

Notes for Classical Mechanics 2, CMI, Spring 2023
Govind S. Krishnaswami, April 25, 2023

Please let me know at govind@cmi.ac.in of any comments or corrections

Course website <http://www.cmi.ac.in/~govind/teaching/cm2-e23>

Contents

0.1	Degrees of freedom and instantaneous configurations	2
0.2	Phase space, dynamical variables, conserved quantities	2
0.3	Lagrangian formulation and principle of extremal action	4
0.4	Nonuniqueness of Lagrangian	9
0.5	Conjugate momenta, their geometric meaning and cyclic coordinates	9
0.6	Coordinate invariance of the form of Lagrange's equations	11
0.7	Hamiltonian and its conservation	12
0.8	Symmetries to conserved quantities: Noether's theorem	13
0.9	Hamilton's equations	16
0.10	Legendre transform: Hamiltonian from Lagrangian	18
0.11	Variational principles for Hamilton's equations	21
0.12	Canonical Poisson brackets	23
0.13	Canonical transformations	28
0.13.1	From point transformations to canonical transformations	28
0.13.2	CTs preserve basic PBs, Time evolution as a CT	30
0.13.3*	Form of Hamilton's equations is preserved iff PBs are preserved	31
0.13.4	Canonical transformations and area-preserving maps	34
0.13.5*	CTs preserve formula for PB of any pair of observables	35
0.13.6	Generating function for finite canonical transformations from variational principle	37
0.14	Kepler's problem of two gravitating point masses	40
0.14.1	CM and relative variables, conservation laws	40
0.14.2	Planetary orbits	42
0.14.3	Time period of elliptical orbits	47
0.15	Small oscillations and normal modes	47
0.15.1	Normal modes of two weakly coupled pendula	48
0.15.2	Normal modes of a diatomic molecule	49
0.16	Constrained extremization via Lagrange multipliers	51
0.17	Rigid body mechanics	52
0.17.1	Lab and comoving frames	53
0.17.2	Configuration space and degrees of freedom	54
0.17.3	Infinitesimal displacement and angular velocity of rigid body	55
0.17.4	Kinetic energy and inertia tensor	56
0.17.5	Types of rigid bodies	58
0.17.6	Angular momentum of a rigid body	60
0.17.7	Equations of motion of a rigid body	61
0.17.8	Force-free motion of rigid bodies	64

0.1 Degrees of freedom and instantaneous configurations

A point particle moving in a central force field in three-dimensional space has three *degrees of freedom*, we need three coordinates to specify the location of the particle at any instant of time (and in particular at the initial time, say $t = 0$). The number of degrees of freedom is the dimension of the space of possible instantaneous locations of the particle. The number of degrees of freedom does not depend on the nature of forces. So a free particle moving in three-dimensional space also has three degrees of freedom. On the other hand, a bead that is constrained to move along a fixed wire (say, because the wire passes through the bead) has only one degree of freedom if it is not allowed to spin. A particle constrained to move on a fixed spherical surface (such as a bob at the end of a rigid rod whose other end is attached to a pivot) has only two degrees of freedom. These two are examples of so-called holonomic constraints. For a particle moving in 3d Euclidean space, a constraint of the form $C(x, y, z, t) = 0$ is called a holonomic constraint. We see that each such constraint reduces the number of degrees of freedom by one. Two point masses moving in three dimensional space (e.g., the Sun and the Earth¹) have six degrees of freedom. One does not need to know about the nature of trajectories to count the number of degrees of freedom. A fluid consisting of $N \sim 10^{24}$ molecules in a bucket has a very large number² of degrees of freedom, which can be taken to be the $3N$ Cartesian coordinates needed to specify the locations of the N molecules, treated as point masses. An *instantaneous configuration* of a system of two point particles is any possible location of the two particles. The set of all instantaneous configurations of a mechanical system is called its configuration space \mathcal{Q} . For a pair of point particles moving in three dimensions (\mathbb{R}^3), \mathcal{Q} is the space (more accurately, *manifold*) \mathbb{R}^6 with coordinates given, for instance, by the Cartesian components of the positions of each of the particles x_1^i, x_2^j for $i, j = 1, 2, 3$. Conventionally, the component indices on coordinates on \mathcal{Q} are denoted by superscripts. The number of degrees of freedom is the dimension of the configuration space. It is important to bear in mind that \mathcal{Q} is in general *not* a linear space and typically cannot be given the structure of a vector space³. For a particle constrained to move on the surface of a sphere embedded in three-dimensional Euclidean space, the configuration space is the so-called two-sphere, denoted S^2 . The latter is not a vector space, we do not have a notion of ‘adding’ points on the sphere to get points on the sphere.

0.2 Phase space, dynamical variables, conserved quantities

Phase space. Being 2nd order in time, Newton’s equation requires both the initial position \mathbf{r} and velocity $\dot{\mathbf{r}}$ (or momentum \mathbf{p}) as initial conditions. Knowledge of current

¹Here we do not restrict to a particular orbit of the Earth around the Sun (treated as point masses) but ask how many coordinates are needed to specify all possible initial locations of the Sun and the Earth *prior* to solving Newton’s equations of motion.

²A fluid is often treated as a *field*, i.e., as a system with infinitely many degrees of freedom.

³A real vector space is a set of vectors $\mathbf{v}, \mathbf{w}, \dots$ that is closed under linear combinations, i.e., $\alpha\mathbf{v} + \beta\mathbf{w}$ is also a vector for any real numbers α, β . In other words, we may add vectors and multiply them by real scalars.

position and momentum determines the subsequent trajectory via Newton's 2nd law⁴. We say that the *state* of the particle is specified by giving its instantaneous position *and* momentum. The set M of possible instantaneous states of the particle is called its state space or *phase space*. For a particle moving in 3d space, its configuration space is $\mathcal{Q} = \mathbb{R}^3$ and its phase space is $M = \mathbb{R}^6$ (locations and momenta).

Phase portrait. The path of the particle $\mathbf{r}(t)$ (satisfying Newton's equation and initial conditions) in configuration space is called its trajectory. A trajectory is a directed curve, the direction being that of increasing time. Also of interest is the trajectory in phase space $(\mathbf{x}(t), \mathbf{p}(t))$. Consider the phase plane trajectories for a free particle moving on the real line ($m\ddot{x} = \dot{p} = 0$) displayed in Fig. 1. Since energy is conserved, these trajectories must lie inside level sets⁵ of energy $E = p^2/2m$. But in general, an energy level set may be a union of several trajectories. For $E > 0$, the energy level sets of a free particle consist of precisely two trajectories: the horizontal lines of fixed $p = \pm\sqrt{2mE}$. Trajectories are directed toward increasing x for $p > 0$ and vice versa. In general, we will extend trajectories as far forward (and backward) in time as possible. See Fig. 1 for the *phase portrait* of a free particle. Notice that the $E = 0$ level set is a union of infinitely many static solutions ($x = x(0), p = 0$).

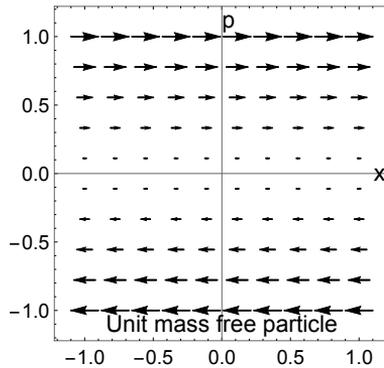


Figure 1: Phase portrait of particle subject to no forces, free to move on the real line. Every point on the x axis is a static solution with $p = 0$; together, they form the zero energy level set. The lengths of the arrows are proportional to the speed of the particle.

Dynamical variables and conserved quantities. Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is called the moment of linear momentum⁶. Its evolution is governed by the moment of the force or torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$:

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = m\dot{\mathbf{x}} \times \dot{\mathbf{r}} + \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}. \quad (1)$$

⁴Thus, mere specification of a point on \mathcal{Q} does *not* determine the trajectory. Interestingly, in the quantum theory (see §??), for reasons having to do with the Heisenberg uncertainty principle, the Schrödinger wave function $\psi(q)$ is a function on the classical configuration space rather than the phase space!

⁵By a level set of energy we mean the set of points in phase space where the energy takes a specific numerical value.

⁶The word moment has a different meaning in probability theory: the n^{th} moment of the probability distribution $\rho(x)$ is $\langle x^n \rangle = \int x^n \rho(x) dx$.

Notably, both \mathbf{L} and $\boldsymbol{\tau}$ depend on the chosen origin for the radius vector \mathbf{r} .

The components of position, momentum, angular momentum and total energy $E = \mathbf{p}^2/2m + V(\mathbf{r})$ are interesting physical quantities associated with the dynamics of a particle. They are examples of *dynamical variables* or *observables*. In general, a dynamical variable is a smooth (infinitely differentiable, denoted C^∞) or at least sufficiently differentiable real-valued function on phase space. For a single particle, dynamical variables may be regarded as functions $f(\mathbf{r}, \mathbf{p})$. The potential $V(\mathbf{r})$ is a function on configuration space and so also a function on phase space independent of momenta. While x^i are called coordinate functions on configuration space, the components of position and momentum x^i and p_j are called coordinate functions on phase space. *Conserved quantities* are dynamical variables that are constant along *every* trajectory. Of course, the value of a conserved quantity may differ⁷ from trajectory to trajectory. Conserved quantities are also called constants or integrals of motion.

0.3 Lagrangian formulation and principle of extremal action

Hamilton's variational principle of extremal action (1835) provides a powerful reformulation of Newton's equations of motion, especially for systems with conservative forces. In this case, it leads to Lagrange's equations of motion, which are equivalent to Newton's equations. One advantage of Lagrange's equations is that unlike Newton's equations, they retain the same form in all systems of coordinates on configuration space (see §0.6). What is more (for conservative systems), a single real-valued function (the Lagrangian) can be used to generate all the equations of motion simply by differentiating it with respect to the various coordinates and velocities. This also makes it easier to change from one coordinate system to another since one only needs to transform one function rather than each component of the vector forces. Constraints among the various coordinates (such as those that force a particle to move on a curve or surface) can also be accounted for without worrying about the forces that are responsible for enforcing the constraints. However, all this is not free of cost: (a) the Lagrangian formulation does not directly apply to dissipative systems, unlike the Newtonian formulation where a frictional force can easily be added and (b) the principle of extremal action relies on the calculus of infinitely many variables, called the variational calculus of paths on configuration space, unlike Newton's equation that is based on the calculus of finitely many variables parameterizing points on configuration space. Nevertheless, once one learns the methods of the calculus of variations, it can be applied to other optimization problems like Plateau's problem on finding the shapes of soap films or the Brachistochrone problem (Prob. ??). A still deeper benefit is that the concept of the action plays a central role in the formulation of the quantum theory via Feynman's path integral.

Extremal action principle. The idea of the action or variational principle is as follows. A static solution (time-independent trajectory) of Newton's equation for a par-

⁷The value of a conserved quantity must vary smoothly or continuously (rather than discontinuously) from one trajectory to a neighboring trajectory. If we dropped this condition, we could cook up a limitless supply of conserved quantities for any system simply by assigning a randomly chosen real number to each trajectory on phase space.

ticle in a potential ($m\ddot{x} = -V'(x)$) occurs when the particle is located at an extremum of the potential. The action principle gives a way of identifying (possibly) time-dependent trajectories as extrema of an action function. However, unlike the potential, the relevant action is not a function on configuration space but a function on the *space of differentiable paths* on configuration space⁸. This is quite natural since instead of identifying one (or more) among the points of \mathcal{Q} as a static solution, we wish to select one (or more) among the paths on \mathcal{Q} as a trajectory. Suppose $q^i(t)$ for $t_1 \leq t \leq t_2$ is a differentiable path on \mathcal{Q} between specified initial and final times⁹. Then the action is typically a functional¹⁰ of the form

$$S[q] = \int_{t_1}^{t_2} L(q^i, \dot{q}^i) dt. \quad (3)$$

Here, $L(q^i, \dot{q}^i)$ is called the Lagrangian of the system, a function of coordinates and velocities. For a suitable L (invariably the difference between kinetic and potential energies, $L = T - V$), we will see that Newtonian trajectories are extrema of S .

In other words, we consider the problem of determining the trajectory that a particle might take if it is at q_1 at t_1 and at q_2 at t_2 . Instead of specifying the initial position and velocity (as in an initial value problem), we give the initial and final positions at these times. Which among all the paths that connect these points solves Newton's equation? The action principle says that classical trajectories are extrema of S . Note that unlike the initial value problem where $q^j(t_1), \dot{q}^j(t_1)$ are specified, this initial-final value problem (where $q^j(t_1)$ and $q^j(t_2)$ are specified), may not have a unique solution. The action may have more than one extremum.

Euler-Lagrange equations. To understand this idea, we need to determine the conditions for S to be extremal. These conditions are called Euler-Lagrange equations. For the case of static or time-independent solutions discussed above, the condition for $V(x)$ to be extremal is that its change under an infinitesimal shift δx of x must vanish to first order in δx , this turns out to be the condition $V'(x) = 0$.

⁸The space of differentiable paths on configuration space is an infinite-dimensional space. This is already the case for a system with just one degree of freedom, where a path between q_1 at t_1 and q_2 at t_2 is simply a differentiable function of time $q(t)$ with these initial and final values. If we imagine expanding $q(t)$ in a Fourier series, the Fourier coefficients are freely specifiable as long as the series converges. These infinitely many coefficients furnish coordinates on the space of such paths. For example, a Fourier series for $q(t)$ for $t_1 \leq t \leq t_2$ with $q(t_1) = q_1$ and $q(t_2) = q_2$ is given by

$$q(t) = q_1 + (q_2 - q_1) \frac{t - t_1}{t_2 - t_1} + \sum_{n=1}^{\infty} a_n \sin(n\pi(t - t_1)/(t_2 - t_1)), \quad (2)$$

where a_1, a_2, \dots are the Fourier coefficients.

⁹It is common to use the symbol q^i (instead of x^i) for coordinates on configuration space. They are called *generalized* coordinates: q^i need not be Cartesian coordinates of particles and need not have dimensions of length, any system of coordinates (with appropriate dimensions) for the degrees of freedom of the system will work.

¹⁰The action S is a function of the function $q(t)$. Such objects are called functionals. However, S does not depend on t , since it is integrated.

**Infinitesimal variation of a path
on configuration space \mathcal{Q}**

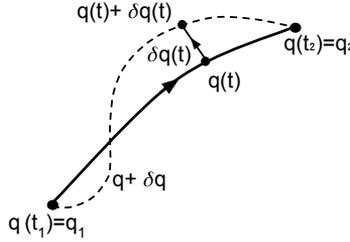


Figure 2: Infinitesimal variation of a path while holding the initial and final configurations $q(t_1) = q_1$ and $q(t_2) = q_2$ fixed.

The Euler-Lagrange equations are got by computing the infinitesimal change δS in action, under a small change in path¹¹ $q^i(t) \rightarrow q^i(t) + \delta q^i(t)$ while holding the initial and final locations $q^i(t_1), q^i(t_2)$ unchanged ($\delta q^i(t_1) = \delta q^i(t_2) = 0$). Assuming the variation (this is what gives the principle its name) in the path is such that $\frac{d\delta q(t)}{dt} = \delta \dot{q}$, we get

$$\begin{aligned} \delta S &= \sum_{i=1}^n \int_{t_1}^{t_2} dt' \left\{ \frac{\partial L}{\partial q^i} \delta q^i(t') + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i(t') \right\} + \mathcal{O}(\delta q)^2 \\ &= \int_{t_1}^{t_2} \delta q^i(t') \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt'} \frac{\partial L}{\partial \dot{q}^i} \right) dt' \\ &\quad + \delta q^i(t_2) \frac{\partial L}{\partial \dot{q}^i(t_2)} - \delta q^i(t_1) \frac{\partial L}{\partial \dot{q}^i(t_1)} + \mathcal{O}(\delta q)^2, \end{aligned} \quad (4)$$

where a sum over the repeated index i is implied. We integrated by parts to isolate the coefficient of δq . The last two ‘boundary terms’ are zero due to the initial and final conditions. For δS to vanish to first order in infinitesimals, the integral must vanish. Now $\delta q^i(t')$ is an arbitrary function for $t_1 < t < t_2$. So the only way the integral can vanish is for the quantity in parenthesis to be identically zero at all such times¹². Thus, we get the *Euler-Lagrange* (EL) (or just Lagrange’s) equations

$$\frac{\partial L}{\partial q^i(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i(t)} = 0 \quad \text{for } i = 1, 2, \dots, n. \quad (5)$$

Now, let us see how the principle of extremal action (for a suitable choice of L) implies Newton’s equation of motion in Cartesian coordinates for a particle in a potential.

¹¹Arbitrary infinitesimal shifts in locations of particles (or generalized coordinates subject to any constraints on the system) are also called virtual displacements since they do not have to be tangent to a trajectory. Such virtual displacements arose in the principle of virtual work and d’Alembert’s principle [?] which were precursors to the principle of extremal action.

¹² $\int_a^b f(t)g(t)dt = 0$ for all $f(t)$ implies that $g(t) = 0$ for $a < t < b$. Intuitively, this says that g is orthogonal (with respect to the L^2 inner product) to all functions f and therefore must vanish. One way to see this is to suppose that $f(t)$ is zero everywhere except in a small neighborhood of t' of width ϵ where it takes the value $1/\epsilon$. Then the value of the integral approaches $g(t')$ as $\epsilon \rightarrow 0$. Thus, $g(t') = 0$ for any $t' \in (a, b)$.

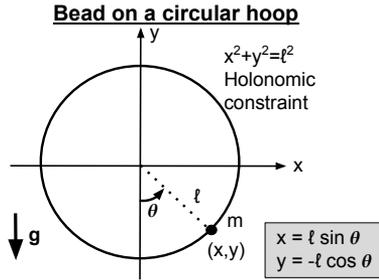


Figure 3: Bead of mass m constrained to move on a circular hoop subject to gravity. A ‘holonomic’ constraint is a relation among coordinates (and possibly time).

Comparing $m\ddot{q} = -V'(q)$ with the EL equation $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ we notice that if we choose¹³ $L = \frac{1}{2}m\dot{q}^2 - V(q)$, then

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \text{and} \quad \frac{\partial L}{\partial q} = -V'(q) \quad (6)$$

and the EL equation reduces to Newton’s equation. This is easily generalized to a particle moving in three dimensions: $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$.

Explicitly time-dependent Lagrangian. Notice that the same EL equations (5) are obtained even if the Lagrangian $L(q, \dot{q}, t)$ depended explicitly on time. The computation of the first variation of S in (4) under an infinitesimal change in path $q \rightarrow q + \delta q$ is unaltered. \square

Particle moving on a vertical hoop. The Lagrangian approach to obtaining equations of motion is particularly convenient when there are constraints which make identifying forces to write down Newton’s equations somewhat tricky. Let us illustrate the use of the Lagrangian with the example of a bead of mass m free to move frictionlessly on a fixed vertical circular hoop (see Fig. 3) of radius l under the influence of a constant acceleration due to gravity g . This problem is physically important: it is equivalent to the simple pendulum. The length of the massless pendulum shaft is l and m is the mass of the bob. Let us use the horizontal and vertical Cartesian coordinates x and y on the plane of the hoop with the origin at its center. The bead is constrained¹⁴ to move on the circle $x^2 + y^2 = l^2$. As above, the kinetic energy is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ while the potential energy is $V = mgy + mgl$ leading to

¹³Lagrangians of physical systems typically do not involve second or higher time derivatives of displacements as they would lead to EOM that are third or higher order in time derivatives. If present in the EOM for a free particle (e.g., $\alpha \ddot{q} = 0$), they would alter Newton’s first law by allowing the particle to have a parabolic rather than straight line trajectory. They would also alter the usual interpretation of Newton’s second law which implies that we need only two ICs for time evolution.

¹⁴Such a constraint is called holonomic. A holonomic constraint is a relation among the generalized coordinates and possibly time, of the form $C(q^1, q^2, \dots, q^n, t) = 0$. A constraint that cannot be expressed in this way (for instance, one that involves generalized velocities, momenta or an inequality) is called a nonholonomic constraint.

the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mg(l + y). \quad (7)$$

We have added the constant $mg l$ to V so that it vanishes at the bottom of the hoop where $y = -l$. The constraint ensures that the bead has only one degree of freedom. We may use the constraint $x^2 + y^2 = l^2$ to eliminate \dot{x} from the Lagrangian¹⁵. Indeed,

$$2x\dot{x} + 2y\dot{y} = 0 \quad \Rightarrow \quad \dot{x}^2 = \frac{y^2\dot{y}^2}{x^2} = \frac{y^2\dot{y}^2}{l^2 - y^2} \quad (8)$$

so that

$$L(y, \dot{y}) = \frac{1}{2}m\dot{y}^2 \left(1 + \frac{y^2}{l^2 - y^2} \right) - mg(l + y). \quad (9)$$

One finds

$$\frac{\partial L}{\partial y} = \frac{ml^2y\dot{y}^2}{(l^2 - y^2)^2} - mg \quad \text{and} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{ml^2(l^2 - y^2)\ddot{y} + 2ml^2y\dot{y}^2}{(l^2 - y^2)^2}. \quad (10)$$

The resulting equation of motion

$$l^2(l^2 - y^2)\ddot{y} + l^2y\dot{y}^2 + g(l^2 - y^2)^2 = 0 \quad (11)$$

looks somewhat formidable. However, it can be simplified by expressing it as an equation for the angle θ that the bead makes with the negative y -axis by putting $y = -l \cos \theta$. We get $\dot{y} = l \sin \theta \dot{\theta}$ and $\ddot{y} = l \cos \theta \ddot{\theta}^2 + l \sin \theta \ddot{\theta}$ leading to the elegantly simple equation

$$\ddot{\theta} = -\frac{g}{l} \sin \theta, \quad (12)$$

which we recognize as being identical to that of the simple pendulum (??). The simplicity of the final equation for θ suggests it may have been easier to eliminate the constraint by expressing the Lagrangian in polar coordinates before deriving the EOM. In fact, putting $x = l \sin \theta$ and $y = -l \cos \theta$ the constraint is identically satisfied and we get

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos \theta), \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -mgl \sin \theta \quad (13)$$

which leads to the same equation (12). Thus, aside from rederiving the equation of motion for a pendulum, we have verified that Lagrange's equations (5) give the same result whether we use rectilinear ($q = y$) or polar ($q = \theta$) coordinates. This freedom to choose generalized coordinates will be further clarified in §0.6. \square

Remark: Given equations of motion for a system, there is no deductive procedure to find a Lagrangian for them. One relies on trial and error and past experience in searching for a Lagrangian. In fact, there are cases (such as the damped motion of a particle) where the equations may not admit any Lagrangian formulation. On the other hand, if a Lagrangian exists, it is not unique, as we shall see in §0.4. \square

¹⁵In fact, if we wanted, we could have taken $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and imposed a second constraint $z = 0$ and used its derivative $\dot{z} = 0$ to eliminate \dot{z} .

0.4 Nonuniqueness of Lagrangian

A Lagrangian for a given system of equations is not uniquely defined. For instance, we may add a constant to $L(q, \dot{q}, t)$ or multiply it by a nonzero real constant without affecting the Euler-Lagrange equations (5). Another source of nonuniqueness arises from the freedom to add the ‘total’ time derivative of any (differentiable) function $F(q, t)$ to the Lagrangian: $L \mapsto L' = L + \dot{F}$. The resulting change in the action (3) is

$$S \mapsto S' \quad \text{where} \quad S' - S = \int_{t_1}^{t_2} \frac{dF}{dt} dt = F(q(t_2), t_2) - F(q(t_1), t_1). \quad (14)$$

When inserted in Hamilton’s variational principle (4), $S' - S$ has zero variation since $t_1, t_2, q(t_1), q(t_2)$ are all held fixed as the path is varied. So the addition of \dot{F} to L does not affect the EL equations. Thus, a strange-looking Lagrangian may give an equivalent description of a familiar system. This freedom to ignore or add time-derivative terms to L can be useful, for instance in the context of motion in boosted or accelerated frames. Moreover, we will exploit this freedom to find generating functions for canonical transformations in §0.13.6.

On the other hand, there is no restriction on the initial and final velocities of the perturbed paths ($\delta\dot{q}_1 \equiv \delta\dot{q}(t_1), \delta\dot{q}_2 \equiv \delta\dot{q}(t_2)$) are not required to vanish) in Hamilton’s variational principle. Suppose $F(q, \dot{q}, t)$ depended on velocities. Then the difference between actions would be $S' - S = F(q_2, \dot{q}_2, t_2) - F(q_1, \dot{q}_1, t_1)$ and its variation is

$$\delta(S' - S) = \frac{\partial F}{\partial q_2} \delta q_2 + \frac{\partial F}{\partial \dot{q}_2} \delta \dot{q}_2 + \frac{\partial F}{\partial t_2} \delta t_2 - \frac{\partial F}{\partial q_1} \delta q_1 - \frac{\partial F}{\partial \dot{q}_1} \delta \dot{q}_1 - \frac{\partial F}{\partial t_1} \delta t_1. \quad (15)$$

Since $\delta\dot{q}_1$ and $\delta\dot{q}_2$ are arbitrary and independent, the only way for the extrema of S' and S to coincide would be for $\frac{\partial F}{\partial \dot{q}_1}$ and $\frac{\partial F}{\partial \dot{q}_2}$ to vanish. This must happen for any initial and final times t_1 and t_2 , so $\frac{\partial F}{\partial \dot{q}}$ must vanish. In other words, for S and S' to lead to the same equations of motion, F must be independent of velocities.

