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Multi-matrix loop equations: algebraic & differential structures and an approximation based on deformation quantization

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Abstract: Large-N multi-matrix loop equations are formulated as quadratic difference equations in concatenation of gluon correlations. Though non-linear, they involve highest rank correlations linearly. They are underdetermined in many cases. Additional linear equations for gluon correlations, associated to symmetries of action and measure are found. Loop equations aren't differential equations as they involve left annihilation, which doesn't satisfy the Leibnitz rule with concatenation. But left annihilation is a derivation of the commutative shuffle product. Moreover shuffle and concatenation combine to define a bialgebra. Motivated by deformation quantization, we expand concatenation around shuffle in powers of q, whose physical value is 1. At zeroth order the loop equations become quadratic PDEs in the shuffle algebra. If the variation of the action is linear in iterated commutators of left annihilations, these quadratic PDEs linearize by passage to shuffle reciprocal of correlations. Remarkably, this is true for regularized versions of the Yang-Mills, Chern-Simons and Gaussian actions. But the linear equations are underdetermined just as the loop equations were. For any particular solution, the shuffle reciprocal is explicitly inverted to get the zeroth order gluon correlations. To go beyond zeroth order, we find a Poisson bracket on the shuffle algebra and associative q-products interpolating between shuffle and concatenation. This method, and a complementary one of deforming annihilation rather than product are shown to give over and underestimates for correlations of a gaussian matrix model.

Keywords: Matrix Models, Quantum Groups, 1/N Expansion, Field Theories in Lower Dimensions.

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1. Introduction

1.1 General Remarks

Approximation methods in physics are often usefully organized as an expansion in a dimensionless parameter. As is well known, at first sight, quantum Yang-Mills theory does not have any such expansion parameter since the dimensionless coupling g^2 of the classical theory is determined in terms of the ratio $\frac{Q^2}{\Lambda^2}$ where Q^2 is the momentum transferred to a hadronic system by an external (say electroweak) current. Λ (say Λ_{QCD}) is the dimensional parameter arising via dimensional transmutation and renormalization. The success of an expansion in inverse (logarithmic) powers of $\frac{Q^2}{\Lambda^2}$ is, however, crucially dependent on the asymptotic freedom of the theory for large values of this parameter [1]. Thus, this expansion (perturbative QCD), which is the analogue of the Born approximation of atomic physics, though spectacularly successful at high momentum transfers, is not particularly useful to describe 'intrinsic' properties of hadrons in the absence of an external probe transferring a large momentum [2].

What about $\hbar$ as an expansion parameter for quantum Yang-Mills theory around its classical limit? This is a bad starting point, since all variables, not just gauge-invariant ones, stop fluctuating in this limit. Since $\hbar$ can be absorbed into g^2, the 'loop' expansion in powers of $\hbar$ around the trivial solution to classical Yang-Mills theory is the same as perturbative QCD. Thus, it is useful only at high momentum transfers.

As observed by 't Hooft [3], $1/N$ of the gauge group $\mathrm{SU}(N)$ is an expansion parameter for quantum Yang-Mills theory, holding $\lambda = g^2 N$ fixed. There are many indications [4] that $N \to \infty$ is a good approximation to the quantum theory. Moreover, it is a classical limit where fluctuations in gauge-invariant variables alone vanish. Despite effort, the $1/N$ expansion has not been as quantitatively successful as perturbative QCD was in the high energy regime. The success of the loop expansion lay in the availability of explicit solutions to classical Yang-Mills theory around which to expand (eg. flat connections, Euclidean instantons). By contrast, we don't know the zeroth order solution of large N Yang-Mills theory around which to perform a $1/N$ expansion. Difficulties are encountered in each of the many ways of formulating the large N limit of Yang-Mills theory: summing an infinite class of planar diagrams [3], solving the Makeenko-Migdal equations for Wilson loops [5 − 7] or solving the factorized Schwinger-Dyson equations for gluon correlations. It would really help to have yet another dimensionless expansion parameter, to organize an approximate solution of $N = \infty$ Yang-Mills theory.

The strategy of looking for an expansion parameter over and above $1/N$ has found success in maximally super-symmetric Yang-Mills theory. In some sectors of the $\mathcal{N} = 4$ theory, an expansion around small values of the ratio of 't Hooft coupling to square of R-charge $(\frac{\lambda}{J^2})$ has been developed [8]. An analog of this for the non-supersymmetric theory would be useful. But since there is no such obvious expansion parameter, we will invent one based on deeper mathematical structures of the theory.

Inspiration for a possible approximation comes from atomic physics, as emphasized by Rajeev [9]. The Hartree-Fock approximation for many-electron atoms is analogous to the $N \to \infty$ limit of Yang-Mills theory, since it can be formulated as the limit in which the number of replicas of each electron (N) tends to infinity [10]. In general, the Hartree-Fock equations are difficult to solve since they involve the electron density matrix, which is a projection operator. However, after $N \to \infty$ it is possible to take a semiclassical limit based on deformation quantization. These limits do not commute. At zeroth order this leads to the Thomas-Fermi non-linear ODE whose solution gives a good first approximation to the charge density of a many-electron atom [9]. Can something similar work for large N Yang-Mills theory?

The approximation method studied in this paper is based on the observation that even in the 'classical' large-N limit, the equations of matrix models and Yang-Mills theory still involve non-commutative concatenation products. It should be possible to take a further 'classical' limit, where they are approximated by commutative products by analogy with deformation quantization. In our case, the parameter controlling this further classical limit is a deformation parameter whose physical value is $q = 1$.

Another lesson from the formulation of Hartree-Fock theory as the limit of a large number of electron replicas, is that the physical value of an expansion parameter need not be small for the expansion to be practically successful. Indeed, the physical number of replicas of the electron is $N = 1$ and yet, Hartree-Fock, which corresponds to $N = \infty$, provides a good first approximation as part of a $1/N$ expansion! A more famous example of an expansion in a parameter whose physical value is 1 is the ϵ-expansion applied to $3 - d$ statistical models in the vicinity of a 2^{nd} order phase transition. Another instance is the δ expansion of Bender and collaborators [11]. Applied to QED, it can be regarded as an expansion in the number of identically charged electron species whose physical value is $\delta = 1$. Yet an expansion in powers of δ is accurate. It has also been successfully applied to a variety of other non-linear equations.

