**Classical Mechanics 1, Autumn 2025 CMI**
Assignment 8
Due by 6pm, 14 Nov, 2025
Kepler problem

1. $\langle \mathbf{3+4+4} \rangle$ **CM and relative variables.** The center of mass and relative vectors in the problem of 2 point particles of masses $m_1$, $m_2$ and positions $\boldsymbol{r}_1, \boldsymbol{r}_2$ are defined as $\boldsymbol{R} = (m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2)/M$ and $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ where $M = m_1 + m_2$. Let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be the momenta of the two particles. Let us define $\mu_{1,2} = m_{1,2}/M$ and the reduced mass $m = m_1 m_2/(m_1+m_2)$. (a) Draw a figure with a suitable origin (not at the center of mass) indicating the vectors $\boldsymbol{R}, \boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2$. (b) Show that

$$\boldsymbol{r}_1 = \boldsymbol{R} - \mu_2 \boldsymbol{r}, \quad \boldsymbol{r}_2 = \boldsymbol{R} + \mu_1 \boldsymbol{r}. \tag{1}$$

Let the CM momentum be defined as $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ and the relative momentum $\boldsymbol{p} = m\dot{\boldsymbol{r}}$. Show that

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2, \quad \boldsymbol{p} = \mu_1 \boldsymbol{p}_2 - \mu_2 \boldsymbol{p}_1 \quad \text{and} \quad \boldsymbol{p}_1 = \mu_1 \boldsymbol{P} - \boldsymbol{p}, \quad \boldsymbol{p}_2 = \mu_2 \boldsymbol{P} + \boldsymbol{p}. \tag{2}$$

(c) Express the total energy of the two-body central force problem (with potential $V(r)$) as a sum of center of mass and relative energies.

2. $\langle \mathbf{3+2+5} \rangle$ Recall that the (relative) energy in the Kepler problem may be expressed as $E = \frac{1}{2}m\dot{r}^2 + V_{\rm eff}(r)$ where $V_{\rm eff} = \frac{l^2}{2mr^2} - \frac{\alpha}{r}$. Here $r$ is the magnitude of the relative vector, $m$ the reduced mass, $\alpha = Gm_1m_2$ and $l$ the $z$ component of relative angular momentum. We showed that for fixed $l \neq 0$ and $E < 0$, the orbit is an ellipse $r = \rho/(1+\epsilon\cos\phi)$ (up to a possible rotation) where $\rho$ is the radius of the circular orbit for that $l$ (or semilatus rectum) and $\epsilon$ the eccentricity. (a) Find $\rho$ in terms of $l, m, \alpha$ and also find the minimum value of the effective potential. (b) Express the length of the semimajor axis $a$ in terms of $\rho, \epsilon$. (c) Express the relative energy of this elliptical orbit in terms of $\rho, \epsilon$ and also in terms of the length of the semimajor axis $a$.