0.5 Conjugate momentum and its geometric meaning, cyclic coordinates

Given a Lagrangian $L(q^i, \dot{q}^i)$, the momentum p_i conjugate to the coordinate q^i is defined as¹⁶

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \quad (16)$$

In general, conjugate momenta do not have the dimensions of mass \times velocity, just as generalized coordinates q^i do not necessarily have dimensions of length. However, since L has dimensions of energy, the product of a coordinate and its conjugate momentum always has dimensions of energy \times time or action. Conjugate momentum is a useful concept. If a coordinate q^j does not appear in the Lagrangian, then

¹⁶An (unsummed) index such as i in a so-called tensorial equation like $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is said to be consistently placed if it appears in all terms as a subscript (or as a superscript throughout). For this purpose, a superscript in the denominator plays the same role as a subscript in the numerator and vice versa. In (16), this ensures that both the LHS and RHS are covariant vectors.

its conjugate momentum is automatically conserved in view of (5):

$$\frac{dp_j}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} = \frac{\partial L}{\partial q^j} = 0. \quad (17)$$

Such a coordinate is called a *cyclic coordinate*. For a free particle moving on a line, $L = \frac{1}{2}m\dot{x}^2$ has the cyclic coordinate x leading to the conservation of the conjugate momentum $p_x = m\dot{x}$, $\dot{p}_x = 0$.

Velocity vectors as tangent vectors

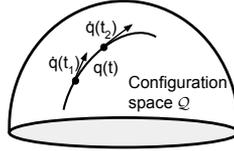


Figure 4: Velocity vector as a tangent vector to a trajectory and to the configuration space. Note that velocity vectors at distinct points along a trajectory live in different tangent spaces and cannot simply be ‘added’.

* **Geometrical interpretation of generalized velocity and momentum.** Geometrically, the trajectory $q^i(t)$ is a curve on configuration space \mathcal{Q} . At any instant of time, $\dot{q}^i(t)$ are the components of the tangent vector to this curve. Thus, generalized velocities are components of tangent vectors to the configuration space. Tangent vectors are *contravariant* vectors and their components are denoted with upper indices. On the other hand, the generalized momenta p_i are components of cotangent vectors (covectors or *covariant* vectors, denoted with lower indices). They are elements of the vector space dual¹⁷ to the tangent space. Tangent and cotangent vectors *at the same point of the configuration space* may be *contracted* via the pairing between a vector space and its dual to get a number. For instance¹⁸ $p_i(t)\dot{q}^i(t)$ is a real number at each point $q^i(t)$ along the curve. This number is independent of what coordinate system we use on configuration space and defines the scalar function $p_i\dot{q}^i$ along the curve. Its integral along the curve is called the abbreviated action. \square

Not every conserved quantity may arise as the momentum conjugate to a cyclic coordinate. For example, if we use Cartesian coordinates for the particle in a central potential on a plane, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2})$, then neither x nor y is

¹⁷ Given a real vector space V , its dual V^* is the space of real-valued linear functions on V . $\phi : V \rightarrow \mathbb{R}$ is a linear function if $\phi(au + bv) = a\phi(u) + b\phi(v)$ for all $u, v \in V$ and all scalars $a, b \in \mathbb{R}$. A momentum covector p (at the point $q \in \mathcal{Q}$) defines a linear function on the tangent space of velocities of curves passing through q . Its value on a velocity vector \dot{q} is $p(\dot{q}) = \sum_i p_i \dot{q}^i$ which is evidently linear in the components of \dot{q} . Thus, momenta live in the space dual to velocities, called the cotangent space. A finite dimensional vector space and its dual are isomorphic, but in general there is no distinguished isomorphism between them. The formula $p_i = \frac{\partial L}{\partial \dot{q}^i}$ provides a canonical isomorphism between tangent and cotangent spaces by associating a specific momentum covector to any given velocity tangent vector.

¹⁸ The repeated index i appearing once as an upper and once as a lower index in a term is summed from 1 to the number of degrees of freedom. A repeated index is called a dummy index and can be replaced with any other repeated symbol $p_i \dot{q}^i = p_j \dot{q}^j$.

cyclic and neither of the momenta ($p_x = m\dot{x}, p_y = m\dot{y}$) is conserved. But as we might suspect from the Kepler problem (and as we will see below), the momentum conjugate to the cyclic angular coordinate $\phi = \arccos(x/\sqrt{x^2 + y^2})$ is conserved. So some physical insight/cleverness/luck may be needed in choosing coordinate systems in which one or more coordinates are cyclic.

0.6 Coordinate invariance of the form of Lagrange's equations

One of the important advantages of Lagrange's equations over Newton's second law is that unlike the latter, they are valid in any system of coordinates on configuration space. Here, we illustrate this through an example and indicate the reason.

Let us begin with an example to show that Newton's equations in Cartesian coordinates $m\ddot{x}^i = f^i$ change their form if we switch to a different coordinate system on configuration space. Consider a free particle moving on the x - y plane. Newton's equations are $m\ddot{x} = 0$ and $m\ddot{y} = 0$ since there is no force. Now let us transform to polar coordinates via $x = r \cos \phi$ and $y = r \sin \phi$. Upon calculating the derivatives, the equations of motion for r and ϕ are found to be

$$m\ddot{r} = mr\dot{\phi}^2 \quad \text{and} \quad mr\ddot{\phi} = -2m\dot{r}\dot{\phi}. \quad (18)$$

There is no force in the \hat{r} or $\hat{\phi}$ directions. Yet, the naive radial and angular 'accelerations' \ddot{r} and $\ddot{\phi}$ are nonzero! Thus, in polar coordinates, Newton's equations do not take the same ' $m\ddot{q}^i = f^i$ ' form¹⁹ that they did in Cartesian coordinates (Nb. by f^θ we mean $\mathbf{f} \cdot \hat{\theta}$ and so on).

Let us now contrast this with Lagrange's equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$ (5) for the same system. In Cartesian coordinates ($q^1 = x, q^2 = y$), $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ leads to the familiar equations $m\ddot{x} = 0$ and $m\ddot{y} = 0$. Upon transforming to plane polar coordinates²⁰, the Lagrangian becomes

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2). \quad (19)$$

The momenta conjugate to r and ϕ are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}. \quad (20)$$

They coincide with the radial component of linear momentum and the z -component of angular momentum. The first of Lagrange's equations is

$$\dot{p}_r = m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\phi}^2. \quad (21)$$

This is the balance of mass \times radial acceleration and the 'centripetal acceleration'. On the other hand, ϕ is a cyclic coordinate and so p_ϕ is conserved:

$$\dot{p}_\phi = \frac{d}{dt}(mr^2\dot{\phi}) = \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad mr\ddot{\phi} = -2m\dot{r}\dot{\phi}. \quad (22)$$

¹⁹All is not lost: the equations $m\mathbf{a} \cdot \hat{\phi} = \mathbf{f} \cdot \hat{\phi}$ and $m\mathbf{a} \cdot \hat{r} = \mathbf{f} \cdot \hat{r}$ do hold.

²⁰This Lagrangian also follows from the square of the Euclidean line element $ds^2 = dr^2 + r^2 d\phi^2$, which implies that the square of velocity is $\dot{s}^2 = (ds/dt)^2 = \dot{r}^2 + r^2\dot{\phi}^2$.

This states the conservation of angular momentum, and involves the Coriolis acceleration term²¹ on the RHS when written out. We see that these equations agree with Newton's equations transformed to polar coordinates (18). Thus, Lagrange's equations in the form $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$ are valid in both Cartesian and polar coordinates, while Newton's equations in the form $m\ddot{q}^i = f^i$ are valid in Cartesian but not polar coordinates.

More generally, Lagrange's equations (5) are valid in any system of coordinates on configuration space. The reason is easy to see: we obtained the EL equations by extremizing the action (3) without making any assumption about what the q^i are. So q^i could be Cartesian or polar or any other coordinates.

Aside from conceptual elegance, the above feature of Lagrange's equations is of significant practical value. We are often interested in studying a system with n degrees of freedom in a particular coordinate system (e.g., for reasons of symmetry) and need to transform the equations from, say, Cartesian coordinates. To transform Newton's equations we need to transform n equations and in each case, calculate second derivatives of the equations of transformation, as we did in arriving at (18). On the other hand, in the Lagrangian approach, it is enough to transform a single Lagrange function (which typically involves only first derivatives in time) and use it to generate the equations in the new coordinate system.

0.7 Hamiltonian and its conservation

Besides the momenta conjugate to cyclic coordinates, the Lagrangian formulation leads automatically to another conserved quantity, the Hamiltonian. For a moment, suppose the Lagrangian depends explicitly²² on time $L = L(q(t), \dot{q}(t), t)$. Then

$$\frac{dL}{dt} = \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} + \frac{\partial L}{\partial t} = \dot{p}\dot{q} + p\ddot{q} + \frac{\partial L}{\partial t} = \frac{d(p\dot{q})}{dt} + \frac{\partial L}{\partial t} \quad (23)$$

which implies $\frac{d(p\dot{q} - L)}{dt} = -\frac{\partial L}{\partial t}$. So, if we define the Hamiltonian $H = p\dot{q} - L$, then $\dot{H} = -\frac{\partial L}{\partial t}$. Thus, if the Lagrangian does not depend explicitly on time, H would be conserved²³. For example, suppose $L = \frac{1}{2}m\dot{q}^2 - V(q)$ for a particle moving along a line. Then $p = m\dot{q}$ and $H = m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + V(q) = \frac{1}{2}m\dot{q}^2 + V(q)$ coincides with the total energy. Thus, the Lagrangian approach gives us a new way of discovering a conserved energy that complements the approach of using an integrating factor in Newton's equation.

²¹The Coriolis *acceleration* is to be contrasted with the Coriolis *force*, which is a fictitious force that appears to act in a rotating frame of reference. The Coriolis acceleration is present in an inertial frame when the particle's r and θ coordinates both vary with time.

²²We say that the Lagrangian $L(q(t), \dot{q}(t))$, does not depend explicitly on time, but only implicitly through $q(t)$ and $\dot{q}(t)$. An example of an explicitly time-dependent Lagrangian is $L = \frac{1}{2}m(t)\dot{q}^2$ where $m(t)$ is a time-dependent mass parameter.

²³Note that the Lagrangian is in general not a conserved quantity even if it does not depend explicitly on time: $\dot{L} = \frac{d(p\dot{q})}{dt}$. For a particle moving in a 1d potential, $\dot{L} = -2pV'(q)/m$ which is not identically zero if there is a force.

Explicitly time-dependent Lagrangian. To see what could happen when L depends explicitly on time, consider $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}k(t)q^2$. This models a particle that is subject to a linear restoring force $-k(t)q$ whose 'spring constant' k varies with time. This could happen, for instance, if the spring 'loses' elasticity with time in a prescribed manner. In this case, the EL equation of motion $m\ddot{q} = -k(t)q$ implies that the Hamiltonian $H = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}k(t)q^2$ evolves according to $\dot{H} = \frac{1}{2}\dot{k}q^2$. Thus, this Hamiltonian ('energy') is not conserved. Another example is an oscillating pendulum that is pushed every time it reaches one of the turning points. In this case, the usual energy of oscillatory motion would be expected to grow with time due to the external driving. \square

Here, we regard the Hamiltonian as a function of generalized coordinates and velocities. In §0.9, when we discuss the Hamiltonian formulation of mechanics, we will eliminate q^i in favor of momenta and regard the Hamiltonian as a function of coordinates and momenta.

If the Lagrangian is not quadratic (bilinear) in velocities, then the Hamiltonian may not coincide with $T + V$. For example, consider a Lagrangian $L = T - V$ with cubic kinetic term $T = \frac{1}{2}m\dot{q}^2 + \frac{1}{3}g\dot{q}^3$ and some smooth potential V . Then the conjugate momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} + g\dot{q}^2. \quad (24)$$

The Hamiltonian is seen not to coincide with $T + V$:

$$H = p\dot{q} - L = (m\dot{q} + g\dot{q}^2)\dot{q} - \frac{1}{2}m\dot{q}^2 - \frac{g}{3}\dot{q}^3 + V = \frac{1}{2}m\dot{q}^2 + \frac{2}{3}g\dot{q}^3 + V \neq T + V. \quad (25)$$

While the Hamiltonian is always conserved (provided $\frac{\partial L}{\partial t} = 0$), $T + V$ may not be conserved and may not be a particularly interesting physical quantity. In fact, henceforth, we will use the word energy for the conserved quantity $p_k \dot{q}^k - L$, rather than for $T + V$, if they are distinct.

0.8 Symmetries to conserved quantities: Noether's theorem

Lagrange's (and Newton's) equations of classical mechanics have been formulated as conditions for the action $S = \int L dt$ to be extremal. Many concepts (such as symmetries) may be formulated more simply in terms of the action or Lagrangian than in terms of the equations of motion.

If a coordinate q^j is cyclic (absent from the Lagrangian), then its conjugate momentum $p_j = \frac{\partial L}{\partial \dot{q}^j}$ is conserved as a consequence of Lagrange's equation $\dot{p}_j = \frac{\partial L}{\partial q^j} = 0$. If the Lagrangian is independent of a coordinate, then it is unchanged when this coordinate is shifted by a constant: $\delta L = 0$ under $q^j \rightarrow q^j + \delta q^j$. We say that translations of a cyclic coordinate q^j are a symmetry of the Lagrangian. This relation between symmetries and conserved quantities is deeper, it goes beyond mere translations of a cyclic coordinate.

A transformation of coordinates $q^i \rightarrow \tilde{q}^i$ is a symmetry of the equations of motion (EOM) if it leaves them unaltered: i.e., the EOM for \tilde{q} is the same as that for q . Symmetries usually allow us to produce new solutions from known ones. For example,

the free particle equation $m\ddot{q} = 0$ is left unchanged by a translation of the coordinate $q \rightarrow \tilde{q} = q + a$ for any constant length a . Now $q = 0$ is one static solution. We may use the symmetry under translations to produce other static solutions, namely $q = a$ for any $a \in \mathbb{R}$, i.e., the particle is at rest at the location with coordinate a rather than at the origin. Incidentally, the momentum of a free particle is conserved in time. We will see that such symmetries are associated with conserved quantities. On the other hand, the equation of motion of a particle attached to a spring $m\ddot{q} = -kq$ is nontrivially modified by a translation of the coordinate $q \rightarrow \tilde{q} = q + a$ since \tilde{q} satisfies a different equation $m\ddot{\tilde{q}} = -k\tilde{q} + ka$. Moreover, $p = m\dot{q}$ is not (in general) conserved for a particle executing simple harmonic motion: the momentum is zero at the turning points and maximally nonvanishing at the point of equilibrium.

Remark. It is important to note that not every transformation of q qualifies as a symmetry of the equations of motion. We have already argued in §0.6 that every transformation of coordinates leaves the form of Lagrange's equations invariant. So here, when we say 'leaves the EOM invariant', we are not referring to the form of Lagrange's equations i.e., $\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ but to the differential equations written out explicitly without any Lagrange function present. \square

A symmetry of the Lagrangian is a transformation that leaves L unchanged (i.e., L remains invariant). E.g., for the free particle, $L = \frac{1}{2}m\dot{q}^2$ is unchanged under the shift $q \rightarrow q + a$. It follows that the action $S[q] = \int_{t_1}^{t_2} \frac{1}{2}m\dot{q}^2 dt$ is also unchanged under the shift $q \rightarrow q + a$. Since the EOM are the conditions for S to be stationary, a symmetry of the Lagrangian must also be a symmetry of Lagrange's equations. Noether's theorem constructs a conserved quantity associated to each *infinitesimal* continuous symmetry of the Lagrangian²⁴. Let us see how. Suppose the infinitesimal change²⁵ $q^i \rightarrow q^i + \delta q^i$ leaves the Lagrangian unchanged to linear order in δq . Then it is automatically an infinitesimal symmetry of the action. Let us explicitly calculate the first variation of the action for paths between the (arbitrary) times t_1 and t_2 , $S[q + \delta q] = S[q] + \delta S$. Up to terms of order $(\delta q)^2$ we get

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left[\delta q^i \frac{\partial L}{\partial q^i} + \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \right] dt = \int_{t_1}^{t_2} \left[\delta q^i \frac{\partial L}{\partial q^i} + \frac{d}{dt} \left(\delta q^i \frac{\partial L}{\partial \dot{q}^i} \right) - \delta q^i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right] dt \\ &= \delta q^i(t_2) \frac{\partial L}{\partial \dot{q}^i}(t_2) - \delta q^i(t_1) \frac{\partial L}{\partial \dot{q}^i}(t_1) + \int_{t_1}^{t_2} \delta q^i \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right] dt. \end{aligned} \quad (26)$$

So far, this is true for any path and for any infinitesimal change²⁶ δq^i . Let us now specialize to infinitesimal changes about a trajectory, so that $q^i(t)$ satisfies Lagrange's equations and the last term vanishes. Furthermore, let us suppose that the transformation is an infinitesimal symmetry of the Lagrangian, so that $\delta S = 0$:

$$0 = \delta S = \delta q^i(t_2) \frac{\partial L}{\partial \dot{q}^i}(t_2) - \delta q^i(t_1) \frac{\partial L}{\partial \dot{q}^i}(t_1). \quad (27)$$

²⁴There is a generalization to the case where L changes by a time derivative.

²⁵Here, δq^i could be functions of all the coordinates.

²⁶We do not require that δq vanish at the initial and final times. A symmetry transformation would not know about these arbitrarily chosen times. For instance, if the symmetry were a translation, $\delta q = a$ would not vanish anywhere!

Since t_1, t_2 are arbitrary, the quantity $\delta q^i \frac{\partial L}{\partial \dot{q}^i}$ must be constant along a trajectory. In other words, an infinitesimal symmetry $q \rightarrow q + \delta q$ of the Lagrangian implies that

$$Q = p_i(t) \delta q^i(t) = \mathbf{p} \cdot \delta \mathbf{q} \quad (28)$$

is a constant of the motion, i.e., the dynamical variable Q has the same value at all points along a trajectory. Q is called the conserved Noether 'charge', in loose analogy with the conservation of electric charge.

Example 1: We already saw that the free particle Lagrangian is translation invariant with $\delta q^i = a^i$ where a^i are the components of an arbitrary infinitesimal vector. It follows that $Q = a^i p_i = \mathbf{p} \cdot \mathbf{a}$ is a conserved quantity. In other words, the component of momentum in *any* direction is conserved. \square

Example 2: Now consider a particle in a central potential $V(q^2)$ so that the Lagrangian is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} - V(\mathbf{q} \cdot \mathbf{q}). \quad (29)$$

Let us first show that L is invariant under rotations of 3d Euclidean space $\mathbf{q} \rightarrow R\mathbf{q}$ where R is any rotation matrix²⁷ ($R^t R = I, \det R = 1$ where t denotes the transpose). Recall that the dot product is defined as $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^t \mathbf{b}$ for any column vectors \mathbf{a}, \mathbf{b} and that $(R\mathbf{a})^t = \mathbf{a}^t R^t$ for any matrix R . Thus,

$$L(R\mathbf{q}, R\dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{q}} R^t R \dot{\mathbf{q}} - V(\mathbf{q}^t R^t R \mathbf{q}) = \frac{1}{2} m \dot{\mathbf{q}}^t \dot{\mathbf{q}} - V(\mathbf{q}^t \mathbf{q}) = L(\mathbf{q}, \dot{\mathbf{q}}). \quad (30)$$

So the Lagrangian is invariant under rotations. Noether's theorem, however, refers to infinitesimal transformations, rotations in this case. So let us find a formula for an infinitesimal rotation.

Suppose we make an infinitesimal counterclockwise rotation of the vector \mathbf{q} about the axis $\hat{\mathbf{n}}$ by a small angle θ , as shown in Fig. 5. Then the vector \mathbf{q} sweeps out a sector of a cone. Suppose \mathbf{q} makes an angle ϕ with respect to $\hat{\mathbf{n}}$, so that the opening angle of the cone is ϕ . Then the rotated vector $\tilde{\mathbf{q}}$ also makes an angle ϕ with respect to the axis $\hat{\mathbf{n}}$. Let $\delta \mathbf{q} = \tilde{\mathbf{q}} - \mathbf{q}$ be the infinitesimal change in \mathbf{q} . By looking at the base of this cone, we find that it is a sector of a circle with radius $q \sin \phi$ and opening angle $\theta \ll 1$. So we find that $|\delta \mathbf{q}| \approx \theta q \sin \phi$. Moreover $\delta \mathbf{q}$ points in the direction of $\hat{\mathbf{n}} \times \mathbf{q}$. Thus, under a counterclockwise rotation about the axis $\hat{\mathbf{n}}$ by a small angle θ , the change in \mathbf{q} is

$$\delta \mathbf{q} = \theta \hat{\mathbf{n}} \times \mathbf{q} \quad \text{and} \quad \delta \dot{\mathbf{q}} = \theta \hat{\mathbf{n}} \times \dot{\mathbf{q}}. \quad (31)$$

In particular, we see that $\delta \mathbf{q}$ and $\delta \dot{\mathbf{q}}$ are orthogonal to \mathbf{q} and $\dot{\mathbf{q}}$ respectively.

²⁷Rotations (or proper rotations) are orientation-preserving linear transformations of \mathbb{R}^3 that preserve the lengths of vectors and angles between vectors. Lengths and angles are preserved if inner products are preserved: $(\mathbf{u}, \mathbf{v}) = (R\mathbf{u}, R\mathbf{v})$, which is the condition $\mathbf{u}^t \mathbf{v} = \mathbf{u}^t R^t R \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Thus, we must have $R^t R = I$, the 3×3 identity matrix. Such matrices form the orthogonal group $O(3)$ and have determinant ± 1 . Those with $\det R = -1$ are orientation-reversing and involve reflections in an essential way. Excluding them leaves the group of proper rotations $SO(3)$ of special orthogonal matrices. Composition of rotations is given by matrix multiplication.

dependent on time. Here $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is the momentum conjugate to q^i . To understand H better, let us compute its differential using Lagrange's equations

$$dH = p_i dq^i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i = \cancel{p_i d\dot{q}^i} + \dot{q}^i dp_i - \dot{p}_i dq^i - \cancel{p_i d\dot{q}^i} = -\dot{p}_i dq^i + \dot{q}^i dp_i. \quad (34)$$

This reveals that the independent variables in H are the generalized coordinates q^i and the generalized momenta p_i , the terms involving the differentials of velocities cancelled out. So we should think of H as $H(q, p)$. On the other hand, by the definition of partial derivatives,

$$dH(q, p) = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i. \quad (35)$$

Comparing (34) and (35), we find that the time derivatives of coordinates and momenta may be expressed in terms of partial derivatives of the Hamiltonian

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (36)$$

These first order ODEs are called Hamilton's equations. They give us yet another way of expressing the equations of time evolution. To make sense of these equations we are first supposed to express $H = p_i \dot{q}^i - L(q^i, \dot{q}^i)$ as a function of the coordinates q and momenta p . This is done by eliminating \dot{q}^i in favor of q, p using the definition of conjugate momenta $p_j = \frac{\partial L}{\partial \dot{q}^j}$.

As in the Lagrangian framework, we have the notion of cyclic coordinates. A cyclic coordinate q^j is one that does not appear in $H(q, p)$. A moment's thought will reveal that if q^j does not appear in L , it cannot appear in any of the relations $p_j = \frac{\partial L}{\partial \dot{q}^j}$ and therefore cannot appear in H . The momentum p_j conjugate to a cyclic coordinate q^j is conserved as a consequence of (36).

