Classical Mechanics 2, Spring 2023 CMI

Assignment 2

Due by 6pm, Saturday Jan 21, 2023

Extremal action principle and Euler-Lagrange equations

1. $\langle \mathbf{4} \rangle$ Show through an example [involving a free particle on a line] that if both q and $p = m\dot{q}$ at initial and final times are specified, there may be no trajectory joining these states.

2. $\langle \mathbf{3+1+2} \rangle$ Suppose the Lagrangian for a system $L(q^1, \cdots, q^n, \dot{q}^1, \cdots, \dot{q}^n)$ is independent of the generalized coordinate q^j for some $1 \leq j \leq n$. (a) Find a dynamical variable that is conserved under time evolution and mention its name. (b) Find the corresponding conserved quantity for $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$ where r, ϕ are plane polar coordinates and $m > 0$ is a fixed real constant. (c) What system does this Lagrangian describe and what is the physical meaning of the obtained conserved quantity?

3. $\langle \mathbf{7+3} \rangle$ Suppose the Lagrangian for a system with one degree of freedom with configuration space $\mathcal{Q} = \mathbb{R}$ is $L = L(q, \dot{q}, \ddot{q})$. Consider the action for paths $q(t)$ with $t_1 \leq t \leq t_2$ between points q_1 and q_2 on $\mathcal{Q}$:
$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q})\, dt. \tag{1}$$
We ask that S be extremal with respect to infinitesimal variations of path holding the initial and final positions and velocities fixed. (a) Find the Euler-Lagrange equation for the stationarity of S. (b) Taking $L = \frac{1}{2}m\dot{q}^2 - V(q) + \frac{1}{2}\alpha\ddot{q}^2$ find the EL equation and mention its order (with respect to time derivatives). Here $m > 0$ and α are real constants and V a real differentiable function.