Classical Mechanics 2, Spring 2023 CMI
Assignment 4
Due by 6pm, Friday Feb 17, 2023
Hamiltonian, Hamilton's equations, Legendre transform

1. ⟨**12**⟩ Consider a mechanical system with configuration space $\mathbb{R}^n$ and generalized coordinates q^i for $i = 1, \ldots, n$. It is governed by a smooth potential energy $V(q)$ and kinetic energy $T = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j$ where m_{ij} are the entries of a fixed real symmetric matrix with nonzero determinant. Repeated indices are summed from 1 to n. Suppose the Lagrangian is $L = T - V$.

   (a) ⟨**3**⟩ Find a simple formula for the momenta p_k conjugate to q^k. Use the notation m^{ij} for the entries of the inverse matrix. What is its defining property?

   (b) ⟨**3**⟩ Find the Hamiltonian $H = p_k \dot{q}^k - L$ and compare it with $E = T + V$. Here, we regard H as a function of q and $\dot{q}$.

   (c) ⟨**3**⟩ Solve for the velocities $\dot{q}^i$ and express the Hamiltonian in terms of coordinates and momenta.

   (d) ⟨**3**⟩ Find Hamilton's equations of motion.

2. ⟨**3+2+3**⟩ Suppose the Lagrangian of a system with one real degree of freedom is $L(q, \dot{q}) = e^{\dot{q}} - V(q)$ where V is some smooth potential. (a) Find the corresponding Hamiltonian $H(q, p)$ by evaluating the Legendre transform of L. (b) Evaluate the inverse Legendre transform of $H(q, p)$ and compare the result with $L(q, \dot{q})$. (c) What do you understand by the term convex function? For any fixed q is $L(q, \dot{q})$ a convex function of $\dot{q}$?