Example: Particle in a 1d potential. Here $L = \frac{1}{2}m\dot{q}^2 - V(q)$ and $p = m\dot{q}$ so $\dot{q} = p/m$. Then $H = p\dot{q} - L = pp/m - p^2/2m + V(q) = p^2/2m + V(q)$. Hamilton's equations are $\dot{q} = \frac{\partial H}{\partial p} = p/m$ which recovers the definition of conjugate momentum and $\dot{p} = -\frac{\partial H}{\partial q} = -V'(q)$ which is Newton's second law. \square

Remark: Time-dependent Hamiltonian. If L depends explicitly on time, then H too will do so. Proceeding as in (34), the differential of H admits two expressions:

$$dH = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t} dt \quad \text{and} \quad dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt. \quad (37)$$

Comparing, we get

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (38)$$

So even for a time-dependent H , Hamilton's equations for coordinates and momenta take the same form as in (36). \square

Significance of the Hamiltonian formulation. The idea of the Hamiltonian $H(q, p)$ and the associated formulation of dynamics has far-reaching consequences. To begin with, the point of the Hamiltonian formulation is not so much as a technical tool for solving the equations of motion²⁸ but as an alternate way of obtaining them and thinking about them. It reveals structural features, especially of conservative systems, that are not obvious in the Newtonian formulation. Just like the Lagrangian, a single Hamilton function can generate all the equations of motion which take the same form in all systems of coordinates on configuration space. However, unlike Newton's or Lagrange's equations which are typically second order in time, Hamilton's equations (36) are first order in time, admit a formulation in terms of vector fields and reveal a certain hidden symmetry between generalized coordinates and momenta. This observation eventually leads to the recognition that Hamilton's equations retain their form under a larger class of changes of variables called canonical transformations (or symplectic transformations). Symmetries of the equations of motion like translations and rotations can be economically formulated as properties of the Hamiltonian (see §0.12). The Hamiltonian is also used to give yet another formulation of the equations of dynamics in terms of the Hamilton-Jacobi partial differential equation. This form turns out to be closely related to the equations of wave optics and the idea of a wavefront. It may also be regarded as the 'point of closest approach' of classical dynamics to the Schrödinger equation of quantum mechanics, which too is formulated in terms of a (quantum) Hamiltonian. On the other hand, Hamilton's ordinary differential equations (36) for trajectories are the mechanical analogue of the equations of ray optics. They are also the point of closest approach of classical dynamics to the Heisenberg equations of quantum mechanics. They may also be used to describe magnetic field lines in a plasma and the mixing of fluids. The Hamiltonian also plays a central role in equilibrium statistical mechanics. In the canonical ensemble at absolute temperature T , the state (q, p) on phase space occurs with relative probability $e^{-H(q,p)/k_B T}$ where k_B is Boltzmann's constant. \square

0.10 Legendre transform: Hamiltonian from Lagrangian

The Legendre transform summarizes the passage²⁹ from Lagrangian $L(q, \dot{q})$ to Hamiltonian $H(q, p)$. It is a way of swapping one independent variable for another: momentum replaces velocity in going from L to H . We begin by observing that the formula for the conjugate momenta $p_j = \frac{\partial L}{\partial \dot{q}^j}$ is the condition for the quantity $p_i \dot{q}^i - L(q, \dot{q})$ to be extremal with respect to small variations in \dot{q}^j . Indeed, differentiating in

²⁸The Hamiltonian formulation *does* lead to new ways of (approximately) solving the equations of motion. For instance, classical perturbation theory is based on the idea of making a succession of 'canonical transformations' to bring the Hamiltonian to a 'normal' form so that the EOM can be solved more easily at each order in perturbation theory. On the other hand, the first order formulation brings to bear techniques from the theory of vector fields and dynamical systems. Accurate schemes to numerically integrate the equations of motion (such as symplectic integrators) are also based on the Hamiltonian formulation.

²⁹The Legendre transform as a way of obtaining the Hamiltonian is relevant to Lagrangians that are quadratic (or of higher degree) in the generalized velocities \dot{q}^i .

\dot{q}^j and equating to zero,

$$\frac{\partial}{\partial \dot{q}^j} (p_i \dot{q}^i - L(q, \dot{q})) = p_i \delta_j^i - \frac{\partial L}{\partial \dot{q}^j} = 0 \quad \text{we get} \quad p_j = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^j}. \quad (39)$$

In finding the condition for an extremum, p is regarded as an independent variable, not as a function of q and \dot{q} . It is only at the extremum that p becomes a specific function of q and \dot{q} , as determined by the Lagrangian.

Moreover, the extremal value of the quantity $p\dot{q} - L$ is the Hamiltonian $H(q, p)$. Thus, we may write

$$H(q, p) = \text{ext}_{\dot{q}} [p_i \dot{q}^i - L(q, \dot{q})]. \quad (40)$$

The extremization is carried out with respect to *all* the generalized velocities. Equation (40) is the definition of the Legendre transform and we say that H is the Legendre transform of L .

Changing independent variables. It is noteworthy that q plays a passive role in the Legendre transform from $L(q, \dot{q})$ to $H(q, p)$. The Legendre transform allows us to use p as an independent variable in place of \dot{q} . We may regard \dot{q} and p as living in dual vector spaces, the tangent and cotangent spaces at the point q on the configuration manifold (see Footnote 17). Consequently, the Legendre transform may be viewed as a recipe for transforming a function on a vector space to a function on the dual vector space. Thus, though we introduced the Legendre transform to systematize the passage from Lagrangian to Hamiltonian, it can be used more generally to change independent variables in functions. Good examples are furnished by the transformations among internal energy, Helmholtz free energy, enthalpy and Gibbs free energy in thermodynamics where we switch between the conjugate variables temperature and entropy or pressure and volume.

Existence and singlevaluedness of Legendre transform. The key step in the Legendre transform is to solve for the (real) velocities \dot{q} in terms of the momenta and coordinates using $p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i}$. This is not always possible. For a quadratic $L = \frac{1}{2}m\dot{q}^2$ with $m \neq 0$, this is easily done since (39) gives us a linear equation for \dot{q} : $p = m\dot{q}$. This allows us to eliminate $\dot{q} = p/m$ in favor of p giving $H(q, p) = pp/m - \frac{1}{2}m(p/m)^2 = \frac{p^2}{2m}$. However, more generally, it could happen that there is none or more than one real solution \dot{q} to (39) for given q, p . If so, H may not exist³⁰ or may be a multivalued function of coordinates and momenta. Conditions that guarantee that the Legendre transform exists as a singlevalued function is convexity³¹ (or concavity) of the Lagrangian as a twice differentiable function of \dot{q} for all q, \dot{q} along with the requirement that $|\frac{\partial L}{\partial \dot{q}}| \rightarrow \infty$ as $\dot{q} \rightarrow \pm\infty$ for any fixed q . The convexity and

³⁰When we cannot solve for the velocities in terms of coordinates and momenta, we say that the Lagrangian is singular.

³¹A function $f(x)$ of one real variable x is convex if the graph of f for $x_1 < x < x_2$ lies below the chord joining $f(x_1)$ to $f(x_2)$ for any $x_1 < x_2$. It is concave if the graph lies above the chord. Viewed from below, a sagging clothesline is convex while an arch is concave.

growth conditions are seen to be satisfied by $L = \frac{1}{2}m\dot{q}^2$ if $m > 0$ ³². Let us see how these conditions help. L is convex provided the 2nd derivative or Hessian $\frac{\partial^2 L}{\partial \dot{q}^2} > 0$ for all q, \dot{q} . Suppose this is the case, then $\Pi(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}}$ must be a strictly monotonically increasing function of \dot{q} for any q . The growth condition implies that $\Pi(q, \dot{q}) \rightarrow \pm\infty$ as $\dot{q} \rightarrow \pm\infty$. Thus the horizontal line at height p (for any real p) cuts the graph of $\Pi(q, \dot{q})$ as a function of \dot{q} at precisely one point. It follows that the equation $p = \frac{\partial L}{\partial \dot{q}}$ has a unique solution \dot{q} for any given values of q, p . In other words, under the given hypotheses, the Legendre transform of H exists for all q, p and is singlevalued.

If the growth condition is violated, $\Pi(q, \dot{q})$ could asymptote to a finite value as $\dot{q} \rightarrow \pm\infty$ resulting in no solution to $\frac{\partial L}{\partial \dot{q}} = p$ for sufficiently large p or $-p$.

To see what can go wrong when convexity fails, let us attempt to compute the Legendre transform of $L = \frac{1}{4}\dot{q}^4 - \frac{1}{2}\dot{q}^2$. The Hessian $L''(\dot{q})$ is $3\dot{q}^2 - 1$. It is negative for $|\dot{q}| < 1/\sqrt{3}$ and positive for $|\dot{q}| > 1/\sqrt{3}$ so L is neither convex nor concave everywhere. It is shaped like a double-well and we expect to run into trouble. Indeed, if we try to solve for \dot{q} in $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}^3 - \dot{q}$, there are three real solutions for any given $|p| < 2/3\sqrt{3}$. In this case, the Legendre transform H is not singlevalued. It is worth noting that we are concerned with real velocities, so nonreal complex roots of this cubic equation are inadmissible. For instance, reversing the sign of \dot{q}^2 leads to the Lagrangian $L = \frac{1}{4}\dot{q}^4 + \frac{1}{2}\dot{q}^2$ which is convex everywhere and satisfies the growth condition. What is more, as is seen from the graph, the equation $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}^3 + \dot{q}$ has a unique real solution \dot{q} for any given p , and the Legendre transform is defined as a singlevalued function.

Legendre transform for more degrees of freedom. For several³³ degrees of freedom, the Legendre transform is defined if L is convex and satisfies a growth condition: $|\frac{\partial L}{\partial \dot{q}^i}| \rightarrow \infty$ as $|\dot{q}^j| \rightarrow \infty$ for each i, j . Note that L is convex provided the 2nd derivative (Hessian) matrix with entries $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is a positive matrix for all \dot{q}^i and $q \in \mathcal{Q}$.

Graphical interpretation. Suppose $L(v)$ is a convex (or concave) function of a velocity and $H(p) = \text{ext}_v [pv - L(v)]$ is its Legendre transform, a function of momentum. The Legendre transform may be obtained by a graphical construction, which is a recipe for finding H given a value of p . Consider the graph of L as a function of v (see Fig. 6). Given a value of p we draw the line pv through the origin with slope p . The vertical distance between the graph of $L(v)$ and this straight line is $pv - L(v)$ (or its negative). Given p , we find the point(s) $v(p)$ at which this distance is an extremum. There is precisely one such point if L is convex and satisfies the growth condition³⁴. Evidently, $(v(p), L(v(p)))$ is the point on the graph where the tangent to the graph is parallel to the straight line pv . Then we define the value of the Legendre transform as

³²If $m < 0$, L is concave and if $m = 0$ it is neither and the equation $p = 0$ cannot be used to solve for \dot{q} in terms of p .

³³For more than 1 degree of freedom, one may choose to perform a partial Legendre transform, replacing some velocities in the Lagrangian by their conjugate momenta and retaining other velocities. The resulting function that replaces the Hamiltonian is called the Routhian.

³⁴If the growth condition is violated but L is convex, there can either be zero or one such v .

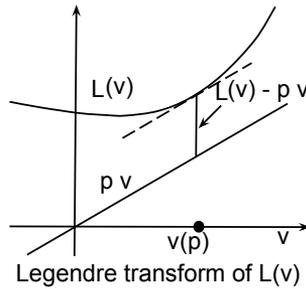


Figure 6: Legendre transform $H(p) = p v(p) - L(v(p))$. We find $v(p)$ by extremizing the vertical distance between $L(v)$ and the straight line through the origin with slope p . The extremum occurs when the slope $L'(v) = p$.

$$H(p) = p v(p) - L(v(p)).$$

Inverse Legendre transform and involutive property. Finally, when the Legendre transform is defined, the Lagrangian can be re-obtained from $H(q, p)$ by another Legendre transform

$$L(q, \dot{q}) = \text{ext}_p [p_i \dot{q}^i - H(q, p)]. \tag{41}$$

For example, taking $H = p^2/2m$ one finds $L = \frac{1}{2}m\dot{q}^2$. Thus, applying the Legendre transform twice to a convex function produces the identity transformation. The Legendre transform is its own inverse and we say that it is an involution.

0.11 Variational principles for Hamilton's equations

We seek a variational principle for Hamilton's equations (36), just as we had one in §0.3 for Lagrange's equations: $S[q] = \int L dt$ and $\delta S = 0$. Inspired by what worked for Lagrange's equations we consider

$$S[q, p] = \int_{t_1}^{t_2} [p_i \dot{q}^i - H(q, p)] dt. \tag{42}$$

The phase space action $S[q, p]$ is a functional of a path $(q(t), p(t))$ on *phase space*, unlike $S[q] = \int L(q, \dot{q}) dt$ which is a functional of a path on *configuration space*³⁵. We are forced to view it as a functional of a path on phase space since $H(q, p)$ is a function on phase space and $p\dot{q}$ involves both the position and momentum along the path. Along such a path, \dot{q} is not an independent quantity, it is obtained simply by differentiating $q(t)$. To obtain Hamilton's equations, we ask that $S[q, p]$ be stationary with respect to small variations in the phase path $(q^i(t), p_j(t)) \rightarrow (q^i(t) + \delta q^i(t), p_j(t) + \delta p_j(t))$ while holding the initial and final configurations and times fixed: $\delta q(t_1) = 0$ and $\delta q(t_2) = 0$. Note that we do not constrain $\delta p(t_1)$ or $\delta p(t_2)$. That would be an over-specification³⁶.

³⁵As always, $\dot{q}(t)$ is simply the time derivative of $q(t)$ and is not an independent quantity. This is why $S[q] = \int L(q, \dot{q}) dt$ is a functional of a path on configuration space and it would *not* make sense to write it as $S[q, \dot{q}]$.

³⁶In general, there would not be *any* trajectory joining specified values of q and p at *both* t_1 and t_2 .

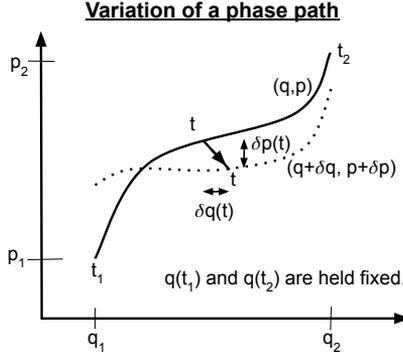


Figure 7: Infinitesimal variation of a phase path. Note that $p(t_1)$ and $p(t_2)$ are *not* held fixed.

Now,

$$\delta S = \int_{t_1}^{t_2} \left[\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right] dt + \dots \quad (43)$$

We find upon integrating by parts in the second term and using $\delta q(t_{1,2}) = 0$,

$$S[q+\delta q, p+\delta p] = S[q, p] + \int_{t_1}^{t_2} \left[\dot{q}^i \delta p_i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right] dt + \dots \quad (44)$$

The action must be stationary with respect to arbitrary infinitesimal independent variations δp , δq subject to $\delta q(t_1) = \delta q(t_2) = 0$. So the coefficients of δp and δq must individually vanish. Thus we recover Hamilton's equations (36) at all times $t_1 < t < t_2$:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (45)$$

Hamilton's equations treat position and momentum on an equal footing except for a sign. But the above boundary conditions treat them asymmetrically since we fix q at t_1 and t_2 but allow p to vary at these times. Could we do the opposite, i.e., fix the initial and final momenta but allow the positions to vary? This is a clue that there is *another* variational principle for Hamilton's equations. Consider the functional of a path on phase space³⁷

$$\tilde{S}[q, p] = \int_{t_1}^{t_2} [-q^j \dot{p}_j - H(q, p)] dt \quad (46)$$

which we extremize with respect to small variations δq , δp while holding $\delta p_j(t_1) = \delta p_j(t_2) = 0$. Then integrating by parts,

$$\delta \tilde{S} = \int_{t_1}^{t_2} \left[-\dot{p}_j \delta q^j - q^j \delta \dot{p}_j - \frac{\partial H}{\partial q^j} \delta q^j - \frac{\partial H}{\partial p_j} \delta p_j \right] dt$$

³⁷A way to remember \tilde{S} is that it is obtained from S through integration by parts while ignoring the boundary terms.

$$= \int_{t_1}^{t_2} \left[- \left(\dot{p}_j + \frac{\partial H}{\partial q^j} \right) \delta q^j + \left(\dot{q}^j - \frac{\partial H}{\partial p_j} \right) \delta p_j \right] dt. \quad (47)$$

So $\delta \tilde{S} = 0$ also implies Hamilton's equations. We will exploit both these variational principles while studying canonical transformations.

0.12 Canonical Poisson brackets

In §0.10, we defined the Hamiltonian as the Legendre transform of the Lagrangian and used it to express the equations of motion as 1st order ODEs (36). Now, we introduce another way of obtaining and thinking about Hamilton's equations: via Poisson brackets. We will see that when expressed in terms of Poisson brackets, Hamilton's equations take the same form for any observable and for any choice of coordinates on phase space. Poisson brackets endow Hamiltonian mechanics with an elegant algebraic and geometric structure which has been hidden from view thus far. They turn out to be a very useful conceptual and technical tool (e.g., to study conserved quantities, symmetries, discuss solvability of equations and so on.) and played a central role in Dirac's formulation of quantum mechanics via their quantum counterpart: the commutator of matrices/operators³⁸.

Consider a particle (or system of particles) with configuration space \mathbb{R}^n , generalized coordinates q^i and generalized momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$. To motivate the idea of Poisson brackets (PBs), let us use Hamilton's equations ($\dot{q}^i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q^i}$) to find the time evolution of any dynamical variable $f(q, p; t)$. In general, f is a function on phase space, which could depend explicitly on time. By the chain rule, we get

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{dq^i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} \right) \\ &= \frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = \frac{\partial f}{\partial t} + \{f, H\}. \end{aligned} \quad (48)$$

Here, we introduced Poisson's bracket $\{\cdot, \cdot\}$ of f with the Hamiltonian (in the expression $\{\cdot, \cdot\}$, the dots are 'placeholders' for any pair of functions). More generally, the

³⁸Dirac recalls thinking about this “(uv – vu)” one Sunday in October 1925. In words that nearly match those of Dirac [?], he noticed that there seemed to be a close similarity between the Poisson bracket and the quantum commutator. He did not remember very well the precise formula for a Poisson bracket and only had some vague recollections. He thought he might be getting to some big idea. However, he could not find any reference to Poisson brackets in the notes he had taken at various lectures or in the text books he had at home. There was nothing he could do, because it was Sunday evening and the libraries were all closed. He had to wait impatiently through that night. The next morning, he hurried along to one of the libraries as soon as it was open, and then looked up Poisson brackets in Whittaker's *Analytical Dynamics*. He found that they were just what he needed!

canonical PB of 2 dynamical variables³⁹ gives another dynamical variable defined as⁴⁰

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \quad (49)$$

In the Poisson bracket $\{f, g\}$, we refer to f as the function in the first slot or entry and g as occupying the second. So the time derivative of any observable is given by its Poisson bracket with the Hamiltonian (aside from any explicit time-dependence). From here on, we will restrict to observables that are not explicitly time-dependent (i.e., depend on time only via $q(t)$ and $p(t)$), unless otherwise stated. Hamilton's equations for time evolution may now be written

$$\dot{q}^i = \{q^i, H\} \quad \text{and} \quad \dot{p}_j = \{p_j, H\}. \quad (50)$$

Verify that upon use of (49), these equations reduce to Hamilton's equations (36). If H isn't explicitly dependent on time, then time does not appear explicitly on the RHS of Hamilton's equations. In this case, we say that the ODEs for q and p are an autonomous system.

One advantage of Poisson brackets is that the time evolution of *any* observable $f(q, p)$ is given by an equation of the same sort $\dot{f} = \{f, H\}$. We say that the Hamiltonian generates infinitesimal time evolution via the Poisson bracket:

$$f(q(t + \delta t), p(t + \delta t)) = f(q(t), p(t)) + (\delta t) \{f, H\} + \mathcal{O}((\delta t)^2). \quad (51)$$

If $\{f, g\} = 0$, we say that f 'Poisson commutes' with g . In particular, f is a constant of motion iff it Poisson commutes with the Hamiltonian, $\dot{f} = 0 \Leftrightarrow \{f, H\} = 0$. We begin to see the utility of the Poisson bracket in the study of conserved quantities.

Properties of the Poisson bracket. The Poisson bracket has some notable properties. The PB of any dynamical variable with a constant is zero. The Poisson bracket is *linear* in each entry. Verify that $\{f, \mu g\} = \mu \{f, g\}$ and $\{f, g + h\} = \{f, g\} + \{f, h\}$, where μ is a real constant. More generally, $\{f, \lambda g + \mu h\} = \lambda \{f, g\} + \mu \{f, h\}$ for $\lambda, \mu \in \mathbb{R}$ and any 3 dynamical variables f, g, h .

The Poisson bracket (49) is *antisymmetric* in the dynamical variables $\{f, g\} = -\{g, f\}$. In particular, the PB of any observable with itself vanishes $\{h, h\} = 0$. A special case of this encodes the conservation of energy. Assuming H isn't explicitly dependent on time,

$$\frac{dH}{dt} = \{H, H\} = 0. \quad (52)$$

Since formula (49) for the PB involves only first order derivatives of f , the PB satisfies the *Leibniz* or *product rule* of differentiation. In fact, one verifies that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \quad \text{and} \quad \{f, gh\} = \{f, g\}h + g\{f, h\}. \quad (53)$$

³⁹The Poisson bracket $\{f, g\}$ has dimensions of $[fg/\text{action}]$.

⁴⁰Some authors (e.g., Landau & Lifshitz) define the PB with an overall minus sign relative to our definition. When this is done, the equation of motion becomes $\dot{f} = \{H, f\}$.

Antisymmetry ensures that the Leibniz rule applies to the second entry once it holds for the first entry.

The basic Poisson brackets are between the basic dynamical variables, namely coordinates and momenta. The above formulae give for one degree of freedom

$$\{q, p\} = 1 \quad \text{or} \quad \{p, q\} = -1, \quad \text{while} \quad \{q, q\} = 0 \quad \text{and} \quad \{p, p\} = 0. \quad (54)$$

The last 3 equations are consequences of the antisymmetry of the PB. For n -degrees of freedom, the basic PB among coordinates and momenta follow from (49):

$$\{q^i, p_j\} = \delta_j^i \quad \text{and} \quad \{q^i, q^j\} = \{p_i, p_j\} = 0 \quad \text{for} \quad 1 \leq i, j \leq n. \quad (55)$$

These are called the canonical ('standard') PB relations between coordinates and conjugate momenta. The noun *canon* and the adjective *canonical* refer to something that is standard or conventional. For the Cartesian coordinates and momenta of a particle moving in 3d, the nonvanishing canonical PBs (up to antisymmetry) are

$$\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1. \quad (56)$$

Poisson's theorem: A remarkable feature of the Poisson bracket is that it can be used to produce new conserved quantities from a pair of existing ones. Poisson's theorem states that if f and g are conserved, then so is $\{f, g\}$. Let us first illustrate this with a couple of examples. For a free particle moving on a plane we know that p_x and p_y are both conserved. Their Poisson bracket is $\{p_x, p_y\} = 0$, which is of course trivially a conserved quantity. As a second example, consider a particle moving in three dimensions under the influence of a central potential. We know that $L_x = yp_z - zp_y$ and $L_y = zp_x - xp_z$ are both conserved. We compute $\{L_x, L_y\}$ by using bilinearity, the Leibniz rule and the basic PBs (56) and find $\{L_x, L_y\} = L_z$:

$$\begin{aligned} \{L_x, L_y\} &= \{yp_z - zp_y, zp_x - xp_z\} \\ &= \{yp_z, zp_x\} - \{yp_z, xp_z\} - \{zp_y, zp_x\} + \{zp_y, xp_z\} \\ &= y\{p_z, z\}p_x + p_y\{z, p_z\}x = -yp_x + xp_y = L_z. \end{aligned} \quad (57)$$

And indeed, we know that L_z is also a conserved quantity. Similarly we check that

$$\{L_x, L_y\} = L_z, \quad \{L_y, L_z\} = L_x \quad \text{and} \quad \{L_z, L_x\} = L_y. \quad (58)$$

This is called the angular momentum Poisson algebra.

Jacobi's identity: More generally, Poisson's theorem is a consequence of the Jacobi identity. For any three dynamical variables f, g and h , the following cyclic sum of twice-iterated Poisson brackets vanishes:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \quad (59)$$

Using antisymmetry, we could rewrite the Jacobi identity as

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \quad (60)$$

Before we prove the Jacobi identity, let us use it to establish Poisson's theorem. Suppose f, g are conserved so that each of them Poisson commutes with the Hamiltonian $h = H$, i.e., $\{f, H\} = \{g, H\} = 0$. Then the Jacobi identity implies that

$$\{\{f, g\}, H\} = 0 \quad \Rightarrow \quad \frac{d}{dt}\{f, g\} = 0. \quad (61)$$

So the PB of two conserved quantities is again a conserved quantity.

Poisson's tensor: To prove the Jacobi identity, it is convenient to introduce a compact notation. Let us combine the positions and momenta into a $2n$ -component 'grand' coordinate ξ on phase space and write its components with upper indices:

$$\xi = (\xi^1, \xi^2, \dots, \xi^n, \xi^{n+1}, \dots, \xi^{2n}) = (\mathbf{q}, \mathbf{p}) = (q^1, \dots, q^n, p_1, \dots, p_n). \quad (62)$$

Then check that the basic Poisson bracket relations may be expressed in terms of ξ^i

$$\{\xi^i, \xi^j\} = r^{ij} \quad \text{where} \quad r^{\text{row, column}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (63)$$

Here, r is a $2n \times 2n$ block matrix with $n \times n$ blocks proportional to the identity and zero matrices as indicated. The constant matrix with components r^{ij} is called the Poisson tensor of the canonical PB relations. At present, r^{ij} may simply be regarded as a short-form for $\{\xi^i, \xi^j\}$. We will soon understand its properties and begin to appreciate its utility, for instance in proving the Jacobi identity in (70).

The PB of any pair of observables may now be written in terms of the 'basic' PB between coordinates and momenta. Beginning with $n = 1$, show that

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) = \sum_{i,j=1}^{2n} \frac{\partial f}{\partial \xi^i} \frac{\partial g}{\partial \xi^j} \{\xi^i, \xi^j\} = \sum_{i,j=1}^{2n} r^{ij} \partial_i f \partial_j g. \quad (64)$$

Here, we denote partial differentiation by $\partial_i f = \frac{\partial f}{\partial \xi^i}$. Henceforth, we will suppress the summation symbol with a sum over repeated indices being implied. All properties of the canonical Poisson brackets are encoded in the Poisson tensor. Of particular importance to us is the antisymmetry of r^{ij} (equivalent to antisymmetry of the PB) and the constancy of the components r^{ij} .