Another possible expansion parameter is the inverse number of space-time dimensions $1/d$. However, we do not yet know of any useful formulation of the $d \to \infty$ limit of large N Yang-Mills theory that is a simplification. This is again motivated by atomic physics, where the $d \to \infty$ limit in the zero angular momentum sector is a non-relativistic $O(d)$ vector model for position vectors of electrons. This provides a spectacularly good approximation to the binding energies of many-electron atoms in a $1/d$ expansion, as shown by Herschbach and collaborators [12].

1.2 Loop Equations of Large-N Matrix Models

A primary aim in the study of a Euclidean large-N multi-matrix model is to determine its factorized correlations. They satisfy quantum corrected equations of motion, which are factorized Schwinger-Dyson or loop equations (LE). We formulate these in a way that makes manifest some algebraic and differential structures they share with the Makeenko-Migdal equations of $N = \infty$ Yang-Mills theory [5–7]. In particular, they are not differential equations, due to a mismatch between the differential and product structures. Though infinite in number and quadratically non-linear, we show that they have a hierarchical structure whereby the highest rank correlations in any equation only appear linearly. However, we show they are underdetermined in many interesting cases. We identify additional equations which a naive passage to the large N limit misses. They are conditions implied by invariance of matrix integrals for correlations, under transformations leaving both action and measure invariant, possibly up to $1/N^2$ corrections (eg. BRST transformations). However, the additional equations are not implemented, so the underdeterminacy of the loop equations is not satisfactorily resolved. On the other hand, we exploit the algebraic and differential structures to propose an approximation scheme for a class of Λ-(multi)-matrix models motivated by the Lagrangian of Yang-Mills theory,

$$\mathcal{L} = \operatorname{tr} \left\{ \frac{1}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) - ig \partial_\mu A_\nu [A^\mu, A^\nu] - \frac{g^2}{4} [A_\mu, A_\nu][A^\mu, A^\nu] \right. \\ \left. + \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + \partial_\mu \bar{c}\, \partial^\mu c - ig \partial_\mu \bar{c}\, [A^\mu, c] \right\}. \tag{1.1}$$

The primary virtue of the scheme is that at zeroth order, it turns the non-linear loop equations into linear PDEs. Prominent in this class of models are those whose action is a linear sum of

$$S_G = \frac{1}{2} \operatorname{tr} C^{ij} A_i A_j, \\ S_{CS} = \frac{2i\kappa}{3} \operatorname{tr} C^{ijk} A_i [A_j, A_k] \ \ \& \ S_{YM} = -\frac{1}{4\alpha} \operatorname{tr} [A_i, A_j][A_k, A_l] g^{ik} g^{jl}. \tag{1.2}$$

In the first two cases, we allow A_i to denote either gluon (hermitian complex) or ghost (grassmann) matrices[1]. Though they arise from terms with $2, 1$ and 0 derivatives in the Yang-Mills action, these matrix models may be called Gaussian, Chern-Simons and Yang-Mills models since they also include the zero momentum limits of the corresponding field theories. The indices i, j, k, l are short for position and polarization quantum numbers, while color indices are suppressed. It may be possible to fruitfully think of Yang-Mills theory as a grand limiting case of such matrix models for appropriate integral kernels C^{ij}, C^{ijk} and g^{ij} when the indices become continuous. Matrix models and field theories of this type also arise in dimensional reductions of Yang-Mills theory to 2 or fewer space-time dimensions. Here we consider bosonic matrix models, the extension of our results to models with ghost matrices will be treated in [13].

Summary of results and organization: In section 2.1 we obtain the large-N loop equations[2] $|iJ| S^{Ji} G_{JI} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}$ for gluon correlations $G_I = \langle \frac{1}{N} \operatorname{tr} A_I \rangle$ of a hermitian multi-matrix model with action $\operatorname{tr} S(A) = \operatorname{tr} S^I A_I$. In section 2.2 we show that the loop

[1]We assume there are an equal number of ghost and anti-ghost matrices in each term, as in Yang-Mills theory.

[2]Small letters i denote single indices, capitals denote multi-indices $I = i_1 i_2 \cdots i_n$ and $|I|$ denotes the number of indices in a multi-index. Repeated upper and lower indices are summed. δ_J^I is 1 if $I = J$ and zero otherwise.

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equations are underdetermined in some interesting cases, though they determine infinitely many higher rank correlations in terms of lower rank correlations. In section 2.3 we obtain additional equations associated with symmetries of both measure and action, which are easily overlooked in passing to the large-N limit. In section 2.4 the loop equations are reformulated in terms of the series $G(\xi) = G_I \xi^I$, where ξ^i are non-commuting sources:

$$\sum_{n\geq 0}(n+1)S^{j_1\cdots j_n i}D_{j_n}\cdots D_{j_1}G(\xi) = G(\xi)\xi^i G(\xi) \quad \text{or} \quad \mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi). \tag{1.3}$$

The linear term (variation of action) is written in terms of left annihilation operators D_i. The quadratic term in gluon correlations involves the concatenation product. It is the variation of the matrix model measure and is universal, independent of the action. However, left annihilation does not satisfy the Leibnitz rule with respect to concatenation, and to make things worse, concatenation is non-commutative. Due to this mismatch, the loop equations are not differential equations in the ordinary sense. On the other hand, there is another natural product between gluon correlations, the shuffle product (section 2.5), which arises from the expectation value of point-wise products of Wilson loops. It turns out that left annihilation is a derivation of the shuffle product. Moreover, there is a democratic version of left annihilation, full annihilation, that is a derivation of concatenation (section 2.6). Furthermore, concatenation and shuffle combine to form a bialgebra (appendices B and C).

These algebraic and differential structures along with ideas from deformation quantization suggest a possible approximation scheme for the loop equations. The idea is to remedy the above mismatch by expanding the non-commutative concatenation product in a series around the commutative shuffle product so that at zeroth order, concatenation is replaced by shuffle and the loop equations become quadratically non-linear inhomogeneous PDEs in an infinite dimensional space spanned by words in Λ letters. Thus, the approximation scheme involves the introduction of a deformation parameter controlling the amount by which the loop equations for gluon and ghost correlations fail to be partial differential equations. The physical value of our dimensionless expansion parameter q is 1.

