Classical Mechanics 2, Spring 2023 CMI
Assignment 3
Due by 8pm, Monday Feb 6, 2023
Hamiltonian, Extremum action principle

1. $\langle \mathbf{1+2+2+7+2} \rangle$ Consider the Lagrangian $L = T - V$ for a system with 1 real degree of freedom q. Suppose $T = \frac{1}{2}m\dot{q}^2 + \frac{1}{3}g\dot{q}^3$ and $V(q)$ is some smooth nonconstant potential function and m, g are nonzero real constants. (a) Find the momentum p conjugate to q. (b) Find the Euler-Lagrange equation of motion as a 2nd order ODE. (c) Express the Hamiltonian as a function of q and $\dot{q}$ and comment on whether it is equal to $J = T + V$. (d) Use the EL equation to check whether H and $J = T + V$ are conserved. (e) Briefly comment on what you may infer from this example.

2. $\langle \mathbf{16} \rangle$ **Nature of extremum of action.** Consider a particle of mass m in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Suppose $x(t)$ is a trajectory between $x(t_i) = x_i$ and $x(t_f) = x_f$ and let $x(t) + \delta x(t)$ be a neighboring path with $\delta x(t_i) = \delta x(t_f) = 0$.

   (a) $\langle \mathbf{4} \rangle$ Write the action of the path $x + \delta x$ for small δx as a quadratic Taylor polynomial in δx. Show that you get the following expression. What is S_1?

   $$\begin{aligned} S[x+\delta x] &\approx S_0 + S_1 + S_2 = S[x] - \int_{t_i}^{t_f} (m\ddot{x} + m\omega^2 x)\, \delta x \, dt \\ &+ \int_{t_i}^{t_f} \left[\frac{1}{2} m (\delta\dot{x})^2 - \frac{1}{2} m\omega^2 (\delta x)^2 \right] dt. \end{aligned} \tag{1}$$

   (b) $\langle \mathbf{2} \rangle$ For what κ is $x(t) + \delta x(t)$ a legitimate neighboring path for the variation $\delta x(t) = \epsilon \, \sin \kappa (t - t_i)$?

   (c) $\langle \mathbf{3} \rangle$ Evaluate $S_2[\delta x]$ for all the allowed values of κ.

   (d) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10\text{s}$ and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *less* than that of $x(t)$. [Assume we may ignore higher order corrections in the Taylor approximation.]

   (e) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10\text{s}$ and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *more* than that of $x(t)$. [Assume we may ignore higher order corrections in the Taylor approximation.]

   (f) $\langle \mathbf{1} \rangle$ What sort of an extremum of action is the classical trajectory?