Classical Mechanics 2, Spring 2014 CMI

Problem set 7

Due by the beginning of lecture on Monday Feb 10, 2014

Poisson brackets

1. ⟨**4**⟩ Find the *unequal* time p.b. $\{q(0), q(t)\}$ for a free particle of mass m moving on a line.

2. ⟨**23**⟩ Consider a particle moving on the plane $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$. x, y, p_x, p_y are the usual coordinates and momenta on phase space satisfying canonical Poisson bracket relations. Define the dynamical variables (plane polars) $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \arctan(y/x)$. Recall that the Lagrangian in terms of polar coordinates is $\tilde{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \tilde{V}(r, \theta)$. We wish to compute the p.b. among polar coordinates and their conjugate momenta using the definition $\{f, g\} = f_x g_{p_x} - f_{p_x} g_x + f_y g_{p_y} - f_{p_y} g_y$. Here subscripts denote partial derivatives.

   (a) ⟨**3**⟩ Express the conjugate momenta p_r, p_θ as functions on phase space (i.e., in terms of x, y, p_x, p_y). Show that you get

   $$p_r = \frac{x}{r(x,y)}p_x + \frac{y}{r(x,y)}p_y \quad \text{and} \quad p_\theta = xp_y - yp_x. \tag{1}$$

   (b) ⟨**4**⟩ Find the partial derivatives (denoted by subscripts) $r_x, r_y, \theta_x, \theta_y$. Show that you get

   $$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad \theta_x = -\frac{y}{r^2}, \quad \theta_y = \frac{x}{r^2}. \tag{2}$$

   (c) ⟨**1**⟩ Find the partial derivatives of r, θ with respect to p_x and p_y: $r_{p_x}, r_{p_y}, \theta_{p_x}, \theta_{p_y}$.

   (d) ⟨**8**⟩ Find the partial derivatives of p_r, p_θ with respect to x, y, p_x, p_y. You must give 8 formulae, two of which must be shown to be

   $$(p_r)_x = \frac{p_x}{r} - \frac{x^2}{r^3}p_x - \frac{xy}{r^3}p_y \quad \text{and} \quad (p_r)_y = \frac{p_y}{r} - \frac{y^2}{r^3}p_y - \frac{xy}{r^3}p_x. \tag{3}$$

   (e) ⟨**6**⟩ Find the 6 (after accounting for anti-symmetry) p.b. among polar coordinates and momenta (i) $\{r, \theta\}$, (ii) $\{r, p_r\}$, (iii) $\{r, p_\theta\}$, (iv) $\{\theta, p_\theta\}$, (v) $\{\theta, p_r\}$ and (vi) $\{p_r, p_\theta\}$.

   (f) ⟨**1**⟩ Comment on the result.