A further remarkable simplification occurs in models whose action is such that $\mathcal{S}^i$ is a derivation of the shuffle product. These are models in which $\mathcal{S}^i$ is a linear combination of iterated commutators of D_i and include the zero-momentum Gaussian, Chern-Simons and Yang-Mills models as well as their field theoretic counterparts as examples (section 2.7). In these cases, the passage from $G(\xi)$ to its shuffle-reciprocal $F(\xi) = F_I\xi^I$ turns the non-linear PDEs into a system of linear equations for the F_I (section 4.1). We obtain an explicit formula for G_I in terms of F_J so that once the linear equations are solved, the $\mathcal{O}(q^0)$ gluon correlations can be obtained. This is illustrated for the zero-momentum Gaussian (section 4.1.1), Chern-Simons (section 4.1.2) and Yang-Mills (section 4.1.3) multi-matrix models. For the Gaussian, the linear equations have a unique solution which provides a first approximation to the exact large N correlations. But for the other examples, the equations are underdetermined just as the original loop equations were and we exhibit infinite classes of solutions. It remains to find and implemented the additional constraints on correlations, such as those associated to symmetries of action and measure.

In section 4.3 we take the first steps to extend the approximation scheme beyond zeroth order. This requires us to find an expansion for concatenation around the shuffle product. Such a formula would be loosely analogous to the associative $*$-product expressions of deformation quantization. We obtain two partial results in this direction. First, we find a one parameter family of associative q-products that interpolates between commutative shuffle ($q = 0$) and non-commutative concatenation ($q = 1$). Moreover, by taking q to be infinitesimal, we obtain a Poisson bracket on the shuffle algebra.

In sections 4.2 and 4.3.2 we briefly investigate another approximation scheme for the loop equations that involves expanding the left annihilation around full annihilation, holding the concatenation product fixed. Though similar in spirit to the main approximation scheme of the paper, it has the potential to give a complementary estimate for correlations as shown by its application to 1-matrix models.

Section 3, is devoted to 1-matrix models. In this case, both concatenation and shuffle are commutative, and an explicit 'star product' formula is obtained for the expansion of the former around the latter (section 3.2). In section 3.3 an expansion for the left annihilation as a series in powers of full annihilation is obtained. These lead to two different approximation methods for the 1-matrix loop equations, involving either a deformation of the product or the annihilation operator. Both schemes are applied to the Gaussian (section 3.4), which is the only 1-matrix model for which $\mathcal{S}^i$ has the derivation property. While deforming the product overestimates correlations, deforming the annihilation operator underestimates them.

Background on Literature: There are several complementary approaches to the loop equations of matrix models. First, they are formulated in different ways: resolvents of matrices, gluon correlations, planar diagrams, Wilson loops etc. Different approaches to multi-matrix models can be broadly categorized by the mathematical structures that play a significant role. A major portion of the literature (eg. [14 − 17]) is devoted to exact solutions for certain observables of specific (e.g. 1-, 2- and chain-type) matrix models, their multi-cut solutions and summing their $1/N$ expansion. This involves connections to integrable systems, algebraic geometry and conformal field theory. Another approach exploits the connections to non-commutative probability theory (eg. [18 − 21]). Yet another point of view seeks to exploit a hidden BRST symmetry [22]. A cohomological interpretation of the loop equations and a variational principle for them was presented in [20]. The viewpoint in this paper is distinguished by its use of algebraic and differential structures and connections to deformation quantization. Its physics roots lie in the early work of Makeenko and Migdal [5, 6], Cvitanovic et. al. [23, 24], loop space formalism for gauge theories [25 − 28], and the more recent investigations of Rajeev and coworkers [29 − 31, 20, 21, 9]. Some structures used in our constructions (eg. shuffle products and their deformations) appear in the mathematics literature on calculus of loop space due to Chen [32], the theory of free Lie algebras [33] and the deformation theory of (Hopf) algebras [34, 35]. A feature of the present work is that we do not make any a priori restriction to a subclass of correlations (eg. 'mixed' or 'unmixed') as is often assumed in the literature.

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2. Algebraic structure of loop equations of multi-matrix models

2.1 Factorized loop equations for gluon correlation tensors

We begin by obtaining the loop equations of a bosonic multi-matrix model in terms of gluon correlation tensors. This is convenient to study their algebraic structures and permits treatment of all factorized $N = \infty$ correlations without restriction. Consider a Euclidean Λ-matrix model with polynomial action $\text{tr}\, S(A) = \text{tr}\, S^J A_J$. Let $\Phi_I = \frac{1}{N}\,\text{tr}\, A_I$ denote the 'loop' variable. The partition function and gluon correlations are

$$Z = \int \Pi_j dA_j e^{-N\,\text{tr}\, S(A)} \quad \text{and} \quad \langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle = \frac{1}{Z} \int \Pi_j dA_j e^{-N\,\text{tr}\, S(A)} \Phi_{K_1} \cdots \Phi_{K_n}. \quad (2.1)$$

$G_K = \lim_{N\to\infty} \langle \Phi_K \rangle$ are the gluon correlations of interest in the large-N limit. Here $A_i = A_i^\dagger$, $1 \le i \le \Lambda$ are $N \times N$ hermitian matrices. The tensors S^I are the 'coupling tensors' defining the theory. Due to the trace, the only part of S^I that contributes is its cyclic projection, so assume that S^I are cyclically symmetric, $S^{Ii} = S^{iI}$ for all i, I. Gluon correlation tensors G_I are also cyclically symmetric. Additionally, assume S^I are chosen such that $(S^I)^* = S^{\bar{I}}$ where $\bar{I}$ is the word with indices reversed[3]. This, along with hermiticity of A_i ensures that $\text{tr}\, S(A)$ is real. In turn, this implies that $G_I^* = G_{\bar{I}}$. To see this, recall that for any complex matrix M, $(\text{tr}\, M)^* = \text{tr}\, M^\dagger$ and apply this to $M = A_I$ and use hermiticity of A_i. For the Gaussian, all $S^I = 0$ except S^{ij} which may be taken as a (positive) real-symmetric matrix.

The Schwinger-Dyson equations(SDE) are constraints on $\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle$ implied by invariance of the matrix integral under an infinitesimal (but non-linear) change of integration variable

$$[A_i]^b_a \mapsto [A'_i]^b_a = [A_i]^b_a + v_i^I [A_I]^b_a, \quad \text{where} \quad v_i^I \quad \text{are infinitesimal real parameters.} \quad (2.2)$$

Under this change of variable, the infinitesimal changes in Φ_K, the action and the measure are

$$\begin{aligned}
\Phi_K &\mapsto \Phi_K + \delta_K^{LiM} v_i^I \Phi_{LIM}, \\
e^{-N\,\text{tr}\, S^J A_J} &\mapsto e^{-N\,\text{tr}\, S^J A_J}(1 - N^2 v_i^I S^{J_1 i J_2} \Phi_{J_1 I J_2}), \\
\det\left(\frac{\partial [A'_i]^a_b}{\partial [A_j]^c_d}\right) &= 1 + N^2 v_i^I \delta_I^{I_1 i I_2} \Phi_{I_1} \Phi_{I_2}.
\end{aligned} \quad (2.3)$$

Invariance of $\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle$ to linear order in v_i^I implies the SDE[4]

$$\begin{aligned}
& v_i^I S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \rangle = v_i^I \delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \rangle + \frac{v_i^I}{N^2} \sum_{p=1}^n \delta_{K_p}^{L_p i M_p} \langle \Phi_{L_p I M_p} \rangle, \\
& \forall \;\; K_p \;\; \text{and} \;\; n = 0, 1, 2, \ldots
\end{aligned} \quad (2.4)$$

[3]This is satisfied by examples such as the Gaussian, Yang-Mills and Chern-Simons theories, see Sec 2.7.

