Classical Mechanics 1, Autumn 2025 CMI
Assignment 2
Due by 6pm Friday 22 Aug, 2025
Levi-Civita symbol, spherical polar coordinates

1. $\langle \mathbf{4+3} \rangle$ (a) Starting from $\hat{x} \times \hat{y} = \hat{z}$ and its cyclic permutations, express the Cartesian components of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ in terms of those of $\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and $\boldsymbol{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$. (b) Use this to obtain an expression for the scalar triple product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ in terms of the components of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Mention a geometric interpretation of the scalar triple product.

2. $\langle \mathbf{3} \rangle$ Define the Levi-Civita symbol (a real number) ϵ_{ijk} for integers $1 \le i, j, k \le 3$ (not necessarily distinct) by the conditions (i) $\epsilon_{123} = 1$ and (ii) ϵ_{ijk} reverses sign under exchange of any pair of indices [e.g. $\epsilon_{ijk} = -\epsilon_{jik}$]. Find ϵ_{213}, ϵ_{312}, ϵ_{321}, ϵ_{112}, ϵ_{313} and ϵ_{222}.

3. $\langle \mathbf{4} \rangle$ Verify that $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \sum_{1 \le i,j,k \le 3} \epsilon_{ijk} a_i b_j c_k$ where a_i, b_j, c_k are the Cartesian components of three vectors in $\mathbb{R}^3$. Here $a_1 = a_x, a_2 = a_y, a_3 = a_z$, etc.

4. $\langle \mathbf{4+4} \rangle$ Recall the spherical polar coordinates in 3d $z = r\cos\theta, x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$. Let $\hat{r}, \hat{\theta}, \hat{\phi}$ be defined as

$$\begin{aligned} \hat{r} &= \cos\theta\hat{z} + \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}), \\ \hat{\theta} &= -\sin\theta\hat{z} + \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) \quad \text{and} \\ \hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y}. \end{aligned} \tag{1}$$

   (a) With the help of figures (3d and suitable projections), explain why the formula for $\hat{\theta}$ gives a unit vector in the direction of increasing θ holding r, ϕ fixed. (b) Verify that $(\hat{r}, \hat{\theta}, \hat{\phi})$ is a right-handed orthonormal system.