Classical Mechanics (PG), Autumn 2013 CMI

Problem set 15

Due at the beginning of lecture on Wednesday Nov 6, 2013

Action-angle variables and Hamilton-Jacobi equation

1. ⟨**5**⟩ Complete the calculation outlined in the lecture to show that the deflection angle θ and angular momentum p_θ (old canonical variables) of a pendulum can be expressed in terms of angle Θ and action variables as follows

$$\begin{aligned} \theta(t) &= 2\arcsin\left[k\,\mathrm{sn}\left(\frac{2K(k)\Theta(t)}{\pi},k\right)\right] \\ p_\theta(t) &= 2ml^2\omega k\,\mathrm{cn}\left(\frac{2K(k)\Theta(t)}{\pi},k\right). \end{aligned} \tag{1}$$

   Where does the dependence on action variable I enter?

2. ⟨**5**⟩ What happens to the above formulae in the limit of low energy (small oscillations)? Give the leading limiting behavior of $\Theta(t)$ as well as $\theta(t)$ and $p_\theta(t)$ and comment on whether they are as physically expected.

3. ⟨**5**⟩ Consider the function of one complex variable $z \mapsto w = e^z$. Given $w \neq 0$, identify a countably infinite collection of complex numbers z such that $e^z = w$. Display a typical such w in the interior of the first quadrant of the complex plane and the corresponding values of z in a figure.

4. ⟨**5**⟩ Consider a particle free to move on a line with hamiltonian $H = \frac{p^2}{2m}$. Work out the 'complete' solutions of the time-dependent HJ equation and find Hamilton's principal function. But unlike in the lecture, make a different choice for the new momentum $P = \alpha$ (such as E or $2mE$). For such a choice, the new 'momentum' may not have the physical dimensions of momentum, but that is ok. Express the new constant coordinate and momentum Q and P in terms of the initial values of the old ones $q(0), p(0)$. Show that Q, P are canonically conjugate even with this altered choice. Use the HJ approach to find the trajectory $q(t), p(t)$ in terms of $q(0)$ and $p(0)$ and show that you get the expected answer even with this altered choice of new coordinates and momenta.