[4]Sometimes called Virasoro constraints in string models or Ward identities. Ward identities seems more appropriate to the special case where the change of integration variable was a gauge or BRST transformation.

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So far we have not made any approximation. In the large N limit, expectation values of U(N) invariants factorize $\langle \Phi_{I_1}\Phi_{I_2}\rangle = \langle \Phi_{I_1}\rangle\langle \Phi_{I_2}\rangle$ [7]. Naively, the leading factorized Schwinger-Dyson or loop equations (LE), which are a closed system for G_I, are

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = v_i^I \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \quad \forall \;\; v \tag{2.5}$$

These infinitesimal changes of variable are associated to vector fields $L_v = v_i^I L_I^i$ whose action on G_J is given by $L_I^i G_J = \delta_J^{J_1 i J_2} G_{J_1 I J_2}$. In particular, choosing the components of the vector fields v_i^I to be non-vanishing only for a single (i, I), we get the loop equations

$$S^{J_1 i J_2} G_{J_1 I J_2} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \quad \forall \;\; I, \;\; i. \tag{2.6}$$

Using cyclicity of S^I and G_I we get

$$|iJ| \;\, S^{Ji} G_{JI} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \;\; \forall \;\; I, \; i. \tag{2.7}$$

LE (2.7) relate changes in (expectation values of) action and measure under the action of L_I^i. However, there may be vector fields L_v (i.e. choices of v_i^I) for which both sides of (2.5) vanish[5]. In that case, the leading equation in the large N limit is different from (2.7) (see section 2.3).

We seek solutions to (2.7) among cyclic symmetric tensors G_I satisfying $G_I^* = G_{\bar{I}}$ and $G_\emptyset \equiv G_0 = 1$, where $\emptyset$ is the empty string. Note that the LE may make sense even when the matrix integrals don't seem to converge, as for a cubic action. When analogues of (2.7) are formulated for Wilson loops in a gauge theory [5], they are called Makeenko-Migdal equations (notice the resemblance between (2.7) and (2.8))

$$\delta_\mu^x \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \oint_C dy_\nu \delta^{(4)}(x-y) W(C_{yx}) W(C_{xy}). \tag{2.8}$$

2.2 Underdetermined nature of loop equations and examples

Given an action $S(A)$, G_I are uniquely defined by (2.1) provided the integrals converge. As examples below show, the large-N LE (2.7) do not determine G_I uniquely in general. In section 2.3 we obtain additional large-N SDE involving G_I that were not accounted for in the passage from (2.4) to (2.7). But even these may not be sufficient to fix the G_I.

Consider first $\Lambda = 1$ matrix models whose LE are got by restricting (2.7) to a single matrix. Suppose $\text{tr}\, S(A) = \text{tr} \sum_{l=1}^m S_l A^l$ is an m^{th} order polynomial, then if $G_k = \langle \frac{\text{tr}}{N} A^k \rangle$

$$\sum_{l=1}^m l S_l G_{k+l} = \sum_{r,s \geq 0, \; r+s=k} G_r G_s, \quad for \;\; k = -1, 0, 1, \ldots. \tag{2.9}$$

The LE listed sequentially are

$$\begin{aligned} k = -1 : &\;\; S_1 + 2S_2 G_1 + \cdots + m S_m G_{m-1} = 0, \\ k = 0 : &\;\; S_1 G_1 + 2 S_2 G_2 + \cdots + m S_m G_m = 1, \end{aligned}$$

[5] Note that this may happen even if there is no (i, I) for which both sides of (2.7) vanish.

$$k=1: \ S_1G_2+2S_2G_3+\cdots+mS_mG_{m+1}=2G_1,$$
$$k=2: \ S_1G_3+2S_2G4+\cdots+mS_mG_{m+2}=2G_2+G_1^2, \qquad \ldots \tag{2.10}$$

We see that in the $k^{\rm th}$ equation, the highest rank correlation G_{m+k} appears linearly ($S_m \neq 0$) and may be determined in terms of lower rank correlations. For a Gaussian ($m=2$) (2.9) determine all moments. More generally, the LE determine higher moments $G_{m-1}, G_m, G_{m+1}, \ldots$ in terms of $m-2$ undetermined lower moments $G_1, \ldots G_{m-2}$. However, among $G_1, \ldots G_{m-2}$, the odd ones must vanish if the action is even. Observe that this is associated with the $[A]^a_b \mapsto -[A]^a_b$ symmetry of an even action and of the measure if $N \to \infty$ through even values. Such transformations provide additional equations missed out by the LE.

For multi-matrix models, suppose $S(A)$ is an $m^{\rm th}$ order polynomial, i.e $S^J=0$ if $|J|>m$ and $\exists\ J$ with $|J|=m$ such that $S^J \neq 0$. Then the loop equation $|iJ|S^{Ji}G_{JI} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}$ for any fixed I and i involves correlations with highest rank $(|I|+m-1)$ only linearly. Of course, there are several correlations with a given rank and several equations for fixed $|I|$. If all G_K up to $|K| \leq r$ are known, we have a system of inhomogeneous linear equations for correlations of rank $r+1$. For the Gaussian $\ {\rm tr}\ S(A) = \frac{1}{2}\ {\rm tr}\ C^{ij}A_iA_j$, these are just recursion relations $G_{iI} = C_{ij}\delta_I^{I_1 j I_2} G_{I_1} G_{I_2}$ where $C_{ij}C^{jk} = \delta_i^k$. Their unique solution for all correlations is given by the planar version of Wick's theorem, which is a sum over all non-crossing partitions of iI into pairs. But for many interesting cubic and higher order actions, the LE are underdetermined even by comparison with 1-matrix models. Not only are G_K for $|K| \leq m-2$ left undetermined, many higher rank correlations are also not determined in terms of them. Consider two examples: a quartic 2-matrix model and the Chern-Simons 3-matrix model.