Remarks on Leibniz rule and derivation property. Eq. (64) leads to some useful formulae for PBs. For instance, taking $f = \xi^k$,

$$\{\xi^k, g(\xi)\} = \frac{\partial \xi^k}{\partial \xi^i} \frac{\partial g}{\partial \xi^j} \{\xi^i, \xi^j\} = \frac{\partial g}{\partial \xi^j} \{\xi^k, \xi^j\}. \quad (65)$$

In particular, taking the PB of a function of position with a momentum coordinate produces the corresponding partial derivative:

$$\{p_k, g(q)\} = \frac{\partial g}{\partial q^j} \{p_k, q_j\} + \frac{\partial g}{\partial p_j} \{p_k, p_j\} = -\frac{\partial g}{\partial q^k}. \quad (66)$$

This is again a reflection of the Leibniz rule: the PB acts as a ‘derivation’. In fact, a useful way of expressing this derivation property is

$$\{f, g\} = \frac{\partial f}{\partial \xi^i} \{\xi^i, g\}. \quad (67)$$

The proof follows by noticing that both sides are equal to (64). On the other hand, suppose f depends on ξ^i only through some function $h(\xi)$, then

$$\{f(h(\xi)), g\} = f'(h) \frac{\partial h}{\partial \xi^i} \{\xi^i, g\} = f'(h) \{h, g\}. \quad (68)$$

Proof of Jacobi’s identity: We wish to evaluate the cyclic sum

$$J = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \quad (69)$$

We use the Poisson tensor and the Leibniz rule to write the first term of J as

$$\{\{f, g\}, h\} = (f_i g_j r^{ij})_k h_l r^{kl} = [f_{ik} g_j h_l + f_i g_{jk} h_l] r^{ij} r^{kl}. \quad (70)$$

Here, $f_i = \frac{\partial f}{\partial \xi^i}$ denotes the partial derivative. In expressions such as these, r^{ij} are entries of a real matrix and h_l are real-valued functions, so they can be written in any order. Adding its cyclic permutations,

$$J = [f_{ik} g_j h_l + f_i g_{jk} h_l + g_{ik} h_j f_l + g_i h_{jk} f_l + h_{ik} f_j g_l + h_i f_{jk} g_l] r^{ij} r^{kl}. \quad (71)$$

If J has to vanish for any sufficiently smooth functions f, g, h on phase space, then the terms involving 2^{nd} derivatives of f must mutually cancel as must those involving 2^{nd} derivatives of g or h . So let us consider the two terms involving second derivatives of f , and call their sum J_f . We find

$$\begin{aligned} J_f &= f_{ik} g_j h_l r^{ij} r^{kl} + f_{jk} g_l h_i r^{ij} r^{kl} = f_{ik} g_j h_l r^{ij} r^{kl} + f_{ik} g_l h_j r^{j i} r^{kl} \\ &= f_{ik} g_j h_l r^{ij} r^{kl} + f_{ik} g_j h_l r^{li} r^{kj} = f_{ik} g_j h_l r^{ij} r^{kl} + f_{ki} g_j h_l r^{lk} r^{ij} \\ &= f_{ik} g_j h_l r^{ij} r^{kl} - f_{ik} g_j h_l r^{ij} r^{kl} = 0. \end{aligned} \quad (72)$$

We relabelled indices of summation $i \leftrightarrow j$, $j \leftrightarrow l$ and $i \leftrightarrow k$ in the three successive equalities and finally used the equality of mixed partial derivatives $\frac{\partial^2 f}{\partial \xi^i \partial \xi^k} = \frac{\partial^2 f}{\partial \xi^k \partial \xi^i}$ and antisymmetry of the Poisson tensor $r^{kl} = -r^{lk}$. Thus, we have shown that $J_f = 0$. By cyclic symmetry, $J_g = J_h = 0$. Thus, $J = J_f + J_g + J_h = 0$ and the Jacobi identity has been established. As a corollary, we obtain Poisson’s theorem on the conservation of the PB of any two conserved quantities.

* **Symplectic form.** The canonical Poisson tensor is a $2n \times 2n$ invertible matrix with entries r^{ij} . Its inverse $\omega = r^{-1}$ is called the canonical symplectic tensor or form. Its components are determined by

$$\omega_{ij} r^{jk} = \delta_i^k \quad \text{where} \quad \delta_i^k \text{ is the Kronecker symbol.} \quad (73)$$

Antisymmetry of r implies that ω too is antisymmetric. As a matrix,

$$\omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}_{2n \times 2n}. \quad (74)$$

0.13 Canonical transformations

Recall that canonical variables are a set of position-type and momentum-type coordinates q^i, p_i on phase space such that the equations of motion take the standard Hamiltonian form $\dot{q}^i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q^i}$ (36) for a suitable Hamiltonian $H(q, p)$ and $i = 1, \dots, n$, where n is the number of degrees of freedom. Roughly, we will define a canonical transformation as a change of phase space variables $(q^i, p_j) \mapsto (Q^i(q, p), P_j(q, p))$ such that the equations of motion, when transformed to the new variables, continue to take a Hamiltonian form. Canonical transformations bear a resemblance to some other transformations we encounter in mathematical physics. An isomorphism L between vector spaces is one that preserves linear combinations: $L(au + bv) = aL(u) + bL(v)$ for any vectors u, v and scalars a, b . In other words, it preserves the linear structure of the space. An automorphism of a group is a map from a group to itself that preserves the group composition law. An isometry is a map between spaces that preserves the distances between points. In a somewhat similar fashion, a canonical transformation is a change of coordinates on phase space that preserves the canonical Hamiltonian form of the equations of motion (§0.13.1).

0.13.1 From point transformations to canonical transformations

Recall that the space of generalized coordinates and momenta (q, p) is called the phase space. Hamilton's equations $\dot{q}^i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q^i}$ (36) may be easier to solve (or understand qualitatively) in some systems of coordinates and momenta compared to others. For instance, some symmetries may be more obvious in certain coordinate systems rather than others and be manifested via the presence of more cyclic coordinates.

Example: Cartesian to plane polar coordinates. For a particle in a central potential $V(r)$ on the plane, rotations about the origin are a symmetry and the angle θ is a cyclic coordinate. This makes the EOM simpler to handle in polar coordinates r and θ rather than in Cartesian coordinates x, y . In fact, from the Lagrangian

$$\begin{aligned} L(x, y, \dot{x}, \dot{y}) &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2}) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) = \tilde{L}(r, \theta, \dot{r}, \dot{\theta}), \end{aligned} \quad (75)$$

we see that θ is a cyclic coordinate and its conjugate momentum $p_\theta = \frac{\partial \tilde{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is conserved. As a consequence, the radial equation

$$\dot{p}_r = \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{r}} = \frac{\partial \tilde{L}}{\partial r} \quad \text{or} \quad m\ddot{r} = p_\theta^2/mr^3 - V'(r) \quad (76)$$

decouples from the angular one, making them easier to solve. On the other hand, neither $p_x = \partial L/\partial \dot{x} = m\dot{x}$ nor $p_y = \partial L/\partial \dot{y} = m\dot{y}$ is conserved and the Cartesian position coordinates satisfy coupled equations. Now, the Hamiltonian is given by

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(\sqrt{x^2 + y^2}) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) = \tilde{H}(r, \theta, p_r, p_\theta). \quad (77)$$

One checks that the equations of motion take the same form in Cartesian and polar coordinates

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{y} &= \frac{\partial H}{\partial p_y}, & \dot{p}_x &= -\frac{\partial H}{\partial x}, & \dot{p}_y &= -\frac{\partial H}{\partial y} \\ \Leftrightarrow \dot{r} &= \frac{\partial \tilde{H}}{\partial p_r}, & \dot{\theta} &= \frac{\partial \tilde{H}}{\partial p_\theta}, & \dot{p}_r &= -\frac{\partial \tilde{H}}{\partial r}, & \dot{p}_\theta &= -\frac{\partial \tilde{H}}{\partial \theta}. \end{aligned} \quad (78)$$

Since the EOM take the same Hamiltonian form, we say that the transformation from Cartesian coordinates and conjugate momenta (x, y, p_x, p_y) to polar ones

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}}, \quad p_r = \frac{xp_x + yp_y}{\sqrt{x^2 + y^2}} \quad \text{and} \quad p_\theta = xp_y - yp_x, \quad (79)$$

is a canonical transformation (CT). For future reference, we notice that in this transformation, the new positions (r, θ) do not depend on the old momenta (p_x, p_y) . This is because the transformation arose from a change of variables on the $x - y$ configuration space with conjugate momenta changing in a manner specified by the Lagrangian. We also check that the basic Poisson brackets among coordinates and momenta (55) are preserved:

$$\begin{aligned} \{x, p_x\} &= \{y, p_y\} = 1, & \{x, p_y\} &= \{y, p_x\} = \{x, y\} = \{p_x, p_y\} = 0 \quad \text{and} \\ \{r, p_r\} &= \{\theta, p_\theta\} = 1, & \{r, p_\theta\} &= \{\theta, p_r\} = \{r, \theta\} = \{p_r, p_\theta\} = 0. \end{aligned} \quad (80)$$

We will soon see that this connection between preserving the form of Hamilton's equations and that of the PBs is generally valid. \square

Definition of a canonical transformation. Suppose we start with a system of coordinates q^i and conjugate momenta p_i , in which the EOM take the standard Hamiltonian form $\dot{q}^i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$ for $i = 1, \dots, n$. Such a system of coordinates and momenta is said to be canonical. Consider an invertible transformation from old variables (q^i, p_i) to new ones $(Q^i(q, p), P_i(q, p))$. Suppose the EOM in the new variables can be expressed in the Hamiltonian form $\dot{Q}^i = \frac{\partial \tilde{H}}{\partial P_i}$, $\dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q^i}$ where $\tilde{H}(Q, P) = H(q(Q, P), p(Q, P))$ is the transformed Hamiltonian. If this can be done for any choice of old Hamiltonian H , then we will call the transformation canonical. In other words, a CT is a transformation between two sets of canonical variables, it preserves the form of Hamilton's equations. If we make a change of variables that is canonical, we do not need to transform the EOM. The latter are guaranteed to take the Hamiltonian form, so we may write them down once we obtain the transformed Hamiltonian. This is useful since there are ways to guarantee the canonicity of a transformation without any reference to the Hamiltonian or Hamilton's equations.

Transformations that are not canonical. To be sure, not every choice of coordinates and momenta is canonical. For example, notice that Hamilton's equations treat coordinates and momenta on a nearly equal footing. So suppose we simply swap them by defining $Q = p$ and $P = q$. Then the Hamiltonian may be written in terms

of the new variables $H(q, p) = H(P, Q)$. The new Hamiltonian is then defined as $\tilde{H}(Q, P) = H(q(Q, P), p(Q, P)) = H(P, Q)$. We find that

$$\dot{Q} = \dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial \tilde{H}}{\partial P} \quad \text{and} \quad \dot{P} = \dot{q} = \frac{\partial H}{\partial p} = \frac{\partial \tilde{H}}{\partial Q}. \quad (81)$$

So the EOM in the new variables *do not* have the form of Hamilton's equations (36), they are off by a sign. So $(q, p) \mapsto (p, q)$ is *not* a canonical transformation. We may also check that the transformation *does not* preserve the basic PB

$$\{q, p\} = 1 \quad \text{while} \quad \{Q, P\} = \{p, q\} = -1. \quad (82)$$

Point transformation. Any change of coordinates alone ('point transformation' on configuration space) $q^i \rightarrow Q^i$, with the associated 'induced' change in momenta $P_i = \frac{\partial \tilde{L}}{\partial \dot{Q}^i}$ is automatically canonical. An example of such a canonical transformation is the one from Cartesian to polar coordinates for a particle on a plane. Interestingly, there are canonical transformations that are *more general* than those resulting from changes of coordinates on the configuration space \mathcal{Q} . Perhaps the simplest such examples are (1) $Q = p, P = -q$ and (2) $Q = -p, P = q$ which exchange coordinates and momenta up to a sign for one degree of freedom. Check that Hamilton's equations retain their form, as do the basic PBs (54).

We may of course compose CTs $(q, p) \rightarrow (Q, P) \rightarrow (\tilde{Q}, \tilde{P})$ to make new CTs and also invert a CT. Thus, canonical transformations form a group with identity given by $Q = q, P = p$. The group of CTs is generally infinite dimensional (even for a particle with one degree of freedom).

0.13.2 CTs preserve basic PBs, Time evolution as a CT

In the above examples of CTs, along with Hamilton's equations, the basic PBs among coordinates and momenta are preserved. This is generally true. It is worth noting that a transformation is canonical irrespective of what the Hamiltonian is. The form of Hamilton's equations must be unchanged for *any* smooth $H(q, p)$. This suggests it should be possible to state the condition of canonicity without reference to the Hamiltonian. This is indeed the case. In fact, a transformation $[(q, p) \mapsto (Q, P)]$ preserves the form of Hamilton's equations (i.e., is canonical) if and only if it preserves the basic Poisson brackets, i.e.,

$$\{Q^i, P_j\} = \delta_j^i, \quad \{Q^i, Q^j\} = \{P_i, P_j\} = 0. \quad (83)$$

We will skip the proof of this assertion here (for one degree of freedom, the proof may be found in §0.13.3*) but will discuss some consequences and examples.

Time evolution as a canonical transformation Perhaps the most important examples of canonical transformations are furnished by time evolution by any fixed Hamiltonian H (which could depend explicitly on time). Let us indicate why this is true. Suppose $q(0), p(0)$ are canonical phase space variables describing the possible state

of a particle at $t = 0$. Then $\{q(0), p(0)\} = 1$ while $\{q(0), q(0)\} = 0 = \{p(0), p(0)\}$. After a small time δt , the position and momentum are given by

$$q(\delta t) = q(0) + \delta t \frac{\partial H}{\partial p} + \mathcal{O}(\delta t^2) \quad \text{and} \quad p(\delta t) = p(0) - \delta t \frac{\partial H}{\partial q} + \mathcal{O}(\delta t^2), \quad (84)$$

where we made use of Hamilton's equations (36). To show that infinitesimal time evolution is a CT, we must show that $\{q(\delta t), p(\delta t)\} = 1$ and $\{q(\delta t), q(\delta t)\} = 0 = \{p(\delta t), p(\delta t)\}$. The latter follow by anti-symmetry. We evaluate the former keeping terms up to order δt :

$$\begin{aligned} \{q(\delta t), p(\delta t)\} &= \left\{ q(0) + \delta t \frac{\partial H}{\partial p}, p(0) - \delta t \frac{\partial H}{\partial q} \right\} + \dots \\ &= \{q(0), p(0)\} + \delta t \left(-\{q(0), \frac{\partial H}{\partial q}\} + \left\{ \frac{\partial H}{\partial p}, p(0) \right\} \right) + \dots \\ &= 1 + \delta t (-H_{pq} + H_{qp}) + \mathcal{O}(\delta t)^2 = 1 + \mathcal{O}(\delta t)^2. \end{aligned} \quad (85)$$

On the RHS all quantities are evaluated at $t = 0$. More generally, the equal-time Poisson brackets of coordinates and momenta

$$\{q^i(t), p_j(t)\} = \delta_j^i \quad \text{and} \quad \{q^i(t), q^j(t)\} = \{p_i(t), p_j(t)\} = 0 \quad (86)$$

are valid at all times t . Thus, the transformation

$$(q^i(t_1), p_i(t_1)) \mapsto (q^i(t_2), p_i(t_2)), \quad (87)$$

which is a map from phase space to itself, is canonical for any two times t_1 and t_2 . Keeping t_1 fixed and allowing t_2 to vary, Hamiltonian time evolution gives us a 1-parameter family of canonical transformations labelled by time t_2 .

Passive and active viewpoints. It is noteworthy that in this discussion, we have changed our perspective. In §0.13.1 we spoke of a change of phase space coordinates $(q, p) \mapsto (Q, P)$ as being canonical if the PBs are preserved. Here, we extend the concept to maps from phase space M to itself. Under a coordinate change, a physical point on phase space gets a new address. By contrast, under a map from M to M , a point (q, p) on phase space moves to a possibly different point (q', p') on M . We declare that the map is a CT if the primed quantities satisfy the same canonical PBs as the unprimed ones. Time evolution from t_1 to t_2 is evidently a map from M to M , since it takes any phase point to another phase point (along a trajectory). Thus (86), considered at two times t_1 and t_2 , would imply that time evolution is a CT in this new sense. In what follows, we will use the term canonical transformation both for *passive* changes of coordinates on phase space and for *active* transformations that rearrange points on phase space while preserving the canonical PB relations.

0.13.3* Form of Hamilton's equations is preserved iff PBs are preserved

Here we give the proof of an assertion made at the beginning of §0.13.2. It is to provide additional details but is not officially part of the course.

Recall that a transformation is canonical irrespective of what the Hamiltonian is. The form of Hamilton's equations must be unchanged for *any* smooth $H(q, p)$. This suggests it should be possible to state the condition of canonicity without reference to the Hamiltonian. This is indeed the case.

Proposition. A transformation preserves the form of Hamilton's equations (i.e., is canonical) if and only if it preserves the basic Poisson brackets.

Proof: For simplicity, we consider the case of one degree of freedom, though the argument can be extended, along similar lines, to more degrees of freedom. Suppose we make a smooth invertible change of coordinates and momenta $q \mapsto Q = Q(q, p)$ and $p \mapsto P = P(q, p)$. The inverse transformation expresses the old coordinates and momenta in terms of the new ones $q = q(Q, P)$ and $p = p(Q, P)$. The old coordinates satisfy canonical Poisson brackets $\{q, p\} = 1$, $\{q, q\} = \{p, p\} = 0$. Under this change, the old Hamiltonian $H(q, p)$ transforms into a new Hamiltonian $\tilde{H}(Q, P) = H(q(Q, P), p(Q, P))$. Suppose the above transformation preserves the form of Hamilton's equations. Denoting partial derivatives by subscripts,

$$\dot{q} = H_p \quad \text{and} \quad \dot{p} = -H_q \quad \text{while} \quad \dot{Q} = \tilde{H}_P \quad \text{and} \quad \dot{P} = -\tilde{H}_Q. \quad (88)$$

Then we will show that $\{Q, P\} = 1$. The other two Poisson brackets $\{Q, Q\}$ and $\{P, P\}$ vanish by antisymmetry.

By Hamilton's equations and the chain rule,

$$\dot{Q} = \tilde{H}_P = H_q q_P + H_p p_P = -\dot{p} q_P + \dot{q} p_P. \quad (89)$$

We also know that $\dot{Q} = Q_q \dot{q} + Q_p \dot{p}$. We want to equate these two expressions and extract information about the PB $\{Q, P\} = Q_q P_p - Q_p P_q$. As it is the partial derivatives of Q and P that appear in this PB, it is of interest to express q_P, p_P etc., in terms of partial derivatives of Q and P using the following Lemma.

Lemma. The partial derivatives of the new coordinates and momenta are related to those of the old coordinates and momenta via

$$q_P = -\frac{Q_p}{\{Q, P\}}, \quad p_P = \frac{Q_q}{\{Q, P\}}, \quad p_Q = -\frac{P_q}{\{Q, P\}} \quad \text{and} \quad q_Q = \frac{P_p}{\{Q, P\}}. \quad (90)$$

Proof: Recall that for an invertible map of one variable, $x \rightarrow X(x) = f(x)$ with inverse $x = x(X) = f^{-1}(X)$ the derivatives are reciprocally related $\frac{dX}{dx} = \left(\frac{dx}{dX}\right)^{-1}$. This arises from differentiating the identity $(f \circ f^{-1})(X) = f(f^{-1}(X)) = X$ with respect to X . We get $f'(x)x'(X) = 1$ or $X'(x)x'(X) = 1$.

What is the analogous formula for the map $F : x^i = (q, p) \mapsto X^i = (Q, P)$? Here

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(q, p) = (Q, P) \quad \text{and} \quad F^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F^{-1}(Q, P) = (q, p). \quad (91)$$

In other words, $F^{-1}(\mathbf{X}) = \mathbf{x}$ or $F^i(\mathbf{x}) = X^i$. As before, $F \circ F^{-1}$ is the identity, so

$$F(F^{-1}(\mathbf{X})) = \mathbf{X} \Rightarrow F^i(\mathbf{x}(\mathbf{X})) = X^i, \text{ and differentiating, } \frac{\partial F^i}{\partial x^j} \frac{\partial x^j}{\partial X^k} = \delta_k^i. \quad (92)$$

Here, the Jacobian matrices of first partials are

$$\frac{\partial F^i}{\partial x^j} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \quad \text{and} \quad \frac{\partial x^j}{\partial X^k} = \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix}. \quad (93)$$

The above condition says that the matrix product must be the identity

$$\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (94)$$

We get a system of 4 inhomogeneous linear equations for the ‘unknowns’ q_Q, q_P, p_Q and p_P :

$$Q_q q_P = -Q_p p_P, \quad P_q q_P + P_p p_P = 1, \quad P_q q_Q = -P_p p_Q, \quad \text{and} \quad Q_q q_Q + Q_p p_Q = 1. \quad (95)$$

These are, in fact, two pairs of linear equations in two unknowns each.

$$\begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} \begin{pmatrix} q_P \\ p_P \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} \begin{pmatrix} p_Q \\ q_Q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (96)$$

Their solution involves the reciprocal of the determinant of the coefficient matrix, which is just the PB $\{Q, P\} = Q_q P_p - Q_p P_q$:

$$q_P = -\frac{Q_p}{\{Q, P\}}, \quad p_P = \frac{Q_q}{\{Q, P\}}, \quad p_Q = -\frac{P_q}{\{Q, P\}}, \quad q_Q = \frac{P_p}{\{Q, P\}}, \quad (97)$$

as desired. \square

Armed with this Lemma, we return to (89) and the fact that the EOM retains its form in the new variables:

$$\dot{Q} = \tilde{H}_P = \frac{[\dot{p}Q_p + \dot{q}Q_q]}{\{Q, P\}} = \frac{\dot{Q}}{\{Q, P\}}. \quad (98)$$

For this equation to be satisfied we must have $\{Q, P\} = 1$. Similarly,

$$\begin{aligned} \dot{P} &= -\tilde{H}_Q = -H_q q_Q - H_p p_Q = \dot{p} \frac{P_p}{\{Q, P\}} + \dot{q} \frac{P_q}{\{Q, P\}} \\ &= \frac{1}{\{Q, P\}} [P_q \dot{q} + P_p \dot{p}] = \frac{\dot{P}}{\{Q, P\}}. \end{aligned} \quad (99)$$

For this to hold, again we need $\{Q, P\} = 1$. Thus, we have shown that if Hamilton’s equations retain their form in the new coordinates, then the transformation also preserves the form of the basic PBs.

Let us now prove the converse, namely, if the transformation $(q, p) \mapsto (Q, P)$ preserves the form of the PB, i.e., if $\{Q, P\} = 1$, then it must preserve the form of Hamilton's equations. So we must show that $\frac{\partial \tilde{H}}{\partial P} = \dot{Q}$ and $-\frac{\partial \tilde{H}}{\partial Q} = \dot{P}$. Let us compute

$$\frac{\partial \tilde{H}}{\partial P} = \frac{\partial H}{\partial q} q_P + \frac{\partial H}{\partial p} p_P = -\dot{p}q_P + \dot{q}p_P = \dot{p}Q_p + \dot{q}Q_q = \dot{Q}. \quad (100)$$

In the penultimate equality we used the Lemma and the assumption that $\{Q, P\} = 1$. So the 1st of Hamilton's equations holds in the new variables! Similarly, we show that the 2nd of Hamilton's equations also holds in the new variables

$$\frac{\partial \tilde{H}}{\partial Q} = \frac{\partial H}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial Q} = -\dot{p}q_Q + \dot{q}p_Q = -\dot{p}P_p - \dot{q}P_q = -\dot{P}. \quad (101)$$

So we showed that if the transformation preserves the PB, then it is canonical (i.e., preserves the form of Hamilton's equations). \square

A similar result holds for several degrees of freedom. Thus, the transformation as well as the new coordinates and momenta are said to be canonical provided Hamilton's equations retain their form

$$\dot{Q}^i = \frac{\partial \tilde{H}}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q^i} \quad (102)$$

or equivalently, if the Poisson brackets are preserved:

$$\{Q^i, P_j\} = \delta_j^i \quad \text{and} \quad \{Q^i, Q^j\} = 0 = \{P_i, P_j\} \quad \text{for all } i, j. \quad (103)$$

0.13.4 Canonical transformations: Area-preserving maps

Here, we give a geometric interpretation of canonical transformations on the phase plane and then briefly discuss one generalization to higher dimensional phase spaces. For one degree of freedom, we have just one coordinate q and one canonically conjugate momentum p , which together parametrize the phase plane. There is only one nontrivial canonical PB $\{q, p\} = 1$ since $\{q, q\} = \{p, p\} = 0$ by antisymmetry. Thus, the map $(q, p) \mapsto (Q, P)$ is canonical iff $\{Q, P\} = 1$. This condition turns out to be equivalent to the condition that the Jacobian determinant for the change of variables be equal to one. On the other hand, we learn in calculus that the Jacobian determinant is precisely the factor that relates the old to the new area element on the phase plane. Thus, CTs for one degree of freedom coincide with transformations that preserve areas on the phase plane.

