_{2\ \text{on}\ 3}$. Thus, we may model the force as a vector satisfying the parallelogram law of addition. This is the principle that allows us to extend Coulomb's law to deal with several charges by reducing the force to a sum of two-body forces.