2.2.1 Quartic 2-Matrix Model

Suppose $\ {\rm tr}\ S(A) = \ {\rm tr}\ [cA_1A_2 + \frac{g}{4}(A_1^4 + A_2^4)]$. The matrix integrals converge and the cyclic coupling tensors are $S^{1111} = S^{2222} = \frac{g}{4}$ and $S^{12} = S^{21} = \frac{c}{2}$. The LE for each I are

$$cG_{2I} + gG_{111I} = \delta_I^{I_1 1 I_2} G_{I_1} G_{I_2} \text{ and } cG_{1I} + gG_{222I} = \delta_I^{I_1 2 I_2} G_{I_1} G_{I_2}. \tag{2.11}$$

Since the action is an $m=4^{\rm th}$ order polynomial, the LE do not fix G_i, G_{ij}. They determine an infinite number of higher rank correlations in terms of these, but also leave an infinite number undetermined. For $I=\emptyset$ the two LE give $G_{111} = -\frac{c}{g}G_2$ and $G_{222} = -\frac{c}{g}G_1$. The other rank-3 correlations G_{112}, G_{122} are left undetermined. For $I=i_1$, the LE determine 4 of 6 correlations leaving G_{1122} and G_{1212} undetermined:

$$G_{1111} = G_{2222} = \frac{1}{g}(1-cG_{12}), \quad G_{1112} = -\frac{c}{g}G_{22}, \quad G_{1222} = -\frac{c}{g}G_{11}. \tag{2.12}$$

For $I=i_1i_2$, the LE are

$$cG_{2i_1i_2} + gG_{111i_1i_2} = \delta^1_{i_2}G_{i_1} + \delta^1_{i_1}G_{i_2} \text{ and } cG_{1i_1i_2} + gG_{222i_1i_2} = \delta^2_{i_2}G_{i_1} + \delta^2_{i_1}G_{i_2}. \tag{2.13}$$

They determine 6 of the 8 rank-5 correlations in terms of lower rank ones

$$G_{11111} = \frac{1}{g}(2G_1 - cG_{112}), \ G_{11112} = \tfrac{1}{g}(G_2 - cG_{122}), \ G_{11122} = \frac{c^2}{g^2}G_1,$$

$$G_{22222} = \frac{1}{g}(2G_2 - cG_{122})\ G_{12222} = \tfrac{1}{g}(G_1 - cG_{112}),\ G_{11222} = \frac{c^2}{g^2}G_2, \tag{2.14}$$

while leaving G_{12121} and G_{21212} undetermined. In this manner, by choosing longer words I, we can fix an infinite number of higher rank correlations in terms of lower rank ones, but at each step a few correlations remain undetermined. The number of undetermined correlators may be significantly reduced by the $A_1 \leftrightarrow A_2$ symmetry of $S(A)$ which implies $G_I = G_J$ if I can be obtained from J by $1 \leftrightarrow 2$ and a cyclic permutation. Notice that this is also a symmetry of the integration measure. The same applies to the change of variables $A_1 \mapsto -A_1, A_2 \mapsto -A_2$.

2.2.2 Chern-Simons Model

The LE of the CS model $\operatorname{tr} S(A) = \frac{2i\kappa}{3}\epsilon^{ijk} \operatorname{tr} A_i A_j A_k$ are

$$2i\kappa\epsilon^{ijk}G_{Ijk} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}. \tag{2.15}$$

They leave rank-1 correlations G_i undetermined ($m = 3$). For $|I| = 0$ and arbitrary i, the LE are $\epsilon^{ijk}G_{jk} = 0$ which do not give any constraints not already implied by cyclic symmetry of G_{jk}. Thus $G_{12}, G_{13}, G_{23}, G_{11}, G_{22}, G_{33}$ are all left undetermined. For $|I| = 1$ with arbitrary $I = i_1$ and i, the LE are $2i\kappa\epsilon^{jki}G_{jki_1} = \delta^i_{i_1}$. From 9 possible (complex) equations we get only 1 independent condition after accounting for cyclicity and hermiticity: the imaginary part of

$$G_{123} - G_{132} = \frac{1}{2i\kappa}. \tag{2.16}$$

This allows us to fix only one parameter in the $c(3, \Lambda = 3) = 11$ dimensional space of 3^{rd} rank cyclic hermitian tensors (see appendix A). For $I = i_1 i_2$ and i arbitrary, the LE are

$$2i\kappa\epsilon^{ijk}G_{i_1 i_2 jk} = \delta^i_{i_2} G_{i_1} + \delta^i_{i_1} G_{i_2}. \tag{2.17}$$

Of the 27 possible equations, there are actually only 9 independent ones that do not follow from cyclicity[6]. Three 'homogeneous' ones $G_{1212} = G_{1122}$, $G_{1313} = G_{1133}$, $G_{2323} = G_{2233}$ and six 'inhomogeneous' ones

$$\begin{aligned}
2i\kappa(G_{1123} - G_{1213}) &= G_1, & 2i\kappa(G_{1213} - G_{1132}) &= G_1 \\
2i\kappa(G_{1223} - G_{1232}) &= G_2, & 2i\kappa(G_{1232} - G_{1322}) &= G_2 \\
2i\kappa(G_{1323} - G_{1332}) &= G_3, & 2i\kappa(G_{1233} - G_{1323}) &= G_3.
\end{aligned} \tag{2.18}$$

Nevertheless, these conditions are not enough to fix the $c(4, \Lambda = 3) = 24$ independent cyclic and hermitian 4^{th}-rank tensors (see appendix A). This underdetermined nature of the LE persists for correlations of higher rank. Notice also that by $A_1 \to A_2 \to A_3 \to A_1$ symmetry of the action and measure, we have $G_1 = G_2 = G_3$ etc, but this is not a consequence of the LE and still leaves the common value of these undetermined.

[6] The fact that many of the loop equations are not independent of each other indicates there are vector fields v_i^I for which both sides of (2.5) vanish identically.