In more detail, suppose we map (q, p) to a new point with position $Q(q, p)$ and momentum $P(q, p)$. For the transformation to be canonical, the new variables must satisfy the same PB, i.e.,

$$1 = \{Q, P\} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q}. \quad (104)$$

The quantity that appears above is in fact the determinant of the Jacobian matrix of 1st partials

$$\det J = \det \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q}. \quad (105)$$

We recall from calculus that the Jacobian determinant is the factor relating oriented area elements on phase space

$$dQ dP = (\det J) dq dp \quad (106)$$

So a CT for a system with one degree of freedom is simply a transformation that preserves the area element on phase space. Such a transformation is called area preserving. Conversely, any area preserving transformation (i.e., one with $\det J = 1$) is also a CT (i.e., with $\{Q, P\} = 1$). Note that if $\det J = -1$, the transformation preserves magnitudes of areas but is orientation reversing.

Pictorially, what is an area-preserving map? The map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ specified by $q \rightarrow Q(q, p)$ and $p \rightarrow P(q, p)$ maps points on the plane to points on the plane. For example, it could be the translation map $q \rightarrow q + 1, p \rightarrow p + 2$ or a rotation. Under such a transformation, any domain $D \subset \mathbb{R}^2$ is mapped to a new one $D' = F(D)$. The map is area preserving if for any D with finite area, $\text{Ar}(D) = \text{Ar}(D')$ i.e., $\iint_D dq dp = \iint_{D'} dq dp$. The above condition guarantees this, since

$$\text{Ar}(D) = \iint_D dq dp = \int_{F(D)} \frac{dQ dP}{\det J} = \iint_{D'} dQ dP = \iint_{D'} dq dp = \text{Ar}(D'). \quad (107)$$

In the 2nd equality, we changed variables of integration, in the third we used $\det J = 1$ and finally relabelled the dummy variables of integration $Q \rightarrow q$ and $P \rightarrow p$.

What we have shown above is that CTs of the phase plane preserve areas. Reversing the steps, we also see that if a transformation preserves areas (and in particular, the infinitesimal area element), it must be canonical. Thus, area-preserving maps of the phase plane coincide with canonical transformations for one degree of freedom. Area-preserving maps include (but are not restricted to) rigid motions like translations and rotations of the *phase plane*. For example, time evolution by a generic Hamiltonian is a canonical transformation which in general will morph a disk on the phase plane into a complicated region having the same area.

Liouville's theorem We just argued that for one degree of freedom, CTs preserve areas in phase space. This is a special case of Liouville's theorem. For n degrees of freedom, it says that CTs preserve $2n$ -dimensional 'volumes' in phase space. In other words, suppose a $2n$ -dimensional region in phase space $D \subset \mathbb{R}^{2n}$ is mapped by a CT to a new region $D' \subset \mathbb{R}^{2n}$. Then $\text{Vol}(D) = \text{Vol}(D')$. Alternatively, it says that the volume element on phase space is invariant under a CT:

$$dP_1 dQ^1 \cdots dP_n dQ^n = dp_1 dq^1 \cdots dp_n dq^n. \quad (108)$$

0.13.5* CTs preserve formula for PB of any pair of observables

This section is for general understanding, not officially part of the course. It may be skipped.

In §0.13.3*, we showed that CTs preserve the basic PB between coordinates and momenta. What about the PB between arbitrary observables f, g ? In the old coordinates,

$$\{f, g\} \equiv \{f, g\}_{q,p} = \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \quad (109)$$

It turns out that if $(q, p) \rightarrow (Q, P)$ is canonical (i.e., preserves the basic PB) then (and only then), the formula for $\{f, g\}$ may also be expressed as⁴¹

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial Q^i} \frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial g}{\partial Q^i} \right). \quad (110)$$

Let us show this for one degree of freedom (though the same calculation also works for n degrees of freedom by putting in the indices.) Now by definition, the chain rule and rearranging terms,

$$\begin{aligned} \{f, g\} &= f_q g_p - f_p g_q \\ &= (f_Q Q_q + f_P P_q) (g_Q Q_p + g_P P_p) - (f_Q Q_p + f_P P_p) (g_Q Q_q + g_P P_q) \\ &= f_Q g_P (Q_q P_p - Q_p P_q) + f_P g_Q (P_q Q_p - P_p Q_q) \\ &\quad + f_Q g_Q (Q_q Q_p - Q_p Q_q) + f_P g_P (P_q P_p - P_p P_q) \\ &= (f_Q g_P - f_P g_Q) \{Q, P\} + f_Q g_Q \{Q, Q\} + f_P g_P \{P, P\}. \end{aligned} \quad (111)$$

The last two terms are identically zero by antisymmetry of the PB, but we displayed them as they help in writing the corresponding formula for n degrees of freedom:

$$\{f, g\} = (f_{Q^i} g_{P_j} - f_{P_j} g_{Q^i}) \{Q^i, P_j\} + f_{Q^i} g_{Q^j} \{Q^i, Q^j\} + f_{P_i} g_{P_j} \{P_i, P_j\}. \quad (112)$$

Thus, we see that

$$\{f, g\}_{q,p} = \sum_{i=1}^n \left(\frac{\partial f}{\partial Q^i} \frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial g}{\partial Q^i} \right) = \{f, g\}_{Q,P} \quad (113)$$

iff the new coordinates and momenta satisfy canonical PB relations, i.e.,

$$\{Q^i, P_j\} = \delta_j^i \quad \text{and} \quad \{Q^i, Q^j\} = 0 = \{P_i, P_j\}. \quad (114)$$

Thus, a transformation is canonical iff the PB of any pair of observables is given by the same sort of formula, whether computed using the old or new variables:

$$(q, p) \mapsto (Q, P) \quad \text{is a CT iff} \quad \{f, g\}_{q,p} = \{f, g\}_{Q,P} \quad \text{for all } f, g. \quad (115)$$

⁴¹We are being economical with notation here. When we differentiate with respect to Q and P we regard $f = f(q(Q, P), p(Q, P))$.

0.13.6 Generating function for finite canonical transformations from variational principle

Transformations between different sets of canonical coordinates and momenta are called canonical transformations. Here we seek to express finite⁴² canonical transformations in terms of generating functions. To do so, we will use the variational principles of §0.11 for Hamilton's equations. We will find that, subject to some mild conditions, any differentiable function of the form $F_1(q, Q, t)$ can be used to generate a finite canonical transformation for any fixed t .

Consider the (possibly explicitly time-dependent) map from $(q^i, p_j) \mapsto (Q^i, P_j)$ with the equations of transformation given by the functions

$$Q^i = Q^i(q, p, t) \quad \text{and} \quad P_i = P_i(q, p, t). \quad (116)$$

Such a change is canonical (for any fixed t) provided there is a new Hamiltonian $K(Q, P, t)$ (called \tilde{H} in (77)) such that the EOM in the new variables take the same form as those in the old variables, i.e.,

$$\dot{Q}^i = \frac{\partial K}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q^i} \quad \text{while} \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (117)$$

When the transformation is not explicitly dependent on time, $K(Q, P)$ is got by expressing q, p in terms of Q, P in the old Hamiltonian $H(q, p)$. We will see that essentially the same thing continues to be true, but with a slight modification. Now, our basic principle of consistency is that these two sets of Hamilton's equations should be equivalent in the sense that if we express Q and P in terms of q and p in the second set, they should reduce to the old Hamilton equations.

Each set of Hamilton's equations follows from the variational principle (42):

$$\delta \int_{t_i}^{t_f} [p_i \dot{q}^i - H(q, p)] dt = 0 \quad \text{and} \quad \delta \int_{t_i}^{t_f} [P_i \dot{Q}^i - K(Q, P, t)] dt = 0 \quad (118)$$

where t_i and t_f are initial and final times. The conditions for extrema of these two functionals $S_{\text{old}}[q, p]$ and $S_{\text{new}}[Q, P]$ are the same equations, just in different coordinates. We will now present a way in which this can happen. Let us define the functional

$$I[q, p, Q, P] = S_{\text{old}}[q, p] - S_{\text{new}}[Q, P] = \int_{t_i}^{t_f} (p\dot{q} - H - P\dot{Q} + K) dt. \quad (119)$$

Suppose I is stationary to first order with respect to small variations of $q^i(t)$, $p_i(t)$, $Q^i(t)$ and $P_i(t)$ holding the initial and final positions $q^j(t_{i,f})$ and $Q^j(t_{i,f})$ fixed. Now, if $\delta I = 0$, then $\delta S_{\text{old}} = \delta S_{\text{new}}$ and it follows that $\delta S_{\text{old}} = 0$ iff $\delta S_{\text{new}} = 0$. Thus, if $\delta I = 0$, then the two sets of Hamilton equations would be equivalent and consequently the transformation would be canonical. Next, we propose a way for $I[q, p, Q, P]$ to be

⁴²The word finite is used to distinguish this from an infinitesimal CT, which is one that departs infinitesimally from the identity CT.

stationary. Since the initial and final configurations are fixed, this can happen if I is independent of $p_j(t_{i,f})$, $P_j(t_{i,f})$ and other features of the paths with the exception of $q^j(t_{i,f})$ and $Q^j(t_{i,f})$. A sufficient condition for this is for the integrand in I to be the total time derivative of some function⁴³ $F_1(q, Q, t)$. For, then

$$I = \int_{t_i}^{t_f} \dot{F}_1 dt = F_1(q(t_f), Q(t_f), t_f) - F_1(q(t_i), Q(t_i), t_i) \quad (120)$$

and $\delta I = 0$ as q and Q are held fixed at the fixed times t_i and t_f . Note that F_1 cannot be taken as a function of p or P since $\delta p(t_i)$, $\delta p(t_f)$, $\delta P(t_i)$, $\delta P(t_f)$ are unconstrained in Hamilton's variational principle and the total time derivative of such an F_1 would violate the constancy of I . In other words, a sufficient condition for the equations in both old and new variables to take the Hamiltonian form is for the relation

$$p_i \dot{q}^i - H = P_i \dot{Q}^i - K + \frac{dF_1}{dt}, \quad (121)$$

to hold for some function $F_1(q, Q, t)$. Multiplying through by dt we get

$$pdq - Hdt = PdQ - Kdt + \frac{dF_1}{dt} dt. \quad (122)$$

That the independent variables in F_1 are q, Q and t is also consistent with the fact that the independent differentials appearing in the rest of the terms above are dt, dq and dQ . So, as an equation among the independent differentials dq, dQ and dt , we have

$$pdq - Hdt = PdQ - Kdt + \frac{\partial F_1}{\partial q} dq + \frac{\partial F_1}{\partial Q} dQ + \frac{\partial F_1}{\partial t} dt. \quad (123)$$

Comparing coefficients, we read off the relations

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q} \quad \text{and} \quad K(Q, P, t) = H(q, p) + \frac{\partial F_1(q, Q, t)}{\partial t}. \quad (124)$$

$F_1(q, Q)$ is called the generator of the CT. The first two equations determine the equations of transformation. The first may be solved to find $Q = Q(q, p, t)$ and using it, the second expresses $P = P(q, p, t)$. The last relation fixes the new Hamiltonian in terms of the old one and the generator F_1 . If F_1 does not depend explicitly on time, then it just says that $K(Q, P) = H(q(Q, P), p(Q, P)) = \tilde{H}(Q, P)$ as in (77). In general, the new and old Hamiltonians differ by the partial time derivative of the generator.

Condition on F_1 to be a generator. However, not every function $F_1(q, Q, t)$ is a legitimate generator. For example, $F_1(q, Q) = q + Q$ would imply $p = 1$ and $P = -1$ which in general cannot be solved to express Q, P in terms of q, p . Similarly, $F_1 = q^2 + Q^2$ also does not generate a CT since it implies $p = 2q, P = -2Q$ which

⁴³We will need to impose some mild conditions of F_1 (including twice differentiability) to ensure that the transformation is invertible. More on this soon. The subscript on F_1 is in anticipation of other possibilities which will involve functions that we will call F_2, F_3 and F_4 .

cannot be solved to express Q, P as functions of q, p . On the other hand, a choice that *does* generate a CT is $F_1(q, Q) = qQ$. In this case, $Q = p$ and $P = -q$ and we have an exchange of coordinates and momenta up to a sign. What CT does $F_1 = -qQ$ generate?

Motivated by these examples, we observe that a necessary condition for $F_1(q, Q)$ to generate a CT is that the ‘Hessian’ of unlike second partials $\frac{\partial^2 F_1}{\partial q \partial Q}$ be nonvanishing⁴⁴. This will allow us to use $p = \partial F_1(q, Q)/\partial q$ to solve for Q in terms of q, p , at least locally. When the second partial is nonvanishing, $\partial F_1(q, Q)/\partial q$ depends nontrivially on Q which can then be solved for and then inserted in $P = -\frac{\partial F_1(q, Q)}{\partial Q}$ to express $P = P(q, p)$. \square

Limitations of the generator $F_1(q, Q, t)$. The identity transformation $Q = q, P = p$ is not expressible via a generating function $F_1(q, Q, t)$. The latter expresses

$$p = \frac{\partial F_1(q, Q)}{\partial q} = p(q, Q) \quad \text{and} \quad P = -\frac{\partial F_1(q, Q)}{\partial Q} = p(q, Q). \quad (125)$$

However, for the identity transformation, it is not possible to express P as a function of Q and q . On the other hand, F_1 is good at generating CTs that are in the vicinity of the one that exchanges coordinates and momenta up to a sign and a possible scaling, $Q = p/\lambda, P = -\lambda q$ that follows from the choice $F_1 = \lambda qQ$ for a nonzero real parameter λ .

Generator of second kind $F_2(q, P, t)$. To find a generator for other canonical transformations, we make use of the second variational principle $\tilde{S}[Q, P]$ (46) for Hamilton’s equations. Here, the momenta are held fixed at the endpoints $\delta P(t_i) = \delta P(t_f) = 0$. For the old Hamilton equations, we use the first variational principle $S[q, p]$ where $\delta q(t_i) = \delta q(t_f) = 0$:

$$\delta \int_{t_i}^{t_f} [p\dot{q} - H(q, p)] dt = 0 \quad \text{and} \quad \delta \int_{t_i}^{t_f} [-Q\dot{P} - K(Q, P)] dt = 0. \quad (126)$$

These two variational principles give the same equations if the integrands differ by the total time derivative of a function $F_2(q, P, t)$ since δq and δP are held fixed at the endpoints. Proceeding as before,

$$pdq - Hdt = -QdP - Kdt + \frac{\partial F_2}{\partial q} dq + \frac{\partial F_2}{\partial P} dP + \frac{\partial F_2}{\partial t} dt. \quad (127)$$

The resulting equations of transformation are

$$p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \quad \text{and} \quad K = H + \frac{\partial F_2}{\partial t}. \quad (128)$$

F_2 is called a generator of the second kind. As before, we will need to assume that $\partial^2 F/\partial P \partial q \neq 0$. It is easily seen that if $F_2(q, P) = qP$, then the resulting transformation is the identity $Q = q, p = P$.

⁴⁴For more degrees of freedom, the Hessian $\frac{\partial^2 F_1}{\partial q^i \partial Q^j}$ of unlike 2^{nd} partials must be invertible.

0.14 Kepler's problem of two gravitating point masses

We now address the *direct* problem of two gravitating point masses. Given Newton's laws of motion and gravity we wish to find the trajectories of the masses. We will use the conservation of momentum to reduce the two-body problem to an effective one-body problem and then use the conservation of energy and angular momentum to determine the elliptical shapes of planetary orbits and their time periods of revolution.

0.14.1 CM and relative variables, conservation laws

We wish to solve Newton's equations of motion for a pair of point masses, say $m_1 = m_s$ and $m_2 = m_e$, that interact via the inverse-square gravitational force to find the trajectory of the Earth or other planet around the Sun. However, we need to take care of the fact that both the Sun and Earth can move.

Center of mass and relative vectors

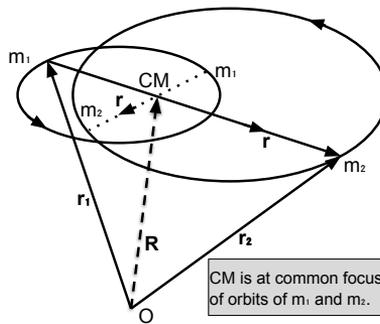


Figure 8: Center of mass (CM) and relative vectors \mathbf{R} and \mathbf{r} as well as position vectors \mathbf{r}_1 and \mathbf{r}_2 of the two masses. For $E < 0$, the ellipses traversed by m_1 and m_2 are indicated, they have a common focus at the CM, which is assumed to be at rest.

Let $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ denote the position vectors of the Sun and Earth relative to the origin of an inertial frame. The Sun-Earth system is one with 6 degrees of freedom, say, the Cartesian components of \mathbf{r}_1 and \mathbf{r}_2 . If we put $\alpha = Gm_1m_2$, then Newton's equations are

$$m_1\ddot{\mathbf{r}}_1 = \frac{\alpha}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1) \quad \text{and} \quad m_2\ddot{\mathbf{r}}_2 = -\frac{\alpha}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1). \quad (129)$$

This is a system of six 2nd order nonlinear ordinary differential equations (ODEs). We wish to solve these equations given the initial values $\mathbf{r}_1(0), \mathbf{r}_2(0), \dot{\mathbf{r}}_1(0)$ and $\dot{\mathbf{r}}_2(0)$. To complicate matters, the evolution of \mathbf{r}_1 and \mathbf{r}_2 are coupled since the RHS of (129) involves $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$: the Sun affects the motion of the Earth and vice versa. It would be helpful to find new variables in terms of which the dynamics decouples. It is clear from (129) that the relative vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ that points from the Sun to the Earth (see Fig. 8) is a natural variable in which to express the gravitational forces. In fact, if we introduce the potential $V(r) = -\alpha/r$, then the EOM become

$$m_1\ddot{\mathbf{r}}_1 = \frac{\alpha}{r^2}\hat{\mathbf{r}} = \nabla_{\mathbf{r}}V(r) \quad \text{and} \quad m_2\ddot{\mathbf{r}}_2 = -\frac{\alpha}{r^2}\hat{\mathbf{r}} = -\nabla_{\mathbf{r}}V(r). \quad (130)$$

What is more, if we add the two equations, the forces cancel and we find that $m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0$. In other words, if we introduce the center of mass vector $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ where $M = m_1 + m_2$ is the total mass, then $M\ddot{\mathbf{R}} = 0$. Thus, the center of mass (CM) of the system may be viewed as moving like a free particle of mass M along a straight line: $\mathbf{R}(t) = \mathbf{R}(0) + \dot{\mathbf{R}}(0)t$. Its dynamics has decoupled from the relative motion of m_1 and m_2 . Henceforth, wherever convenient, we will choose the origin of the inertial frame to lie at the CM so that the CM is stationary.

On the other hand, subtracting the first from the second in (130) gives us an evolution equation for the relative vector \mathbf{r} :

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\nabla V \Rightarrow m\ddot{\mathbf{r}} \equiv \dot{\mathbf{p}} = -\nabla V. \quad (131)$$

We abbreviated $\nabla_{\mathbf{r}} = \nabla$ and defined the relative momentum $\mathbf{p} = m\dot{\mathbf{r}}$ where $m = m_1m_2/(m_1 + m_2)$ is called the reduced mass⁴⁵. Thus, the relative vector \mathbf{r} evolves independently of the CM vector \mathbf{R} . So by switching from $\mathbf{r}_1, \mathbf{r}_2$ to \mathbf{R}, \mathbf{r} , via the transformation

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} \quad \text{and} \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \text{with} \quad \mathbf{r}_{1,2} = \mathbf{R} \mp \frac{m_{2,1}}{M}\mathbf{r}, \quad (132)$$

we have decoupled the equations of motion. We may view the dynamics of \mathbf{r} as a self-contained mechanical system with just three degrees of freedom. The latter system is a one-body problem concerning the motion of a fictitious particle⁴⁶ of mass m located at \mathbf{r} and subject to the central force $-\nabla_{\mathbf{r}}V(\mathbf{r})$.

Our aim is to solve the EOM for \mathbf{r} and then work our way back using (132) to find out how \mathbf{r}_1 and \mathbf{r}_2 vary with time. However, (131) remains a nonlinear coupled system of equations for the three Cartesian components of $\mathbf{r} = (x, y, z)$. To find the corresponding trajectories, we will exploit certain conservation laws.

Momentum. The simplest one to consider is that of momentum. Since $\dot{\mathbf{p}} = -\nabla V \neq 0$, the relative momentum is *not* conserved⁴⁷. However, the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = M\dot{\mathbf{R}}$ is conserved, indeed

$$\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \nabla V - \nabla V = 0. \quad (133)$$

Energy. To obtain a conserved energy, we dot the first equation in (130) by the integrating factor $\dot{\mathbf{r}}_1$ and the second by $\dot{\mathbf{r}}_2$ to get (compare with (??))

$$m_1\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_1 \cdot \nabla_{\mathbf{r}}V(\mathbf{r}) \quad \text{and} \quad m_2\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = -\dot{\mathbf{r}}_2 \cdot \nabla_{\mathbf{r}}V(\mathbf{r}). \quad (134)$$

⁴⁵There is no particle with mass m . But if $m_1 \gg m_2$, as is the case for the Sun and a planet, then the total mass $M = m_1 + m_2 \approx m_1$ is approximately that of the Sun and the reduced mass that of the planet $m \approx m_2$.

⁴⁶For the Sun-Earth system, since $m_1 \gg m_2$, the Sun is located nearly at the CM (which is taken to be the origin) and \mathbf{r} is nearly the position vector of the Earth, whose mass m_2 almost coincides with the reduced mass m .

⁴⁷We recall from Noether's theorem in §0.8 that this is related to the lack of translation invariance with respect to the relative coordinate. When we focus exclusively on relative motion, the center of mass is fixed at the origin and this breaks translation invariance.

Adding these two equations, we get

$$\frac{d}{dt} \left(\frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 \right) = -\dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} V(r) = -\frac{dV(r)}{dt}. \quad (135)$$

Hence, energy is conserved:

$$E_{\text{tot}} = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 + V(r) \quad \text{where} \quad V(r) = -\frac{\alpha}{r}. \quad (136)$$

It is revealing to write this in terms of the center of mass and relative coordinates. One finds

$$E_{\text{tot}} = E_{\text{cm}} + E \quad \text{where} \quad E_{\text{cm}} = \frac{1}{2} M \dot{\mathbf{R}}^2 \quad \text{and} \quad E = \frac{1}{2} m \dot{\mathbf{r}}^2 + V(r). \quad (137)$$

Since $\dot{\mathbf{R}}(t) = \dot{\mathbf{R}}(0)$, E_{cm} is separately conserved. It follows that the energy E of relative motion is also conserved.

Angular momentum. Using (131), one checks that the relative angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved, since the force is central ($\nabla_{\mathbf{r}} V(r) = V'(r) \hat{\mathbf{r}}$)

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \frac{1}{m} \mathbf{p} \times \mathbf{p} - \mathbf{r} \times \nabla_{\mathbf{r}} V(r) = 0. \quad (138)$$

The total angular momentum $\mathbf{L}_{\text{tot}} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$ is of course also conserved

$$\dot{\mathbf{L}}_{\text{tot}} = \mathbf{r}_1 \times \dot{\mathbf{p}}_1 + \mathbf{r}_2 \times \dot{\mathbf{p}}_2 = -(\mathbf{r}_2 - \mathbf{r}_1) \times \nabla V(r) = -\mathbf{r} \times \nabla V(r) = 0, \quad (139)$$

where we used $\dot{\mathbf{r}}_1 \times \mathbf{p}_1 = \dot{\mathbf{r}}_2 \times \mathbf{p}_2 = 0$.

0.14.2 Planetary orbits

In this section, we will use the conservation of relative angular momentum and energy to deduce the shapes of planetary orbits from Newton's laws of motion and gravity. Since the relative angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a conserved vector (138), we may choose $\hat{\mathbf{z}}$ along \mathbf{L} , assuming it is nonzero⁴⁸. Once $\hat{\mathbf{z}}$ is fixed, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ may be chosen so that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ and xyz form a right-handed system. From $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, both the relative vector \mathbf{r} and momentum \mathbf{p} must lie on the $(x-y)$ plane perpendicular to \mathbf{L} . By the right-hand rule for cross products, orbits must have a 'counterclockwise' sense on the $x-y$ plane. We will show that planetary orbits are ellipses with the center of mass at a focus. However, in order to do this, we need to be able to recognize the formula for an ellipse when we encounter it. Thus, we begin with a quick primer on the equation for an ellipse in plane polar coordinates, since that is the way it will arise.

⁴⁸If $\mathbf{L} = 0$, it means the planet is heading radially inwards or away from the Sun. Since planetary orbits are not of this sort, it is safe to assume $\mathbf{L} \neq 0$. If $\mathbf{L} = 0$, then the angular momentum vector does not define a direction.

