Classical Mechanics 2, Spring 2023 CMI

Assignment 5

Due by 6pm, Sun Mar 26, 2023

Canonical Transformations, Kepler problem

1. ⟨**14**⟩ Suppose (r_1, p_1) and (r_2, p_2) are the canonically conjugate Cartesian coordinates and momenta of a pair of point particles of masses m_1, m_2 that move on a line (this is a 1d toy version of the two-body Kepler problem). Define the center of mass and relative positions and momenta by

$$R = (m_1 r_1 + m_2 r_2)/M, \quad r = r_2 - r_1, \quad P = p_1 + p_2, \quad p = m(p_2/m_2 - p_1/m_1). \qquad (1)$$

where $M = m_1 + m_2$ and $m = m_1 m_2 / M$.

   (a) ⟨**1**⟩ Justify the formula for the relative momentum p.

   (b) ⟨**4**⟩ Invert this transformation and express r_1, r_2, p_1, p_2 in terms of R, r, P, p.

   (c) ⟨**3**⟩ Given that the old variables are canonical, what are the 6 (up to antisymmetry) basic PBs among them?

   (d) ⟨**6**⟩ Determine whether this transformation is canonical by evaluating all the 6 relevant PBs among new (CM and relative) variables.

2. ⟨**14**⟩ Consider a system with 1 degree of freedom and real position coordinate q and canonically conjugate momentum p. Suppose $q \mapsto Q(q)$ is a point transformation on configuration space. Let $P(q, p)$ denote the new momentum conjugate to Q.

   (a) ⟨**2**⟩ Does this canonical transformation include the identity transformation of phase space variables as a special case? Say yes/no and why.

   (b) ⟨**2**⟩ We seek a generating function of type 2, $F_2(q, P)$ which, for historical reasons, we will denote $W(q, P)$. Write the 2 equations of transformation in terms of the generating function W.

   (c) ⟨**4**⟩ Integrate one of the two equations to arrive at a formula for $W(q, P)$ including any integration 'constants'. Hint: You need to decide which equation can be integrated easily and with respect to which variable.

   (d) ⟨**2**⟩ Find an expression for the old momentum p in terms of $P, Q(q)$ etc.

   (e) ⟨**2**⟩ Invert this to obtain a formula for the new momentum P in terms of the old quantities.

   (f) ⟨**2**⟩ Earlier, we used a Lagrange function to lift a point transformation on configuration space to a CT on phase space. What roughly encodes the role of the Lagrange function in the above construction?