2.3 Additional equations for gluon correlations

Are there more equations satisfied by G_I that will lessen the underdeterminacy of the LE? In going from finite-N SDE (2.4) to large-N LE (2.7), we overlooked the possibility that both l.h.s. and r.h.s. of (2.5) may vanish for some v. In other words, $A_i \to A_i + v_i^I A_I$ may leave the (factorized expectation value of) action and measure simultaneously invariant at leading order as $N \to \infty$. For such v_i^I the $\mathcal{O}(N^0)$ terms in (2.4) identically vanish and the $\mathcal{O}(1/N^2)$ terms constitute the leading large-N SDE. Denote

$$\langle \Phi_I \rangle = G_I + \frac{G_I^{(2)}}{N^2} + \frac{G_I^{(4)}}{N^4} + \cdots; \quad \langle \Phi_{I_1} \Phi_{I_2} \rangle = G_{I_1} G_{I_2} + \frac{G_{I_1;I_2}^{(2)}}{N^2} + \frac{G_{I_1;I_2}^{(4)}}{N^4} + \cdots \tag{2.19}$$

Then the $\mathcal{O}(1/N^2)$ terms in (2.4) become

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2}^{(2)} = v_i^I \delta_I^{I_1 i I_2} G_{I_1;I_2}^{(2)} + v_i^I \sum_{p=1}^{n} \delta_{K_p}^{L_p i M_p} G_{L_p I M_p} \quad \forall \;\; v, \;\; K_p \;\; \text{and} \;\; n = 1, 2, \ldots \tag{2.20}$$

Unfortunately, (2.20) involve not just the G_I but also $1/N^2$ corrections to single and double-trace correlations. Thus, an attempt to ameliorate the underdetermined nature of the LE seems to open a new can of worms. However, in keeping with the spirit of the large-N limit as an approximation where we retain only the leading large-N contribution to all quantities, it seems reasonable to ignore the $G_{\ldots}^{(2)}$ terms and consider

$$\sum_{p=1}^{n} v_i^I \delta_{K_p}^{L_p i M_p} G_{L_p I M_p} = 0 \Leftrightarrow \sum_{p=1}^{n} v_i^I L_I^i G_{K_p} = 0 \Leftrightarrow \sum_{p=1}^{n} L_v G_{K_p} = 0 \tag{2.21}$$

At first, these equations seem universal, they do not involve the coupling tensors S^I at all! However, for generic v, these are $1/N^2$ contributions to the SDE and should be ignored in the large-N limit. But if v_i^I are such that both r.h.s. and l.h.s. of (2.5) vanish identically, then these become the leading large-N SDE. Thus, these equations are *not* universal, since they must be enforced only for those v_i^I for which the leading change in action and measure vanish identically. To summarize, the additional equations are

$$\sum_{p=1}^{n} L_v G_{K_p} = 0 \;\; \forall \;\; K_1, \ldots, K_n \;\; \text{and} \;\; n = 1, 2, 3 \ldots$$
$$\text{and all} \;\; v_i^I \;\; \text{such that} \;\; v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = v_i^I \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} = 0. \tag{2.22}$$

Are there any such additional equations? This is related to whether there are any transformations that leave both action and measure invariant at leading order as $N \to \infty$. We exhibited several such discrete transformations in sections 2.2, 2.2.1 and 2.2.2. BRST transformations of gauge fixed Yang-Mills theory are also of this sort and lead to Ward or Slavnov-Taylor identities. Are the LE (2.7) consistent with the additional equations (2.22)? This would vindicate our throwing away the subleading $G_{\ldots}^{(2)}$ terms in (2.20). If so, do the LE (2.7) together with (2.22) determine the G_I, or do we need yet more conditions? We postpone investigation of these very interesting issues and focus on the LE in the rest of this paper.

2.4 Loop equation in terms of left annihilation and concatenation

Define the generating series of gluon correlations by the formal sum $G(\xi) = G_I \xi^I$. Here, $\xi^i, 1 \le i \le \Lambda$ are non-commuting variables that can be thought of as sources, and $\xi^{i_1\cdots i_n} = \xi^{i_1}\cdots\xi^{i_n}$. If they did commute, the generating series would only contain information about the symmetric correlations. But since $G_{i_1\cdots i_n}$ are not symmetric in general (only cyclically symmetric), there is no relation between $\xi^i\xi^j$ and $\xi^j\xi^i$. Define the concatenation product *conc* by

$$\xi^I \xi^J = \xi^{IJ} \quad \text{or} \quad F(\xi)G(\xi) = F_I G_J \xi^{IJ} \quad \Rightarrow \quad (FG)_K = \delta_K^{IJ} F_I G_J. \tag{2.23}$$

For example[7],

$$(FG)_0 = F_0 G_0; \quad (FG)_i = F_i G_0 + F_0 G_i; \quad (FG)_{ij} = F_0 G_{ij} + F_i G_j + F_{ij} G_0; \quad \text{etc.} \tag{2.24}$$

In terms of *conc*, the r.h.s. of (2.7) becomes $\delta_I^{I_1 i I_2} G_{I_1} G_{I_2} = [G(\xi)\xi_i G(\xi)]_I$. Also define left annihilation[8]

$$D_j \xi^{i_1\cdots i_n} = \delta_j^{i_1} \xi^{i_2\cdots i_n}. \tag{2.25}$$

D_j eliminates the left most source if $i_1 = j$ and returns zero otherwise. In terms of coefficients,

$$[D_j G]_I = G_{jI}, \quad [D_{j_n} \cdots D_{j_1} G]_I = G_{j_1\cdots j_n I}, \tag{2.26}$$

so that $G_{JI} = [D_{\bar{J}} G]_I$. The LE (2.7), one for each i, can be written as

$$\sum_{n\ge 0} (n+1) S^{j_1\cdots j_n i} D_{j_n} \cdots D_{j_1} G(\xi) = G(\xi)\xi^i G(\xi) \quad \text{or} \quad \mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi). \tag{2.27}$$

We used cyclicity of S^I, G_I in deriving this. Thus, the LE involve left annihilation and *conc* product. The l.h.s. of (2.27) defines the action dependent operator

$$\mathcal{S}^i = \sum_{n\ge 0} (n+1) S^{j_1\cdots j_n i} D_{j_n} \cdots D_{j_1}. \tag{2.28}$$

At first glance, the LE (2.27) look like quadratically non-linear PDEs whose order is one less than that of the action polynomial. However, concatenation in the universal term on the r.h.s. is non-commutative since sources ξ^i do not commute. Further, left annihilation does not satisfy the Leibnitz rule with respect to concatenation, i.e. D_j are *not* derivations of *conc*. This 'mismatch' between product and annihilation make the LE difficult to solve. It turns out there is another natural product between gluon correlation tensors, the shuffle product, with respect to which left annihilation satisfies the Leibnitz rule. We try to exploit the interplay between *conc*, shuffle and their derivations to find an approximation method to solve the LE.