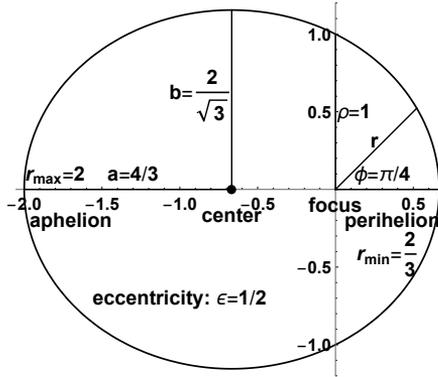


Figure 9: Ellipse $r = \rho/(1 + \epsilon \cos \phi)$ for semilatus rectum $\rho = 1$ and eccentricity $\epsilon = 1/2$ showing the lengths a and b of the semimajor and semiminor axes marked in bold. Note that the origin lies at the right focus rather than at the center, to make formulae simpler.

Equation for an ellipse in polar form. An ellipse centered at the origin of the x - y plane with axes along the x and y axes is defined as the set of points satisfying $x^2/a^2 + y^2/b^2 = 1$. For definiteness, we may take $a \geq b > 0$, they are the lengths of the semiaxes. The eccentricity $\epsilon = \sqrt{1 - b^2/a^2}$, which lies between 0 and 1 is a measure of the departure from a circle, which results when $a = b$ and $\epsilon = 0$. The points $(\pm\sqrt{a^2 - b^2}, 0)$ are called the foci of the ellipse. One verifies that the sum of the distances of any point on the ellipse from the foci is the constant $2a$. For application to planetary orbits, it is clear from Kepler's second law that it is more convenient to place the origin (roughly the Sun) at a focus rather than at the center of the ellipse. Furthermore, just as in circular motion, it is more convenient to use polar coordinates in place of Cartesian coordinates. We will soon show that the equation for an ellipse with the left focus on the x axis and the right focus at the origin of plane polar coordinates is

$$r = \frac{\rho}{(1 + \epsilon \cos \phi)}, \quad \text{where } \rho > 0 \tag{140}$$

is a constant length (the semilatus rectum) that sets the scale and $0 < \epsilon < 1$ is the eccentricity⁴⁹ (see Fig. 9 which shows the semilatus rectum extending from the focus upwards). Why is this an ellipse? When $\epsilon = 0$, it is obviously a circle of radius ρ . Using Cartesian coordinates $x = r \cos \phi$ and $y = r \sin \phi$, (140) becomes $r = \rho - \epsilon x$. Upon squaring and completing the square, we get

$$(1 - \epsilon^2)x^2 + 2\rho\epsilon x + y^2 - \rho^2 = 0 \quad \text{or} \quad \frac{(1 - \epsilon^2)^2}{\rho^2} \left(x + \frac{\rho\epsilon}{1 - \epsilon^2} \right)^2 + \frac{(1 - \epsilon^2)}{\rho^2} y^2 = 1. \tag{141}$$

⁴⁹Closest approach to the Sun (perihelion) occurs at $r = r_{\min} = \frac{\rho}{1 + \epsilon}$ when $\phi = 0$ and aphelion (farthest point from Sun) at $r = r_{\max} = \frac{\rho}{1 - \epsilon}$ when $\phi = \pi$. We have oriented and positioned the axes to ensure the semimajor axis is along \hat{x} and there is a $y \rightarrow -y$ symmetry. The origin $x = y = 0$ is at the right focus. A rotated ellipse results if we take $\frac{\rho}{r} = 1 + \epsilon \cos(\phi - \phi_0)$. By choosing $\phi_0 = \pi$ we may move the origin to the left focus.

This is now in the standard Cartesian form $\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$ of an ellipse centered at the point $(-\rho\epsilon/(1-\epsilon^2), 0)$. The semilatus rectum ρ is related to the length a of the semimajor axis via $\rho = a(1-\epsilon^2)$ while $b = \rho/\sqrt{1-\epsilon^2}$ is the semiminor axis and $\epsilon^2 = 1 - b^2/a^2$ (see Fig. 9 for the semiaxes, they are measured from the center). When $\epsilon > 1$, the coefficients of x^2 and y^2 have opposite signs and we get a hyperbola. For $\epsilon = 1$ we get the parabola $y^2 = \rho^2 - 2\rho x$ opening out to the left. A nice feature of (140) is that it applies not just to ellipses but also to parabolae and hyperbolae simply by extending the range of ϵ . \square

Elliptical orbits from conservation laws and orbit equation. Conservation of relative energy and angular momentum allow us to establish the elliptical shape of planetary orbits. More precisely, we will first show that the relative vector \mathbf{r} traverses an ellipse and then deduce that the planet \mathbf{r}_2 and the Sun \mathbf{r}_1 also follow elliptical paths in a frame where the CM is at rest (see Fig. 8). To begin with, the relative kinetic energy in (137) may be expressed as a sum of radial and angular kinetic energies and the latter absorbed into an ‘effective’ potential energy⁵⁰. If $m = m_1 m_2 / (m_1 + m_2)$ denotes the reduced mass, then

$$E = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}} \quad \text{where} \quad V_{\text{eff}} = \frac{l^2}{2mr^2} - \frac{\alpha}{r} \quad \text{and} \quad \alpha = Gm_1 m_2. \quad (142)$$

We get this from the square of relative angular momentum⁵¹ $\mathbf{L} = \mathbf{r} \times \mathbf{p}$,

$$\mathbf{L}^2 = l^2 = (\mathbf{r} \times \mathbf{p})^2 = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 \quad \Rightarrow \quad \frac{p^2}{2m} = \frac{(\hat{\mathbf{r}} \cdot \mathbf{p})^2}{2m} + \frac{l^2}{2mr^2}. \quad (143)$$

The latter is the promised splitting of kinetic energy into radial and angular contributions. In spherical polar coordinates, $l^2 = (mr^2)^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$ and for motion on the $\theta = \pi/2$ plane, $l = mr^2 \dot{\phi}$. However, despite the appearance of all these derivatives of angles, l^2 is a constant and the angular kinetic energy may be viewed as a function of r alone. It appears as a repulsive angular momentum barrier (sometimes referred to as a centrifugal barrier) $l^2/2mr^2$ in the effective potential V_{eff} , which also includes the attractive gravitational potential $-\alpha/r$. As Fig. 10 indicates, for any fixed E and $l^2 > 0$ the angular momentum term prevents the radial distance from becoming arbitrarily small.

Let us now determine the possible types of radial motion using the graph (Fig. 10) of $V_{\text{eff}}(r)$. For fixed angular momentum $l \neq 0$, the minimum energy orbit is a circle of radius $r = \rho = l^2/m\alpha$ which is the location of the minimum: $V'_{\text{eff}}(\rho) = 0$. The planet executes uniform circular motion $\dot{r} = 0$ and $\dot{\phi} = l/m\rho^2$ with energy $E = -\frac{\alpha}{2\rho}$. On the other hand, if $-\frac{\alpha}{2\rho} < E < 0$, from Fig. 10, we expect the radial distance to

⁵⁰The potential is typically the part of the energy that does not depend explicitly on velocities. The name effective potential is used because it differs from the Newtonian gravitational potential $V(r) = -\alpha/r$ by the term $l^2/2mr^2$. This latter term comes from the angular part of the kinetic energy, but is expressible in a manner independent of velocities due to the conservation of angular momentum.

⁵¹We used $\hat{\mathbf{r}} \cdot \mathbf{p} = m\dot{r}$. To see this, recall that $\hat{\mathbf{r}} = (x, y, z)/r$ and $\mathbf{p} = m(\dot{x}, \dot{y}, \dot{z})$. Thus, $\hat{\mathbf{r}} \cdot \mathbf{p} = (m/r)(x\dot{x} + y\dot{y} + z\dot{z}) = (m/2r)dr^2/dt = m\dot{r}$.

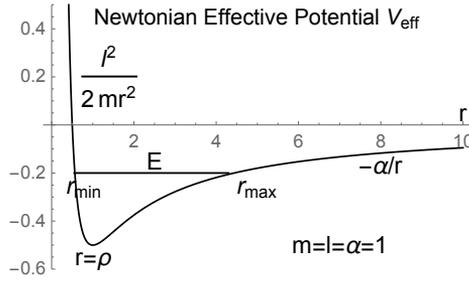


Figure 10: Effective potential $V_{\text{eff}} = l^2/2mr^2 - \alpha/r$ showing radial turning points for an energy $E < 0$.

oscillate about $r = \rho$ with radial turning points at perihelion r_{min} and aphelion r_{max} which are the roots of the quadratic equation

$$V_{\text{eff}}(r) = E \quad \text{or} \quad 2mEr^2 + 2m\alpha r - l^2 = 0. \quad (144)$$

For $E \geq 0$, the orbit is unbounded, though it has a radial turning point at perihelion⁵², resulting in parabolic and hyperbolic orbits. These could be used to model an aperiodic (single apparition) comet which comes in from far away, reaches a point of closest approach to the Sun and then escapes from the solar system. Finally, if $l = 0$, there is no angular momentum barrier and we get straight-line orbits where the reduced mass moves radially towards or away from the center of mass. In what follows, we obtain the shapes of orbits for $E < 0$.

Energy conservation ($E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$) gives a 1st order equation. We could integrate it to find $t(r)$

$$\pm (t - t_0) = \sqrt{\frac{m}{2}} \int_{r_0}^r \frac{ds}{\sqrt{E - V_{\text{eff}}(s)}} \quad (145)$$

and invert to find $r(t)$. Then we may use conservation of $l = mr^2\dot{\phi}$ to find $\phi(t)$

$$\phi(t) - \phi_0 = \int_{t_0}^t \frac{l dt'}{mr^2(t')} = \pm \frac{l}{\sqrt{2m}} \int_{r_0}^{r(t)} \frac{ds}{s^2 \sqrt{E - V_{\text{eff}}(s)}}. \quad (146)$$

In the second equality, we used $dt^2 = (m/2)dr^2/(E - V_{\text{eff}})$ from (142) and renamed r as s . So, at least for $l \neq 0$, the solution of the EOM has been reduced to quadrature.

However, if we are primarily interested in the shape of the orbit (rather than the time-dependence), it is simpler to think of r as a function of ϕ . Then

$$\dot{r} = r'(\phi)\dot{\phi} = r'(\phi)(l/mr^2). \quad (147)$$

A 1st order *orbit equation* is obtained by using energy conservation:

$$E = \frac{l^2}{2mr^4}r'^2 + \frac{l^2}{2mr^2} - \frac{\alpha}{r}. \quad (148)$$

⁵²The $l^2/2mr^2$ angular momentum barrier in V_{eff} prevents the reduced mass from getting too close, the point of closest approach is perihelion. There is no aphelion as it can escape to infinity.

However, this equation is rather nonlinear and difficult to solve directly. The appearance of r'/r^2 suggests the substitution $u = 1/r$ in terms of which the energy becomes

$$E = (l^2/2m) (u^2 + u'(\phi)^2) - \alpha u. \quad (149)$$

However, this is still a nonlinear equation! Fortunately, differentiating in ϕ allows us to eliminate E and leaves us with an inhomogeneous *linear* 2nd order differential equation for the orbit parametrized by ϕ :

$$u'' + u = \frac{m\alpha}{l^2} \equiv \frac{1}{\rho} \quad \text{where } \rho \text{ is the radius of the circular orbit for that } l. \quad (150)$$

This is the equation for a harmonic oscillator (with angular frequency $\omega = 1$ and with u playing the role of the displacement from equilibrium and ϕ that of time) with constant driving force. A particular solution is the constant $u_p = 1/\rho$ and the general solution of the homogeneous equation $u'' + u = 0$ is $u_h = \kappa \cos(\phi - \phi_o)$ where κ may be taken positive (a $-$ sign can be eliminated by advancing ϕ_o by π). In fact, the phase ϕ_o , simply rotates the orbit and will be omitted. The second integration constant κ has dimensions of inverse length and will be related to the energy. Thus, the equation for the orbit reduces to that of a conic section (140) with the origin at a focus if we identify the dimensionless constant $\kappa\rho$ with the eccentricity:

$$\frac{1}{r} = \kappa \cos \phi + \frac{1}{\rho} \quad \text{or} \quad \frac{\rho}{r} = 1 + \epsilon \cos \phi. \quad (151)$$

If $0 \leq \epsilon < 1$, we conclude that the tip of the relative vector \mathbf{r} (from Sun to planet) traces an ellipse with focus at the origin (Sun). By substituting in (149) we express the energy in terms of the parameters of the ellipse (ϵ , ρ or semimajor axis a)

$$E = -\frac{\alpha}{2\rho} (1 - \epsilon^2) = -\frac{\alpha}{2a} \quad \text{where} \quad \rho = \frac{l^2}{m\alpha}. \quad (152)$$

The semilatus rectum ρ for given l is physically interpreted as the radius of the circular orbit for that l . The condition $0 \leq \epsilon < 1$ implies $E < 0$, which ensures the motion is bounded. It is clear that for $E = -\alpha/2\rho$ the eccentricity vanishes and we get a circular orbit. For $-\alpha/2\rho < E < 0$ we have $0 < \epsilon < 1$ and we get elliptical orbits. For $E = 0$, the eccentricity $\epsilon = 1$ and we have an unbound parabolic orbit $y^2 = \rho^2 - 2\rho x$. For $\epsilon > 1$, $E > 0$ and we have unbound hyperbolic orbits.

Finally, we may revert to the position vectors of the Sun and Earth via (132):

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{M} \mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{M} \mathbf{r}. \quad (153)$$

We see that in a frame where the CM \mathbf{R} is at rest, \mathbf{r}_1 and \mathbf{r}_2 trace out rescaled conic sections with the CM as a focus (see Fig. 8). For the Sun-Earth system, $m_2/M \ll 1$, so the Sun's orbit is a tiny ellipse around the center of mass. On the other hand, $m_1/M \approx 1$, so the Earth's orbit relative to the CM is nearly congruent to the ellipse traced by the relative vector \mathbf{r} .

0.14.3 Time period of elliptical orbits

For $E < 0$, since orbits are closed curves (ellipses), the motion is periodic in the sense that position and velocity return to their initial values after a time T . Why? Suppose after time T , \mathbf{r} returns to its initial position $\mathbf{r}(0)$. Energy is unchanged throughout and so the potential energy must return to its initial value (since it depends only on $|\mathbf{r}|$). Thus the kinetic energy and speed must also be unchanged. The direction of velocity is tangent to the same ellipse and is also unchanged. So both the relative position and velocity return to their original values. From (153), the Sun's and Earth's elliptical trajectories also have the same period T . We now wish to find an expression for T in terms of the semimajor axis a and compare with Kepler's 3rd law.

There is a simple way to find the period T of elliptical orbits without evaluating the integral in (145). By the conservation of angular momentum, the areal speed is a constant $\frac{dA_r}{dt} = \frac{l}{2m}$ (Kepler's second law). Integrating over one period, the area of the ellipse is $A_r = lT/2m$. On the other hand, the area is also given by $A_r = \pi ab = \pi a\sqrt{a\rho}$ from (141). Using $\rho = l^2/m\alpha$ (152) to eliminate l and $m = m_1m_2/M$ we get

$$T^2 = \frac{4m^2}{l^2}(A_r)^2 = \frac{4\pi^2m}{\alpha}a^3 = \frac{4\pi^2}{GM}a^3 = \frac{a^3}{\tilde{K}}. \quad (154)$$

We have expressed the period in terms of the semimajor axis and $\tilde{K} = GM/4\pi^2$. We also recover Kepler's 3rd law (not just for circular orbits). Since $M = m_1 + m_2$, \tilde{K} depends on the *sum* of solar and planetary masses. But since all planets are at least a 1000 times lighter than the Sun, $M \approx m_s$. Hence, $\tilde{K} \approx Gm_s/4\pi^2 = K$ reduces to Kepler's constant and is approximately the same for all planets.

0.15 Small oscillations and normal modes

Earlier, we studied small oscillations around a stable equilibrium state of a system with one degree of freedom. This led us to the harmonic oscillator. What if two or more simple harmonic oscillators or pendula executing small oscillations (without damping and driving forces) interact with each other weakly? To uncover some of the new phenomena that can arise, we examine elementary examples of small oscillations in a system with two degrees of freedom. Examples include two pendula coupled by a spring, a diatomic molecule or a single particle that moves on a plane while attached via two springs to walls. The resulting coupled small oscillations are widely encountered in physical systems, typically at low energies or more generally around a stable equilibrium state. The first step in understanding small oscillations is to linearize the equations of motion to obtain a system of linear ODEs with *constant coefficients*. The main complication is that the various degrees of freedom generally satisfy coupled equations. A key idea is to make a linear change of variables to *normal modes*. The latter satisfy *uncoupled* equations that are easier to solve and admit nice physical interpretations. Degrees of freedom that satisfy uncoupled equations evolve independently of each other. If the initial state involves an excitation along one normal mode, then other normal modes cannot get activated during subsequent time evolution via the linearized equations of motion.

0.15.1 Normal modes of two weakly coupled pendula undergoing small oscillations

Consider a pair of identical plane pendula of length ℓ and bob mass m . From our earlier discussions, *small* oscillations of the deflection angles θ_1 and θ_2 are governed by the EOM

$$m\ell\ddot{\theta}_1 = -mg\theta_1 \quad \text{and} \quad m\ell\ddot{\theta}_2 = -mg\theta_2 \quad (155)$$

in the angular (tangential) directions. Now, we let the pendula interact weakly⁵³ by connecting their massless rods by a spring whose natural length is equal to the distance between the two supports. The spring has a force constant k and is attached to the rods a small distance s below the supports (see Fig. 11).

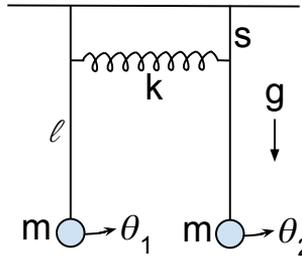


Figure 11: First (left) and second (right) pendula coupled by a spring.

Suppose the 2nd pendulum swings to the right slightly ($\theta_2 > 0$) while $\theta_1 = 0$. Then, the spring expands by a length $\approx s\theta_2$ and exerts a force $\approx ks\theta_2$ to the right on the 1st pendulum and an equal and opposite force $\approx ks\theta_2$ to the left on the 2nd pendulum. Similarly, if $\theta_2 = 0$ and $\theta_1 > 0$, then there is a leftward force $\approx ks\theta_1$ on the 1st pendulum and an equal and opposite rightward force of approximate magnitude $ks\theta_1$ on the 2nd. Thus, the EOM of the coupled pendula are given by:

$$\begin{aligned} m\ell\ddot{\theta}_1 &= -mg\theta_1 - ks(\theta_1 - \theta_2) \quad \text{and} \\ m\ell\ddot{\theta}_2 &= -mg\theta_2 + ks(\theta_1 - \theta_2). \end{aligned} \quad (156)$$

Introducing $\omega = \sqrt{g/\ell}$ and $\bar{\omega} = (ks/m\ell)^{1/2}$, we get the linear equations of motion

$$\ddot{\theta}_1 + (\omega^2 + \bar{\omega}^2)\theta_1 = \bar{\omega}^2\theta_2 \quad \text{and} \quad \ddot{\theta}_2 + (\omega^2 + \bar{\omega}^2)\theta_2 = \bar{\omega}^2\theta_1. \quad (157)$$

The terms on the RHS couple the equations. The evolution of θ_1 depends on θ_2 and vice-versa. The degrees of freedom or ‘modes’ θ_1 and θ_2 are coupled.

Normal modes are variables built from θ_1 and θ_2 that evolve independently of each other. It is easy to find normal modes in this example: we simply add and subtract the two equations. If we denote $\theta_+ = \theta_1 + \theta_2$ and $\theta_- = \theta_1 - \theta_2$, then

$$\ddot{\theta}_+ + \omega^2\theta_+ = 0 \quad \text{and} \quad \ddot{\theta}_- + (\omega^2 + 2\bar{\omega}^2)\theta_- = 0. \quad (158)$$

We see that θ_{\pm} evolve independently: they are normal modes of our coupled pendula. Each satisfies the equation for a harmonic oscillator with the solutions

$$\theta_+(t) = A_+ \cos(\omega t + \phi_+) \quad \text{and} \quad \theta_-(t) = A_- \cos\left(\sqrt{\omega^2 + 2\bar{\omega}^2}t + \phi_-\right). \quad (159)$$

⁵³Replacing the spring by a rigid rod would result in a ‘strong’ interaction between pendula.

Here, ϕ_{\pm} are a pair of initial phases. Each of the normal modes is periodic in time, but the time periods $T_+ = 2\pi/\omega$ for θ_+ and $T_- = 2\pi/\sqrt{\omega^2 + 2\bar{\omega}^2}$ for θ_- are distinct as long as $\bar{\omega} \neq 0$. What is the pattern of oscillation in each of the normal modes?

1. In the θ_+ normal mode, ICs are such that $A_- = 0$ so that $\theta_1 - \theta_2 = 0$ at all times. Thus, both deflection angles are always equal, the pendula swing ‘in phase’ and the spring is never compressed nor elongated. The frequency and time period are the same as for the uncoupled pendula.
2. In the θ_- normal mode, ICs are such that $A_+ = 0$ so that $\theta_1 = -\theta_2$ at all times. The pendula are ‘out of phase’: they swing in opposite directions with the same amplitude. The angular frequency $\sqrt{\omega^2 + 2\bar{\omega}^2}$ exceeds the angular frequency ω of the individual pendula since the spring is either compressed or elongated leading to a larger restoring force than that due to gravity alone. The factor of 2 arises because both bobs simultaneously compress or expand the spring in the θ_- normal mode.

For other initial conditions, both normal modes are excited. In fact, due to the linearity of the EOM, the general solution is a linear combination of normal modes:

$$\begin{aligned}\theta_1 &= \frac{1}{2}(\theta_+ + \theta_-) = (A_+/2) \cos(\omega t + \phi_+) + (A_-/2) \cos\left(\sqrt{\omega^2 + 2\bar{\omega}^2}t + \phi_-\right), \\ \theta_2 &= \frac{1}{2}(\theta_+ - \theta_-) = (A_+/2) \cos(\omega t + \phi_+) - (A_-/2) \cos\left(\sqrt{\omega^2 + 2\bar{\omega}^2}t + \phi_-\right).\end{aligned}\quad (160)$$

It is natural to ask if the general solution is periodic. If the ratio of T_+/T_- is rational, say p/q where p, q are coprime positive integers, then both θ_1 and θ_2 are periodic with period given by the least common integer multiple $T = qT_+ = pT_-$. On the other hand, if T_+ and T_- are incommensurate (i.e., T_+/T_- is irrational or T_{\pm} linearly independent over the integers) then the resulting motion is not periodic and it is as though the least common integer multiple has gone to infinity. Such motion is called quasiperiodic (or conditionally periodic): a linear superposition of incommensurate periodic motions. In the incommensurate case, although the motion is not periodic, $\theta_{1,2}$ will return to values arbitrarily close to their initial values if we wait long enough. This is because the irrational ratio T_+/T_- can be approximated to any desired precision by the ratio p/q of sufficiently large coprime positive integers p and q .

0.15.2 Normal modes of a diatomic molecule

Next, we consider small longitudinal oscillations (without collisions) of a diatomic molecule modeled as a pair of point particles (each of mass m) connected by a spring with force constant k . As shown in Fig. 12, we will choose the x -axis along the ‘logitudinal’ direction defined by the line from the first mass to the second mass in the undisturbed equilibrium state. The undisturbed equilibrium positions of the masses are denoted \bar{x}_1 and \bar{x}_2 (which are independent of time) with $\bar{x}_2 - \bar{x}_1$ being the natural length of the spring.

For simplicity, we restrict to small oscillations along the x axis. If the first mass moves to the right, the spring is compressed resulting in a restoring force on it pointing to the left. If the second mass moves to the right, the spring is expanded and the

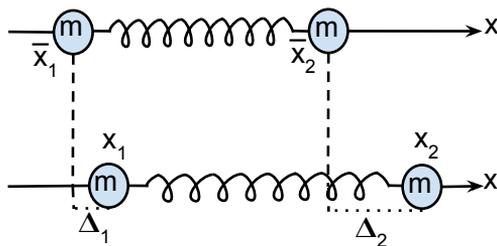


Figure 12: *Undisturbed* (upper) and *disturbed* (lower) configurations of a diatomic molecule executing small oscillations. The first mass is the one to the left.

restoring force on the first mass acts to the right. Thus, by Hooke's law we get the equation of motion for x_1 :

$$m\ddot{x}_1 = -k(x_1 - \bar{x}_1) + k(x_2 - \bar{x}_2). \quad (161)$$

Similarly, the EOM for x_2 is given by

$$m\ddot{x}_2 = k(x_1 - \bar{x}_1) - k(x_2 - \bar{x}_2). \quad (162)$$

This formula is consistent with Newton's 3rd law, which implies that the force on the first mass due to a small displacement of the second must be equal and opposite to the force on the second. These are a pair of coupled inhomogeneous linear ODEs for x_1 and x_2 . It would be nice to decouple them by passing to suitable normal modes.