[7] Note that concatenation of cyclically symmetric tensors is not cyclically symmetric in general.

[8] Left annihilation does not preserve cyclic symmetry of tensors in general.

2.5 Shuffle multiplication from products of Wilson loop expectation values

Here we obtain the shuffle product of gluon correlations induced by expectation values of products of Wilson loops. The expectation value of the Wilson loop $F(\gamma)$ is a complex-valued gauge-invariant function on the space of loops $\gamma : S^1 \to M$, where M is space-time. If $A_\nu(x)$ denotes the components of a gauge field 1-form valued in the Lie algebra of hermitian matrices, we define the path ordered exponent

$$F(\gamma) = \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \left[i \int_0^1 A_\nu(x) \frac{dx^\nu}{ds} ds \right]. \tag{2.29}$$

Parameterized loops on M are denoted $x^\nu(s)$. Wilson loops are typical functions on loop-space and their expectation values can be expanded in iterated integrals of gluon correlations

$$\begin{aligned} \langle F(\gamma) \rangle &= \sum_{m=0}^{\infty} i^m \int_{0 \le s_1 \le \cdots \le s_m \le 1} \langle \frac{1}{N} \operatorname{tr} A_{\nu_1}(x(s_1)) \cdots A_{\nu_m}(x(s_m)) \rangle \frac{dx^{\nu_1}}{ds_1} \cdots \frac{dx^{\nu_m}}{ds_m} ds_1 \cdots ds_m \\ &= \sum_{m=0}^{\infty} i^m \int_{0 \le s_1 \le \cdots \le s_m \le 1} F_{\nu_1 \cdots \nu_m}(x(s_1), \ldots, x(s_m)) \frac{dx^{\nu_1}}{ds_1} \cdots \frac{dx^{\nu_m}}{ds_m} ds_1 \cdots ds_m \end{aligned} \tag{2.30}$$

where the gluon correlation tensors associated to $F(\gamma)$ are

$$F_{\nu_1 \cdots \nu_m}(x(s_1), \ldots, x(s_m)) = \langle \frac{1}{N} \operatorname{tr} A_{\nu_1}(x(s_1)) \cdots A_{\nu_m}(x(s_m)) \rangle. \tag{2.31}$$

The point-wise commutative product of functions on loop-space is defined as $(FG)(\gamma) = F(\gamma)G(\gamma)$. Taking expectation-values and working in the large-N limit, where correlations factorize, we get

$$\langle (FG)(\gamma) \rangle = \langle F(\gamma) G(\gamma) \rangle = \langle F(\gamma) \rangle \langle G(\gamma) \rangle + \mathcal{O}(\frac{1}{N^2}). \tag{2.32}$$

We may expand the l.h.s. in correlation functions associated to the Wilson loop $(FG)(\gamma)$. We call these $(F \circ G)_{\rho_1 \cdots \rho_p}(x(u_1) \cdots x(u_p))$. They are defined as

$$\langle (FG)(\gamma) \rangle = \sum_{p=0}^{\infty} i^p \int_{0 \le u_1 \le \cdots \le u_p \le 1} (F \circ G)_{\rho_1 \cdots \rho_p}(x(u_1) \cdots x(u_p)) \frac{dx^{\rho_1}}{du_1} \cdots \frac{dx^{\rho_p}}{du_p} du_1 \cdots du_p \tag{2.33}$$

Meanwhile, the expansion of the r.h.s. reads

$$\begin{aligned} \langle F(\gamma) \rangle \langle G(\gamma) \rangle = & \sum_{m,n=0}^{\infty} i^{m+n} \int_{\substack{0 \le s_1 \le \cdots s_m \le 1 \\ 0 \le t_1 \le \cdots t_n \le 1}} F_{\nu_1 \cdots \nu_m}(x(s_1), \ldots, x(s_m)) G_{\mu_1 \cdots \mu_n}(x(t_1), \ldots, x(t_n)) \\ & \times \frac{dx^{\nu_1}}{ds_1} \cdots \frac{dx^{\nu_m}}{ds_m} \frac{dx^{\mu_1}}{dt_1} \cdots \frac{dx^{\mu_n}}{dt_n} ds_1 \cdots ds_m dt_1 \cdots dt_n. \end{aligned} \tag{2.34}$$

To make this look like the expansion of the l.h.s. , we collect terms with a common sum $n + m = p$ and then sum from $p = 0$ to ∞. Moreover, we must relabel the ν's and μ's as ρ's and the s's and t's as u's. We must allow every possible relabeling that preserves the order among the s's and t's. When this is done, we read off the relation between the gluon correlations associated to the Wilson loop $(FG)(\gamma)$ and those associated to $F(\gamma)$ and $G(\gamma)$

$$
\begin{aligned}
&(F \circ G)_{\rho_1 \cdots \rho_p}(x(u_1) \cdots x(u_p)) = \\
&\sum_{m+n=p} \sum_{\substack{\sigma \text{ an } (m,n) \\ \text{shuffle}}} F_{\rho_{\sigma^{-1}(1)} \cdots \rho_{\sigma^{-1}(m)}}(x(u_{\sigma^{-1}(1)}), \ldots, x(u_{\sigma^{-1}(m)})) \\
&\times G_{\rho_{\sigma^{-1}(m+1)} \cdots \rho_{\sigma^{-1}(m+n)}}(x(u_{\sigma^{-1}(m+1)}), \ldots, x(u_{\sigma^{-1}(m+n)})).
\end{aligned} \tag{2.35}
$$

An (m,n) shuffle is a permutation of $m+n$ letters $(1, 2, \ldots, m+n)$ such that

$$
\sigma^{-1}(1) < \cdots < \sigma^{-1}(m) \text{ and } \sigma^{-1}(m+1) < \cdots < \sigma^{-1}(m+n). \tag{2.36}
$$

For brevity, we combine the Lorentz μ and space-time x^μ indices into a single index i, then

$$
(F \circ G)_{i_1 \cdots i_p} = \sum_{m+n=p} \sum_{\sigma \text{ an } (m,n) \text{ shuffle}} F_{i_{\sigma^{-1}(1)}, \cdots i_{\sigma^{-1}(m)}} G_{i_{\sigma^{-1}(m+1)}, \cdots i_{\sigma^{-1}(m+n)}}. \tag{2.37}
$$