Since $\ddot{\bar{x}}_{1,2} = 0$, the equations only involve the departures from equilibrium locations. So we introduce the variables $\Delta_1 = x_1 - \bar{x}_1$ and $\Delta_2 = x_2 - \bar{x}_2$, in terms of which the EOM become

$$m\ddot{\Delta}_1 = -k(\Delta_1 - \Delta_2) \quad \text{and} \quad m\ddot{\Delta}_2 = k(\Delta_1 - \Delta_2). \quad (163)$$

These are a pair of coupled homogeneous linear equations for $\Delta_1(t)$ and $\Delta_2(t)$. We may obtain uncoupled equations by introducing analogues of center of mass and relative variables $\Delta_+ = \Delta_1 + \Delta_2$ and $\Delta_- = \Delta_1 - \Delta_2$. These satisfy the equations

$$\ddot{\Delta}_+ = 0 \quad \text{and} \quad \ddot{\Delta}_- = -2(k/m)\Delta_-. \quad (164)$$

Δ_{\pm} are normal modes of our diatomic molecule. Each evolves independently of the other, so ICs may be chosen so that only one of the normal modes is excited.

The normal mode Δ_+ is called a *zero mode* since it has zero angular frequency ω_+ when written as $\ddot{\Delta}_+ + \omega_+^2 \Delta_+ = 0$. It displays the dynamics of a free particle. Moreover, in this mode, $\Delta_- = 0$, so $\Delta_1 = \Delta_2$ and therefore, $x_1 - \bar{x}_1$ and $x_2 - \bar{x}_2$ are equal at all times. Thus, the two masses move together uniformly either to the right or left. Hence, in the Δ_+ 'translational' mode, the molecule as a whole moves uniformly without any internal vibration.

The normal mode Δ_- is an oscillatory mode with angular frequency $\omega_- = \sqrt{2k/m}$. In this mode, $\Delta_+ = \Delta_1 + \Delta_2 = 0$, so the two masses move in opposite directions, either both compressing or expanding the spring. This results in double the restoring force that would arise if just one mass was compressing/expanding the spring. In this mode, the molecule as a whole is at rest and displays purely vibrational motion.

0.16 Constrained extremization via Lagrange multipliers

Here, we introduce the technique of Lagrange multipliers for constrained extremization problems. Consider the problem of extremizing a real-valued function $F(\mathbf{x})$ subject to a constraint $\mathcal{C}(\mathbf{x}) = 0$, where $\mathbf{x} \in \Omega$. Ω could be \mathbb{R}^n or some connected open region in \mathbb{R}^n . We may think of $\mathcal{C}(\mathbf{x}) = 0$ as defining a constraint hypersurface $\Sigma \subset \Omega$ (see Fig. 13). Our task is to find the points on Σ where F is stationary subject to infinitesimal changes in \mathbf{x} tangent to Σ . The method of Lagrange multipliers says that to extremize F subject to the constraint \mathcal{C} is the same as the unconstrained extremization of $F - \lambda\mathcal{C}$ on Ω . Here, the real number λ is called a *Lagrange multiplier*. The value of λ is not known in advance. There are typically several allowed values of λ corresponding to several local extrema of F subject to the constraint \mathcal{C} . Let us see why this prescription works. The condition for extrema of $F - \lambda\mathcal{C}$ is $\nabla F = \lambda\nabla\mathcal{C}$. Geometrically, this says that the gradient ∇F must point along/opposite to $\nabla\mathcal{C}$. Since $\nabla\mathcal{C}$ points normal to the constraint hypersurface Σ , this would ensure that ∇F is also normal to Σ . In other words, F would be stationary on the constraint hypersurface, as desired! If ∇F and $\nabla\mathcal{C}$ were not collinear at a point $\mathbf{x} \in \Sigma$, then one could move along the constraint hypersurface (along/opposite to the projection of ∇F) to an infinitesimally nearby point where F takes a larger/smaller value.

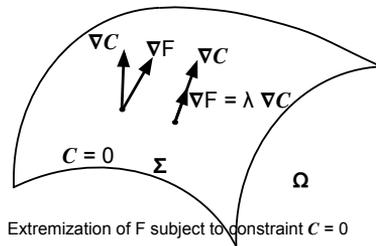


Figure 13: Constrained extremization via Lagrange multipliers. While $\nabla\mathcal{C}$ is always normal to the constraint surface $\mathcal{C} = 0$, ∇F is generally not. At an extremum of F subject to the constraint, ∇F is normal to the surface and is a multiple of $\nabla\mathcal{C}$.

The method of Lagrange multipliers may be extended to the problem of extremizing a function F subject to several constraints $\mathcal{C}_1 = 0, \mathcal{C}_2 = 0, \dots$. Argue that this problem is equivalent to the unconstrained extremization of $F - \lambda_1\mathcal{C}_1 - \lambda_2\mathcal{C}_2 - \dots$.

Lagrangian for a plane pendulum: tension as a Lagrange multiplier. Recall that Newton’s equations for a plane pendulum are given by the \hat{r} and $\hat{\theta}$ components of $m\mathbf{a} = \mathbf{F}$:

$$-m\ell\dot{\theta}^2 = mg \cos \theta - \mathcal{T} \quad \text{and} \quad m\ell\ddot{\theta} = -mg \sin \theta. \tag{165}$$

Here \mathcal{T} is the radially inward tension in the rod while m and ℓ are the bob mass and length of rod. We seek a Lagrangian for these EOM. The Lagrangian $\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos \theta)$ leads to the angular equation but not the radial one, which involves the tension as a new dynamical variable. To account for radial forces and the centripetal acceleration, it is natural to regard $r(t)$ as a dynamical variable that is, however, subject to the constraint $r(t) - \ell = 0$. In the absence of the constraint, we have a particle

(bob) moving in a vertical plane subject to the downward gravitational force. The corresponding equations of motion are the extremality conditions for the action

$$S = \int_{t_i}^{t_f} L dt \quad \text{where} \quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr(1 - \cos \theta). \quad (166)$$

Now, we have one constraint $r(t) - \ell = 0$ for each instant of time. To deal with these (infinitely many) constraints, let us first define the sequence of intermediate times $t_j = t_i + j\Delta$ for $j = 1, \dots, n$ where $\Delta = (t_f - t_i)/n$. The constraint at time t_j may be written as $(r(t_j) - \ell)\Delta = 0$ and may be imposed via the Lagrange multiplier $\lambda(t_j)$. In the limit $n \rightarrow \infty, \Delta \rightarrow 0$ and

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \lambda(t_j)(r(t_j) - \ell)\Delta = \int_{t_i}^{t_f} \lambda(t)(r(t) - \ell)dt. \quad (167)$$

Hence, the extremization of S subject to these constraints is equivalent to the unconstrained extremization of the ‘effective’ action

$$S_{\text{eff}} = \int_{t_i}^{t_f} L_{\text{eff}} \quad \text{where} \quad L_{\text{eff}} = L - \lambda(t)(r - \ell). \quad (168)$$

Thus, we arrive at an effective Lagrangian for the plane pendulum:

$$L_{\text{eff}}(r, \dot{r}, \theta, \dot{\theta}, \lambda) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr(1 - \cos \theta) - \lambda(r - \ell). \quad (169)$$

It remains to verify that L_{eff} leads to the EOM (165) and to identify the physical meaning of the Lagrange multipliers $\lambda(t)$. The EL equations for r, θ and λ are

$$m\ddot{r} = mr\dot{\theta}^2 - mg(1 - \cos \theta) - \lambda, \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr \sin \theta \quad \text{and} \quad r = \ell. \quad (170)$$

We use the third equation to eliminate r and arrive at the pair of equations

$$-m\ell\dot{\theta}^2 = mg \cos \theta - mg - \lambda \quad \text{and} \quad m\ell\ddot{\theta} = -mg \sin \theta. \quad (171)$$

Comparing with (165) we see that these equations agree with the radial and tangential Newton equations if we interpret $\lambda + mg$ as the tension. So up to an additive constant, the Lagrange multiplier function is the tension \mathcal{T} in the rod.

0.17 Rigid body mechanics

A rigid⁵⁴ body is a system of particles such that the distances between any pair is fixed⁵⁵. For instance, we could have four mass points m_a for $a = 1, 2, 3, 4$ arranged at the vertices of a regular tetrahedron and pairwise connected by rigid rods

⁵⁴The word *rigid* is used to distinguish a rigid body from a *deformable* body such as a stretched string, elastic solid or fluid. Indeed, the complexity of rigid body dynamics lies between that of a point particle and that of a deformable body.

⁵⁵These are examples of holonomic constraints.

of negligible mass as shown in Fig. 14a. A continuous distribution of mass is also possible, as in a stone (see Fig. 14b). In this case, the discrete index a is replaced with the continuous location ‘index’ \mathbf{r} and the masses of individual particles are replaced by the mass $dm(\mathbf{r}) = \rho(\mathbf{r})dV$ in any elemental volume dV around \mathbf{r} where $\rho(\mathbf{r})$ is the local mass density. Many commonly encountered objects can be treated as rigid bodies. Examples include a point mass, a dumbbell, a plate, a cricket or golf ball, a boomerang, a top, a gyroscope, an asteroid and a spaceship. The concept is useful in quantum physics as well; e.g., the rotational energy levels (as opposed to the vibrational spectrum) of a molecule can be approximately understood by treating it as a rigid body.

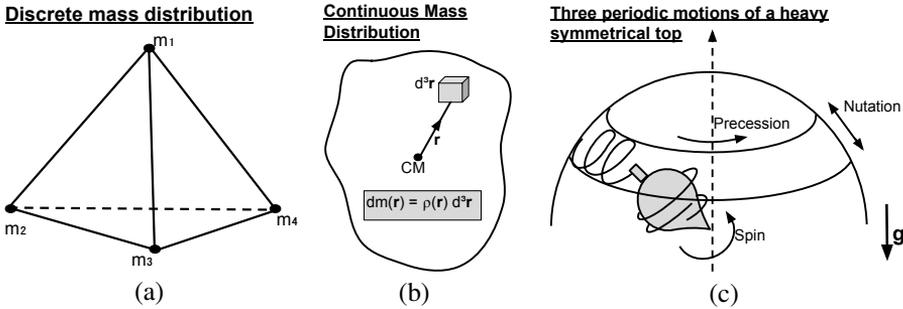


Figure 14: (a) Tetrahedral rigid body (b) Continuous mass distribution in a rock (c) Spinning, precession and nutation (wobble or swaying) of a top in a gravitational field. The locus of the tip of the top is shown on a fictitious spherical surface.

0.17.1 Lab and comoving frames

The motion of points in the rigid body may be described using the lab (‘inertial’ or ‘space’ or ‘fixed’) frame. We will use a system of Cartesian coordinates X, Y, Z for the lab frame. The position vector of the point in the body labelled a , relative to the origin of the lab frame, is called \mathbf{z}_a (see Fig. 15). The components of \mathbf{z}_a relative to the lab axes are $z_a = (X_a, Y_a, Z_a)$. The lab frame coordinates $X_a(t), Y_a(t), Z_a(t)$ (or coordinates in ‘space’) of a point in the body, in general, change as the body moves. The center of mass (CM) of the body is the point $\mathbf{R} = (\bar{X}, \bar{Y}, \bar{Z})$ whose coordinates are

$$\bar{X} = \frac{1}{M} \sum_a m_a X_a, \quad \bar{Y} = \frac{1}{M} \sum_a m_a Y_a, \quad \bar{Z} = \frac{1}{M} \sum_a m_a Z_a \quad \text{where} \quad M = \sum_a m_a. \tag{172}$$

The sum on a is over all the mass points that make up the body of mass M . The CM is a distinguished point associated to a rigid body (however, it need not be inside the body). Thus, it is convenient to consider the location of a mass point labelled a , relative to the center of mass. The corresponding position vector relative to the CM is called \mathbf{r}_a . The components of \mathbf{r}_a relative to the lab axes, in general, change with time. The position vector of the center of mass, relative to the center of mass is obviously the zero vector: $\sum_a m_a \mathbf{r}_a = 0$.

We may also use a moving system of Cartesian coordinates $r_1 = x, r_2 = y, r_3 = z$ referred to the so-called comoving (or body-fixed or corotating) frame whose axes

are rigidly fixed in the body and participate in its motion. The origin O of the comoving frame is most conveniently chosen to lie at the center of mass of the body, so $\sum_a m_a x_a = \sum_a m_a y_a = \sum_a m_a z_a = 0$. The comoving frame components (x_a, y_a, z_a) of a point \mathbf{r}_a in the body are independent of time.

0.17.2 Configuration space and degrees of freedom

A point mass moving in 3d space has three degrees of freedom, its configuration space is 3d Euclidean space \mathbb{R}^3 . If a rigid body is concentrated along a line (e.g., a pair of mass points connected by a slender rod of zero thickness or a straight wire or pen of negligible thickness) it is called a *rigid rotator*. A rigid rotator has three translational degrees of freedom, which may be regarded as specifying the location of the CM. Once the location of the CM has been fixed, the orientation of the pen is determined by the point on a unit sphere⁵⁶ (centered at the CM), at which the ray from the CM to the nib intersects it. Thus, a rigid rotator has five degrees of freedom and configuration space $\mathbb{R}^3 \times S^2$.

A noncollinear rigid body has six degrees of freedom⁵⁷. We need three coordinates to locate the position of the CM. We are then free to rotate the rigid body about its center of mass in any manner. The orientation of the comoving frame relative to the lab frame is determined by such a rotation of 3d space. A rotation is determined by an axis (which can be taken as a unit vector) and an angle of rotation. We need two angles to define an axis and one to specify the angle of rotation. So rotations are a three-parameter family. Thus, a noncollinear rigid body has three translational and three rotational degrees of freedom. Moreover, rotations may be composed and inverted; they form the group of 3×3 special orthogonal matrices⁵⁸. Thus, the configuration space of the rigid body is $\mathbb{R}^3 \times SO(3)$.

* **Remark.** Though we can specify a finite rotation by a vector (along the axis of rotation with length equal to the angle of rotation), the composition of rotations does not correspond to addition of these vectors, since rotations generally do not commute. The composition of rotations is modeled by multiplication of special orthogonal matrices. On the other hand, the

⁵⁶The unit 2-sphere is defined as $S^2 = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x^2 + y^2 + z^2 = 1\}$.

⁵⁷For example, a flat circular disk with a uniform mass distribution has 6 degrees of freedom: 3 translational, two to orient the symmetry axis and one corresponding to rotations about the axis. Though rotations about this axis are a symmetry of the mass distribution, the angular orientation about the axis is a degree of freedom. This is because the particles that make up the disk are classically distinguishable though they may be identical. More concretely, we could mark a point along the rim (without significantly altering the mass distribution) to keep track of the azimuthal orientation of the disk. Moreover, unlike rotations about the axis of a collinear rigid body, rotations of the disk about this axis have an associated rotational kinetic energy.

⁵⁸The group $SO(3)$ of special orthogonal matrices consists of 3×3 real matrices R with $R^t R = R R^t = I$ and $\det R = 1$. Such matrices rotate vectors $\mathbf{x} \rightarrow R\mathbf{x}$. The first condition which involves the transpose, ensures preservation of length while the second excludes ‘improper rotations’ (which involve reflections in an essential way - for example the composition of a reflection in the xy plane followed by a reflection in the xz plane is in fact a rotation by π in the yz plane). Composition of rotations is given by matrix multiplication and in particular, the composition of two reflections (each with determinant -1) is a proper rotation.

composition of infinitesimal rotations is commutative to leading order and we will be able to represent the composition of infinitesimal rotations by vector addition. This is why we will soon be able to express angular velocity (which results from an infinitesimal rotation) as a vector. \square

0.17.3 Infinitesimal displacement and angular velocity of rigid body

We begin our study of rigid body motion with the kinematical concepts of translational and angular velocities. An infinitesimal displacement of the body is expressible as a sum of an infinitesimal translation of the CM (O) to its final location (keeping the orientation of the body & comoving frame fixed) and an infinitesimal rotation (keeping the CM fixed) that orients the body (and comoving frame) appropriately. Moreover, any infinitesimal rotation about O is a rotation about some axis through O.

Position and velocity vectors in lab and comoving frames

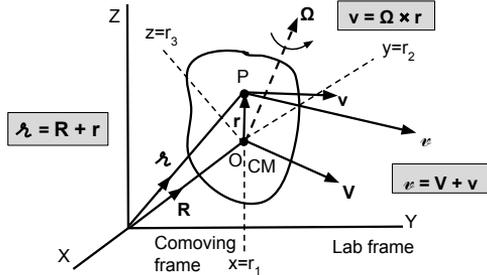


Figure 15: Position and velocity vectors of a marked point P in a rigid body with respect to the lab frame. The comoving frame is also indicated.

Let us denote by z the position vector of a point P in the rigid body, relative to the origin of the lab frame (see Fig. 15). Suppose the same point has the position vector r relative to the CM. Then $z = R + r$ where R is the position vector of the CM relative to the origin of the lab frame. We will regard each of these three vectors as given by their components with respect to the lab axes.

Now, a small displacement δz of P may be written as

$$\delta z = \delta R + \delta \phi \times r. \tag{173}$$

Here, δR is the displacement of the CM. On the other hand, $\delta \phi \times r$ is the infinitesimal change in the position vector r (relative to the lab frame) due to a counterclockwise rotation about the axis $\widehat{\delta \phi}$ along the Euler vector⁵⁹ $\delta \phi$ (passing through the CM) by an angle $\delta \phi = |\delta \phi|$. To see why, we note that the change in r under an infinitesimal counterclockwise rotation by angle $\delta \phi$ about $\delta \phi$ is in magnitude

$$|\delta r| = |r| \sin \theta |\delta \phi|, \tag{174}$$

where θ is the angle between $\delta \phi$ and r . For an infinitesimal rotation, δr is perpendicular to both r and $\delta \phi$, as indicated in Fig. 5. Thus, $\delta r = \delta \phi \times r$.

⁵⁹The Euler vector $\delta \phi$ points along the axis of rotation (which is often taken to be a unit vector) and has magnitude equal to the angle of rotation.

Dividing (173) by the time δt in which the infinitesimal motion took place,

$$\frac{\delta \mathbf{z}}{\delta t} = \frac{\delta \mathbf{R}}{\delta t} + \frac{\delta \phi}{\delta t} \times \mathbf{r}. \quad (175)$$

Now we let $\delta t \rightarrow 0$. If we denote the velocity vector of P relative to the origin of the lab frame by $\boldsymbol{\nu} = d\mathbf{z}/dt$, the translational velocity of the CM by $\mathbf{V} = d\mathbf{R}/dt$ and the angular velocity by $\boldsymbol{\Omega} = \lim_{\delta t \rightarrow 0} \delta\phi/\delta t$, then

$$\boldsymbol{\nu} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r} \quad \text{or} \quad \boldsymbol{\nu} = \mathbf{V} + \mathbf{v} \quad \text{where} \quad \mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}. \quad (176)$$

The angular velocity vector $\boldsymbol{\Omega}$ points along the axis of right-handed rotation ($\widehat{\delta\phi}$) passing through the CM. For example, an upright top that is just spinning uniformly counterclockwise at one radian per second about its symmetry axis has a constant $\boldsymbol{\Omega} = \hat{z}$. The angular velocity may of course change with time both in direction and magnitude.

More generally, for each mass point m_a in the rigid body, we denote by \mathbf{r}_a its position vector relative to the CM,

$$\begin{aligned} \mathbf{z}_a &= \mathbf{R} + \mathbf{r}_a, \quad \text{and differentiating in time,} \quad \boldsymbol{\nu}_a = \mathbf{V} + \mathbf{v}_a \\ \text{where } \mathbf{V} &= \dot{\mathbf{R}} \quad \text{and} \quad \mathbf{v}_a = \dot{\mathbf{r}}_a = \boldsymbol{\Omega} \times \mathbf{r}_a. \end{aligned} \quad (177)$$

Here, $\mathbf{v}_a = \dot{\mathbf{r}}_a$ is the velocity relative to the CM. Multiplying $\boldsymbol{\nu}_a$ by the mass m_a , we get the momentum of the a^{th} particle relative to the lab frame:

$$\not\!/\boldsymbol{\nu}_a = m_a \mathbf{V} + \mathbf{p}_a \quad \text{where} \quad \mathbf{p}_a = m_a \mathbf{v}_a = m_a \boldsymbol{\Omega} \times \mathbf{r}_a. \quad (178)$$

The vector sum of all the momenta in the lab frame coincides with the center of mass momentum

$$\mathbf{P} = \sum_a \not\!/\boldsymbol{\nu}_a = M\mathbf{V} + \boldsymbol{\Omega} \times \sum_a m_a \not\!/\mathbf{r}_a = M\mathbf{V}. \quad (179)$$

The sum vanishes as \mathbf{r}_a are defined relative to the CM. The coordinates and momenta ($\mathbf{z}_a, \not\!/\boldsymbol{\nu}_a$) of all the particles in the body together specify the instantaneous state of the rigid body. To understand its dynamics, we examine its Lagrangian.

0.17.4 Kinetic energy and inertia tensor

Here, we wish to express the kinetic energy T of a rigid body in terms of its CM velocity \mathbf{V} and angular velocity $\boldsymbol{\Omega}$.

In doing so, we will discover a 3×3 matrix called the inertia tensor associated to the mass distribution of a rigid body. The inertia tensor plays a role in the rotational motion of a rigid body similar to that played by the total mass M in its translational motion.

The kinetic energy T of a rigid body is a sum of free particle energies of each constituent mass point, whose velocity vector we have denoted $\boldsymbol{\nu}_a$ relative to the lab frame. So

$$T = \sum_a \frac{1}{2} m_a \boldsymbol{\nu}_a^2 \equiv \sum_a \frac{1}{2} m \boldsymbol{\nu}^2. \quad (180)$$

We will often suppress the index a labelling points. It appears on \mathbf{r} and $\boldsymbol{\omega}$ but not on $\boldsymbol{\Omega}$ or \mathbf{V} , which are properties of the body as a whole. Thus,

$$\begin{aligned} T &= \sum \frac{m}{2} (\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r})^2 = \sum \frac{m}{2} \mathbf{V}^2 + \sum m \mathbf{V} \cdot \boldsymbol{\Omega} \times \mathbf{r} + \sum \frac{m}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2 \\ &= \frac{1}{2} M \mathbf{V}^2 + \mathbf{V} \cdot (\boldsymbol{\Omega} \times \sum m \mathbf{r}) + \frac{1}{2} \sum m [\Omega^2 r^2 - (\boldsymbol{\Omega} \cdot \mathbf{r})^2]. \end{aligned} \quad (181)$$

Above, Ω, r are the magnitudes of $\boldsymbol{\Omega}, \mathbf{r}$. Thus, T may be written as a sum of the translational kinetic energy of a body of mass M located at the CM and the rotational kinetic energy about the CM:

$$T = \frac{1}{2} M \mathbf{V}^2 + \frac{1}{2} \Omega_i \Omega_j \sum m (r^2 \delta_{ij} - r_i r_j) = \frac{1}{2} M \mathbf{V}^2 + \frac{1}{2} \mathcal{I}_{ij} \Omega_i \Omega_j. \quad (182)$$

Inertia tensor. T_{rot} involves a 3×3 matrix called the inertia tensor, with components

$$\mathcal{I}_{ij} = \sum_a m_a (r_a^2 \delta_{ij} - (r_a)_i (r_a)_j). \quad (183)$$

For a body with a continuous mass distribution and density $\rho(\mathbf{r})$,

$$\mathcal{I}_{ij} = \iiint \rho(\mathbf{r}) (r^2 \delta_{ij} - r_i r_j) d^3 \mathbf{r}. \quad (184)$$

The components of the inertia tensor are independent of time if $(r_a)_i$ are the components with respect to the comoving frame. If instead, we use the components of \mathbf{r}_a relative to the lab frame, then \mathcal{I}_{ij} will be generally time-dependent and less convenient. The matrix \mathcal{I}_{ij} is an intrinsic property of the mass distribution of the rigid body, the chosen origin and axes of the comoving frame.

The inertia matrix is a real and symmetric matrix $\mathcal{I}_{ij} = \mathcal{I}_{ji}$. It is a positive-semidefinite matrix in the sense that the associated quadratic form ('rotational kinetic energy') is manifestly nonnegative for any $\boldsymbol{\Omega}$:

$$T_{\text{rot}} = \frac{1}{2} \mathcal{I}_{ij} \Omega_i \Omega_j = \frac{1}{2} \Omega^t \mathcal{I} \Omega = \sum_a \frac{1}{2} m_a (\boldsymbol{\Omega} \times \mathbf{r}_a)^2 \geq 0. \quad (185)$$

We may write out the components of the inertia matrix with respect to the comoving frame,

$$\mathcal{I} = \sum_a \begin{pmatrix} m_a (y_a^2 + z_a^2) & -m_a x_a y_a & -m_a x_a z_a \\ -m_a y_a x_a & m_a (z_a^2 + x_a^2) & -m_a y_a z_a \\ -m_a z_a x_a & -m_a z_a y_a & m_a (x_a^2 + y_a^2) \end{pmatrix}. \quad (186)$$

Evidently, it is the sum of inertia matrices of each mass point in the body. For a continuous mass distribution $\sum m(\dots)$ is replaced with $\int \rho(\mathbf{r})(\dots) dx dy dz$. The diagonal entries $\mathcal{I}_{11} = \sum m(y^2 + z^2)$, \mathcal{I}_{22} and \mathcal{I}_{33} are called the moments of inertia about the first, second and third axes of the comoving frame. In general, given any axis $\hat{\mathbf{n}}$, the moment of inertia about $\hat{\mathbf{n}}$ is defined as $\mathcal{I}_{\hat{\mathbf{n}}} = \sum_a m_a \rho_a^2$ where ρ_a is the perpendicular distance of point a from the axis. The parallel axis theorem relates the

moment of inertia about an axis through the CM to the moment of inertia about a parallel axis \hat{p} : $\mathcal{I}_{\hat{p}} = \mathcal{I}_{\text{cm}} + Md^2$, where d is the distance between axes.

Being a real symmetric matrix, the inertia matrix may be diagonalized by an orthogonal transformation⁶⁰ S : $S^{-1}\mathcal{I}S = D$ where D is the diagonal matrix of eigenvalues. As the matrix is positive-semidefinite, the eigenvalues are nonnegative, they are called the principal moments of inertia, which we may order as $0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq \mathcal{I}_3$. The eigenvectors of the inertia matrix may be chosen orthonormal⁶¹ and are called the principal axes of inertia. If the axes of the moving frame are chosen along the principal axes of inertia, then the inertia matrix is diagonal

$$D = \sum_a \begin{pmatrix} m_a(y'_a{}^2 + z'_a{}^2) & 0 & 0 \\ 0 & m_a(z'_a{}^2 + x'_a{}^2) & 0 \\ 0 & 0 & m_a(x'_a{}^2 + y'_a{}^2) \end{pmatrix} = \begin{pmatrix} \mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 \\ 0 & 0 & \mathcal{I}_3 \end{pmatrix}. \quad (187)$$

Here x'_a, y'_a, z'_a are the components of the position vector of a point in the body with respect to the principal axis frame. Note that the off-diagonal entries vanish due to cancellations $\sum_a m_a x'_a y'_a = 0$ although $m_a x'_a y'_a$ is, in general, nonzero for individual particles in the body. In the principal axis basis, the rotational kinetic energy is particularly simple:

$$T_{\text{rot}} = \frac{1}{2} (\mathcal{I}_1 \Omega_1^2 + \mathcal{I}_2 \Omega_2^2 + \mathcal{I}_3 \Omega_3^2). \quad (188)$$

0.17.5 Types of rigid bodies

As (188) shows, only the eigenvalues of the inertia tensor enter the formula for the rotational kinetic energy of a rigid body. Thus, in the absence of external forces, the pattern of eigenvalues of \mathcal{I} determines the nature of the rotational motion. The remaining details of the mass distribution can play a role in the presence of external torques⁶². Thus, it is useful to classify rigid bodies based on their principal moments of inertia.

If the 3 principal moments of inertia are pairwise unequal, we call it an *anisotropic* rigid body. Most stones are anisotropic rigid bodies. If one pair coincide, it is called a *symmetrical top*⁶³, a uniform right circular cylinder is an example. For definiteness, if $\mathcal{I}_1 = \mathcal{I}_2$, then the corresponding two principal axes may be chosen to be any pair of

⁶⁰An orthogonal transformation is one that satisfies $SS^t = S^tS = \mathbf{1}$. The columns of S are the eigenvectors of \mathcal{I} in the same order as that chosen for the eigenvalues that appear along the diagonal of D .

⁶¹Eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal and can be normalized to have unit lengths. If two eigenvalues coincide, the corresponding eigenvectors can be chosen as any two linearly independent vectors in the corresponding eigenplane; in particular, they can be chosen orthonormal.

⁶²For instance, for a symmetrical top held fixed at a point on its axis in a uniform gravitational field, the only other piece of information on the mass distribution that affects the rotational motion is the distance between the fixed point and the CM.

⁶³Some authors use the word *top* to refer to a rigid body subject to gravity. For us, a top is simply a rigid body. A rigid body subject to gravity will be called a heavy top.

linearly independent vectors in the corresponding x - y eigenplane. If all three eigenvalues coincide, it is called a *spherical top*. A solid spherical ball with uniform mass distribution is a simple example. The principal axes of inertia of a spherical top may be chosen as any three linearly independent vectors. Despite this freedom, we often pick the principal axes to form an orthonormal frame to simplify calculations.

Example: Uniform solid cube. Let us find the principal moments of inertia and principal axes of a uniform solid cube of side L and mass M . To make the determination of the inertia tensor simple, we choose coordinate axes that are parallel to the edges of the cube with origin at the center of the cube (see Fig. 16). Thus, the 6 corners of the cube are at $(L/2)(\pm 1, \pm 1, \pm 1)$. The density of the cube is a constant: $\rho = M/L^3$. The center of mass of the cube is then at $(0, 0, 0)$ since the three integrals $\bar{X} = \rho \int x \, dx \, dy \, dz$, $\bar{Y} = \rho \int y \, dx \, dy \, dz$ and $\bar{Z} = \rho \int z \, dx \, dy \, dz$ vanish on account of the integrands being odd. Next, we evaluate the components of the inertia tensor (186). For instance,

$$\mathcal{I}_{11} = \rho \int (y^2 + z^2) \, dx \, dy \, dz = 2\rho L^2 \int_{-L/2}^{L/2} y^2 \, dy = 4\rho L^2 [y^3/3]_0^{L/2} = \frac{ML^2}{6}. \tag{189}$$

The other diagonal entries are seen to be equal $\mathcal{I}_{11} = \mathcal{I}_{22} = \mathcal{I}_{33}$. The off-diagonal entries vanish as the integrands are odd. E.g., $\mathcal{I}_{12} = -\rho \int xy \, dx \, dy \, dz = 0$. Thus, the inertia tensor of this cube is a multiple of the identity $\mathcal{I} = (ML^2/6)\mathbf{1}$. Consequently, every (nonzero)⁶⁴ vector is an eigenvector of \mathcal{I} with eigenvalue $ML^2/6$. Hence, any three linearly independent vectors through the CM could be chosen to be the principal axes, though it is most convenient to take them parallel to the edges. Moreover, all three principal moments of inertia are equal to $ML^2/6$. It follows that a homogeneous cube is a spherical top!

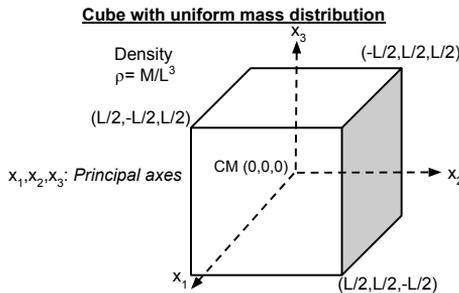


Figure 16: Homogeneous cube with a convenient choice of principal axes. The front, top and right faces are shown. It is a spherical top as the three principal moments of inertia are equal. In fact, the moment of inertia about any axis through the CM is $ML^2/6$.

Collinear rigid body or rigid rotator. If the body is concentrated along a straight line, say the z -axis, then it is called a rigid rotator and has no rotational inertia when

⁶⁴An eigenvector v is a nontrivial solution to the eigenvalue problem $Av = \lambda v$ for a matrix A and complex number λ . The zero vector v is always a solution for any A and for any λ , so it is not considered an eigenvector although it lies in the eigenspace spanned by any eigenvector.

spinning about the z -axis. Note that the mass distribution need not be uniform along the z -axis. Since $x = y = 0$ for all particles, the center of mass lies on the z -axis and the origin $z = 0$ may be chosen at the CM. We must have $\mathcal{I}_3 = 0$ and $\mathcal{I}_1 = \mathcal{I}_2 = \sum_a m_a z_a^2$. The principal axes of inertia can be taken to point along the z -axis and any pair of mutually orthogonal directions in the x - y plane.

0.17.6 Angular momentum of a rigid body

The angular momentum of a rigid body is an important dynamical vector. It is possible to formulate the equations of motion of a rigid body in terms of its linear and angular momenta. The angular momentum, like other dynamical vectors, depends on the choice of reference frame: its components depend on the choice of coordinate axes as well as an origin. For instance, we will find that in the absence of external torques, the components of the rotational angular momentum about the CM with respect to the lab frame are conserved and can therefore be used to restrict the possible types of motion. Furthermore, just as the linear momentum of a particle is linearly related to its velocity through multiplication by its mass, the angular momentum is linearly related to the angular velocity of a rigid body via the inertia tensor. However, since the inertia tensor is generally not a multiple of the identity, the angular momentum and angular velocity vectors need not point in the same direction, unlike the linear momentum and translational velocity. This relation between the two vectors has useful consequences. For instance, in the absence of external torques, the conservation of angular momentum and the form of the inertia tensor may be used to infer the components of angular velocity and thereby deduce the nature of the rotational motion of a rigid body.

Angular momentum of CM motion and of rotation about CM. To begin with, we recall that the angular momentum of a system of particles is defined with respect to an origin. If we use the origin of the lab frame, then the position vector of a point labelled a in the rigid body is $\mathbf{r}_a = \mathbf{R} + \mathbf{r}_a$ where \mathbf{R} is the position vector of the center of mass. If \mathbf{p}_a is the lab-frame momentum of the same particle, we must have $\mathbf{p}_a = m_a \mathbf{v}_a = m_a \mathbf{V} + m_a \boldsymbol{\Omega} \times \mathbf{r}_a$ from (178). Thus, the ‘total’ angular momentum about the origin of the lab frame is

$$\mathbf{L}_{\text{tot}} = \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a \mathbf{R} \times \mathbf{p}_a + \sum_a m_a \mathbf{r}_a \times \mathbf{V} + \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\Omega} \times \mathbf{r}_a). \quad (190)$$

Furthermore, the total momentum $\mathbf{P} = \sum_a \mathbf{p}_a$ coincides with the CM momentum $M\mathbf{V}$ (179). Thus, the total angular momentum about the origin of the lab frame

$$\mathbf{L}_{\text{tot}} = \mathbf{R} \times \mathbf{P} + \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\Omega} \times \mathbf{r}_a) = \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{rot}} \quad (191)$$

is a sum of a center of mass part $\mathbf{L}_{\text{cm}} = \mathbf{R} \times \mathbf{P}$ and a rotational part. \mathbf{L}_{cm} is the angular momentum of the CM about the origin of the lab frame. \mathbf{L}_{rot} also admits a nice interpretation. Consider an instant of time when the CM is moving at velocity \mathbf{V} . Imagine an inertial frame that moves with uniform velocity \mathbf{V} relative to the lab frame and is centered at the CM at the instant considered. The CM is momentarily at

rest in this frame. At this instant, the momentum of particle a relative to this inertial frame is $\mathbf{p}_a = m_a \mathbf{v}_a = m_a \boldsymbol{\Omega} \times \mathbf{r}_a$ (178). Thus, we may interpret \mathbf{L}_{rot} as the angular momentum relative to the CM in this inertial frame. Interestingly, \mathbf{L}_{rot} also coincides with the angular momentum of the body relative to the (moving) CM in the lab frame! In fact,

$$\mathbf{L}_{\text{body}}^{\text{cm-lab}} = \sum_a \mathbf{r}_a \times \cancel{\mathbf{p}_a} = \sum_a \cancel{m_a \mathbf{r}_a} \times \mathbf{V} + \sum_a \mathbf{r}_a \times \mathbf{p}_a = \mathbf{L}_{\text{rot}}. \quad (192)$$

Henceforth, we will denote this rotational part of the angular momentum simply by the symbol \mathbf{L} :

$$\mathbf{L} = \mathbf{L}_{\text{rot}} = \mathbf{L}_{\text{body}}^{\text{cm-lab}} = \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a m_a (r_a^2 \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \mathbf{r}_a) \mathbf{r}_a), \quad (193)$$

where we used the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. In components,

$$L_i = \sum m (r^2 \delta_{ij} - r_i r_j) \Omega_j = \mathcal{I}_{ij} \Omega_j. \quad (194)$$

We see that the rotational angular momentum with respect to the CM is related to the angular velocity via the inertia matrix \mathcal{I} . This is analogous to how the translational momentum of a point particle is related to its velocity via the mass $\mathbf{p} = m\mathbf{v}$. Momentum always points in the same direction as velocity. But in general, angular momentum points in the same direction as angular velocity only for an isotropic rigid body, for which the inertia matrix is a multiple of the identity. For an anisotropic rigid body, \mathbf{L} points in the same direction as $\boldsymbol{\Omega}$ only if a principal axis of inertia can be taken to point along $\boldsymbol{\Omega}$.

* **Inertial vs comoving frames.** We note that \mathbf{L}_{rot} is the angular momentum as defined by an inertial observer instantaneously stationed at the center of mass. It is different from the angular momentum about the CM defined by a noninertial observer who always moves and rotates with the body. According to such an observer, the momentum of each particle in the body is zero, so that the corresponding angular momentum is identically zero. However, there is something nontrivial in between: the components of \mathbf{L}_{rot} referred to a comoving frame.

0.17.7 Equations of motion of a rigid body

The EOM of a rigid body made of N point masses can be written using Newton's second law in the lab (inertial) frame

$$m_a \ddot{\mathbf{x}}_a = \mathbf{f}_a \quad \text{for } a = 1, \dots, N. \quad (195)$$

Here, \mathbf{f}_a is the force acting on the a^{th} particle, which in general is a sum of external forces and internal forces due to other particles in the rigid body. These are $3N$ second order equations. However, a *generic* rigid body has only 6 degrees of freedom, irrespective of how large N may be. So the above equations are somewhat redundant: they do not make manifest the rigidity of the body. We should be able to formulate

the EOM of a generic rigid body more economically in terms of just 6 second order ODEs without reference to individual mass points. We may regard the remaining $3N - 6$ equations (assuming $N \geq 3$) as simply encoding the rigidity of the body due to internal forces.

The configuration of the rigid body may be specified by giving the location \mathbf{R} of the center of mass and a rotation about the CM that brings the body to the desired orientation, relative to a reference orientation. We use (ϕ_1, ϕ_2, ϕ_3) to denote three angles that specify the orientation of the rigid body. We may choose them to be the so-called Euler angles, but we do not need them now.

The EOM for the rigid body may be formulated as 1st order equations for the CM momentum \mathbf{P} and for the angular momentum \mathbf{L} about the CM. We will obtain these equations in the lab frame or any other inertial frame, the equations have the same form in all such frames by Galilean invariance. Note that the comoving frame is in general a noninertial frame, as the body may accelerate and rotate. When we need to, we may transform the EOM to the comoving frame.

Center of mass motion. Working in the lab frame, suppose \mathbf{f}_a is the force on the a^{th} particle, then $\not\!/\mathbf{z}_a = \mathbf{f}_a$. Adding these for all the particles we get $\dot{\mathbf{P}} = \mathbf{F}$ where $\mathbf{P} = \sum_a \not\!/\mathbf{z}_a = M\mathbf{V}$ is the total momentum and $\mathbf{F} = \sum \mathbf{f}_a$ is the total force acting on the body⁶⁵. We need only include the external forces on the particles as the interparticle forces cancel by Newton's 3rd law. $\dot{\mathbf{P}} = \mathbf{F}$ is the equation of motion in the lab frame. By Galilean invariance, it takes the same form in any other inertial frame.

This equation may also be obtained from the Lagrangian⁶⁶

$$L = T - U = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mathcal{I}_{ij}\Omega_i\Omega_j - U(\mathbf{R}, \phi) \quad \text{where} \quad \mathbf{V} = \dot{\mathbf{R}}. \quad (196)$$

The (external) potential U may also be regarded as a function of the positions of all the masses $U = \tilde{U}(\mathbf{z}_a)$. The corresponding EOM for \mathbf{R} are seen to reproduce $\dot{\mathbf{P}} = \mathbf{F}$:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{R}_i} = \frac{d}{dt} \frac{\partial L}{\partial V_i} = \frac{\partial L}{\partial R_i} \Rightarrow M\dot{\mathbf{V}} \equiv \dot{\mathbf{P}} = -\frac{\partial U}{\partial \mathbf{R}} = \mathbf{F} = \text{external force.} \quad (197)$$

Here we used the fact that the change in potential energy under a translation of the center of mass by $\delta\mathbf{R}$ is $\delta U = \sum_a \frac{\partial \tilde{U}}{\partial \mathbf{z}_a} \cdot \delta \mathbf{z}_a = -\sum_a \mathbf{f}_a \cdot \delta \mathbf{R}$ since all particles are translated by the same amount, $\delta \mathbf{z}_a = \delta \mathbf{R}$. So $\delta U = -\mathbf{F} \cdot \delta \mathbf{R}$ whence $\frac{\partial U}{\partial R_i} = -F_i$.

Rotational motion. Next, we compute the time derivative of the rotational angular momentum $\mathbf{L} = \sum_a \mathbf{r}_a \times \mathbf{p}_a$ (193) where \mathbf{r}_a is the position vector of the a^{th} particle relative to the CM and $\mathbf{p}_a = \not\!/\mathbf{z}_a - m_a\mathbf{V} = m_a\dot{\mathbf{r}}_a = m_a\mathbf{v}_a = m_a\boldsymbol{\Omega} \times \mathbf{r}_a$ (178)

⁶⁵In fact, the equation $\dot{\mathbf{P}} = \mathbf{F}$ relating the rate of change of total momentum to the vector sum of external forces is valid for any system of particles (e.g., two gravitating point masses in an external gravitational field) and is not restricted to rigid bodies. It is a consequence of Newton's second and third laws for the individual particles.

⁶⁶The external potential energy U could depend both on the location \mathbf{R} of the center of mass as well as on the orientation of the body, specified via the angular variables ϕ .

its momentum due to the rotation alone. As mentioned in §0.17.6, \mathbf{L} is the angular momentum of the body relative to the CM in the lab frame. Moreover, $\dot{\mathbf{p}}_a = \mathbf{f}_a - m_a \dot{\mathbf{V}}$ where \mathbf{f}_a is the force on this particle in the lab frame. Thus,

$$\dot{\mathbf{L}} = \sum_a (\dot{\mathbf{r}}_a \times \mathbf{p}_a + \mathbf{r}_a \times \dot{\mathbf{p}}_a) = \sum_a \mathbf{r}_a \times \mathbf{f}_a - \sum_a m_a \mathbf{r}_a \times \dot{\mathbf{V}} = \sum_a \boldsymbol{\tau}_a = \boldsymbol{\tau}. \quad (198)$$

Here, $\boldsymbol{\tau}$ is the total torque on the body about the CM (in fact, we may restrict to the external torque, see below). Since $\mathbf{L} = \mathcal{I}\boldsymbol{\Omega}$, we could also write $d(\mathcal{I}\boldsymbol{\Omega})/dt = \boldsymbol{\tau}$. This equation holds in the lab frame or any other inertial frame.

Eqn. (198) for $\dot{\mathbf{L}}$ also follows formally from the Lagrangian (196). At first, we are puzzled since we have (i) not specified the angles ϕ_i , (ii) not clarified the relation between Ω_i and $\dot{\phi}_i$ and (iii) not expressed the Lagrangian in terms of $\dot{\phi}_i$. All this can be done using the Euler angles, but we can manage without them for now. Here, we use the axis-angle representation of rotations and exploit the fact that to obtain the EL equations, it is enough to consider infinitesimal rotations. To begin with, we think of $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ as a vector specifying a rotation from a reference orientation. The unit vector along $\boldsymbol{\phi}$ is the axis of rotation while its magnitude is the counterclockwise angle of rotation. For an infinitesimal rotation, $\boldsymbol{\phi}$ reduces to the Euler vector $\delta\boldsymbol{\phi}$. What is more, $\boldsymbol{\Omega}$ was defined as the limit of $\delta\boldsymbol{\phi}/\delta t$ as $\delta t \rightarrow 0$. Thus, if $\boldsymbol{\phi} = \delta\boldsymbol{\phi}$ represents an infinitesimal rotation, then we are justified in taking $\Omega_i = \dot{\phi}_i$. Bearing this in mind, we evaluate the LHS of Lagrange's equations for the angular variables:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_i} = \frac{d}{dt} \frac{\partial L}{\partial \Omega_i} = \frac{d}{dt} (\mathcal{I}_{ij} \Omega_j) = \dot{L}_i. \quad (199)$$

As for the RHS, we wish to show that $\frac{\partial L}{\partial \dot{\phi}^i}$ or equivalently $-\frac{\partial U}{\partial \phi^i}$ are the components of the torque $\boldsymbol{\tau}$. To begin with, $\frac{\partial U}{\partial \phi^i}$ is defined as the limit of $\delta U/\delta \phi^i$ as the infinitesimal rotation specified by the Euler vector $\delta\boldsymbol{\phi} \rightarrow 0$. The change in potential energy δU due to an infinitesimal rotation specified by $\delta\boldsymbol{\phi}$ (about an axis passing through the CM) is minus the work done by the external forces⁶⁷. Under a rotation about the CM, the center of mass \mathbf{R} does not move, so $\delta \mathbf{z}_a = \delta \mathbf{r}_a = \delta\boldsymbol{\phi} \times \mathbf{r}_a$. Consequently, the change in potential energy is (here \mathbf{f}_a may be restricted to external forces)

$$\delta U = - \sum_a \mathbf{f}_a \cdot \delta \mathbf{z}_a = - \sum_a \mathbf{f}_a \cdot (\delta\boldsymbol{\phi} \times \mathbf{r}_a) = -\delta\boldsymbol{\phi} \cdot \sum_a \mathbf{r}_a \times \mathbf{f}_a = -\delta\boldsymbol{\phi} \cdot \boldsymbol{\tau}. \quad (200)$$

From this we deduce that $\partial U/\partial \boldsymbol{\phi} = -\boldsymbol{\tau}$. So the Euler-Lagrange equations imply the law of evolution of angular momentum (relative to the CM) of the rigid body

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{\phi}}} = \frac{\partial L}{\partial \boldsymbol{\phi}} \Rightarrow \dot{\mathbf{L}} = - \frac{\partial U}{\partial \boldsymbol{\phi}} = \boldsymbol{\tau} = \text{external torque about the CM}. \quad (201)$$

In the absence of any external torque about the CM (e.g., if there are no external forces, so that U can be taken to vanish), the angular momentum of the rigid body about its CM is independent of time: $\dot{\mathbf{L}} = 0$.

⁶⁷Internal forces between particles in the body can do work only if the body is deformed.

0.17.8 Force-free motion of rigid bodies

Now consider a rigid body in the absence of any external forces⁶⁸. If the center of mass was initially at rest, it will remain so. More generally, the CM moves along a straight line since $\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F} = 0$ (197). Let us choose our inertial frame to move at velocity $\mathbf{V} = \dot{\mathbf{R}}$ relative to the lab, so that the CM is always at rest at its origin. Of course, the body could still rotate. Since there are no external forces, there are no external torques either, about any point. So the angular momentum about any point and in particular about the CM, $\mathbf{L} = \mathbf{L}_{\text{rot}}$ must be constant in time⁶⁹ by (198). We may use the conservation of angular momentum and the formula $\mathbf{L} = \mathcal{I}\boldsymbol{\Omega}$ to understand many qualitative and quantitative aspects of the free rotational motion of spherical tops, rigid rotators and symmetrical tops, without attempting to solve the equations of motion explicitly. Some striking qualitative features emerge. For instance, all that a spherical top can do is to spin at a constant angular velocity about an axis that points along the conserved angular momentum vector and passes through the CM. The direction of this \mathbf{L} vector is determined by initial conditions and the rate of spinning is determined in terms of its magnitude and the moment of inertia about the CM. A rigid rotator (collinear rigid body) on the other hand, must rotate uniformly in a plane perpendicular to the conserved \mathbf{L} vector. In both these cases, the angular velocity vector $\boldsymbol{\Omega}$ is also conserved. A symmetrical top can do more *even* in the absence of an external torque: it can spin steadily about its symmetry axis while the symmetry axis itself rotates (i.e., precesses) uniformly about the fixed direction of \mathbf{L} . This can be applied to the precession ('Chandler wobble') of the angular velocity vector of the Earth about its symmetry axis. An anisotropic rigid body can display more intricate motion.

Free motion of a spherical top. For a spherical top, the principal moments of inertia $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ are equal and the inertia tensor $\mathcal{I} = \text{diag}(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ is a multiple of the 3×3 identity matrix. Thus $\mathbf{L} = \mathcal{I}_1\boldsymbol{\Omega}$, so the angular velocity and angular momentum both point in the same direction and are constant in time. In particular, force-free motion of a spherical top consists of uniform rotation about some axis (through the CM, why?) that is fixed in the lab frame. The direction of this axis and the rate of rotation $\Omega = |\boldsymbol{\Omega}|$ are determined by initial conditions. The conserved energy $E = \frac{1}{2}\mathcal{I}_1(\Omega_1^2 + \Omega_2^2 + \Omega_3^2) = \frac{1}{2}\mathcal{I}_1|\boldsymbol{\Omega}|^2 = \frac{|\mathbf{L}|^2}{2\mathcal{I}_1}$ is also determined by initial conditions.

⁶⁸While external forces like gravity are usually present, it may be possible to ignore the effect of the gravitational torque if the body has a large enough rotational kinetic energy.

⁶⁹We note in passing that even in such a force-free situation, the components of \mathbf{L} , relative to a body-fixed frame, will in general depend on time.