The r.h.s. is called the shuffle product (sh). It is commutative. A compact notation for sh is

$$
(F \circ G)_I = \sum_{I = J \sqcup K} F_J G_K. \tag{2.38}
$$

The condition $I = J \sqcup K$ means that J and K are complementary order-preserving subwords of I. The operation $J \sqcup K$ is a riffle-shuffle of two card packs J and K. Some examples are

$$
\begin{aligned}
[F \circ G]_i &= F_i G_0 + F_0 G_i; \qquad [F \circ G]_{ij} = F_{ij} G_0 + F_i G_j + F_j G_i + F_0 G_{ij}; \\
[F \circ G]_{ijk} &= F_{ijk} G_0 + F_{ij} G_k + F_{ik} G_j + F_{jk} G_i \\
&\quad + F_i G_{jk} + F_j G_{ik} + F_k G_{ij} + F_0 G_{ijk}; \\
[F \circ G]_{ijkl} &= F_{ijkl} G_0 + F_{ijk} G_l + F_{ijl} G_k + F_{ikl} G_j + F_{jkl} G_i \\
&\quad + F_{ij} G_{kl} + F_{ik} G_{jl} + F_{il} G_{jk} + F_{jk} G_{il} + F_{jl} G_{ik} + F_{kl} G_{ij} \\
&\quad + F_i G_{jkl} + F_j G_{ikl} + F_k G_{ijl} + F_l G_{ijk} + F_0 G_{ijkl}.
\end{aligned} \tag{2.39}
$$

We notice two properties of sh. If F_I and G_J are cyclically symmetric for all I and J, then so is $(F \circ G)_K$ for all K. To see why this is true in general, observe that $(F \circ G)_K$ is the expectation value of the trace of a product of gluon fields, and the trace makes it cyclically symmetric. Thus sh preserves cyclicity of tensors. Moreover, we notice that if F_I and G_J satisfy the hermiticity properties $F_I^* = F_{\bar{I}}, G_J^* = G_{\bar{J}}$ for all I, J, then so does their shuffle product

$$
(F \circ G)_I^* = (F \circ G)_{\bar{I}} \quad \forall \;\; I. \tag{2.40}
$$

This is a reflection of the relations[9] $F(\gamma)^* = F(\bar{\gamma})$ and $(FG)^*(\gamma) = F^*(\gamma) G^*(\gamma) = (FG)(\bar{\gamma})$ when the path-ordered exponential is expanded out in iterated integrals.

[9] $\bar{\gamma}$ is the loop γ with opposite orientation.

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The shuffle product allows us to reduce manipulations in the commutative algebra of functions on the infinite dimensional space $Loop(M)$ to operations on tensors on the finite dimensional space M. More precisely, start with a manifold M, and denote the space of 1-forms on M by $\Lambda^1(M)$. Then consider the tensor algebra $\mathcal{T}$ on $\Lambda^1(M)$. The shuffle algebra is

$$Sh(M) = \mathcal{T}(\Lambda^1(M)). \tag{2.41}$$

The shuffle algebra is a replacement for the algebra of functions on $Loop(M)$. Let $\xi^{i_1}, \xi^{i_2}, \ldots$ be a basis for $\Lambda^1(M)$ (think of these as $dx^{i_1}, \ldots$), then an element of the shuffle algebra is

$$G = \sum_n G_{i_1 \cdots i_n} \xi^{i_1} \otimes \cdots \otimes \xi^{i_n} \equiv \sum_n G_{i_1 \cdots i_n} \xi^{i_1 \cdots i_n}, \tag{2.42}$$

and is to be regarded as a function on $Loop(M)$. A specific collection of gluon correlations $\{G_{i_1 \cdots i_n}\}_{n=0}^{\infty}$ can encode the information contained in the expectation value of a specific function $G(\gamma)$ on $Loop(M)$[10]. The shuffle product of basis elements is

$$\xi^i \circ \xi^j = \xi^{ij} + \xi^{ji}; \quad \xi^{ij} \circ \xi^k = \xi^{ijk} + \xi^{ikj} + \xi^{kij} \tag{2.43}$$

and in general

$$\xi^{i_1 \cdots i_p} \circ \xi^{i_{p+1} \cdots i_{p+q}} = \sum_{\sigma \text{ a } (p,q) \text{ shuffle}} \xi^{i_{\sigma(1)} \cdots i_{\sigma(p+q)}} \qquad \text{or} \quad \xi^J \circ \xi^K = \delta_I^{J \sqcup K} \xi^I. \tag{2.44}$$

To summarize, we have shown that the commutative point-wise product of Wilson loops induces the commutative, cyclicity and hermiticity preserving shuffle product of gluon correlations[11].

2.6 Derivations of shuffle and concatenation products

Concatenation and shuffle combine to define a pair of dual bialgebras on the vector space $\text{span}(\xi^I)$ (see appendices B and C). Derivations of concatenation and shuffle play a central role in this paper. Recall that the LE (2.27) involved left annihilation D_i defined in (2.25). We show that D_i is a derivation of sh i.e. it satisfies the Leibnitz rule

$$D_i(F \circ G) = (D_i F) \circ G + F \circ (D_i G). \tag{2.45}$$

The proof is by explicit calculation $[D_i(F \circ G)]_I = [F \circ G]_{iI} = \sum_{I_1 \sqcup I_2 = iI} F_{I_1} G_{I_2}$. Now either $i \in I_1$ or $i \in I_2$, so

$$[D_i(F \circ G)]_I = \sum_{I_1 \sqcup I_2 = I} F_{iI_1} G_{I_2} + \sum_{I_1 \sqcup I_2 = I} F_{I_1} G_{iI_2} = \sum_{I_1 \sqcup I_2 = I} [D_i F]_{I_1} G_{I_2} + \sum_{I_1 \sqcup I_2 = I} F_{I_1} [D_i G]_{I_2}$$

[10] The map is not 1-1 since gluon correlations are not gauge invariant in general, unlike Wilson loops. A way to deal with this is to introduce ghosts. When LE are formulated in terms of Wilson loops, gauge fixing and ghost contributions cancel out [7]. But this is not the case if we work with correlation tensors. Extension of this formalism to include ghosts in matrix models will be treated in [13].

[11] This construction generalizes to differential forms on $Loop(M)$, but we do not use it in this paper.

$$= [(D_i F) \circ G]_I + [F \circ (D_i G)]_I. \qquad (2.46)$$

Full annihilation[12] $\mathbf{D}_j$ is a democratic version of left annihilation. It is defined as

$$\mathbf{D}_j \xi^I = \delta^I_{I_1 j I_2} \xi^{I_1 I_2} \quad \text{and} \quad [\mathbf{D}_j F]_I = \delta^{I_1 I_2}_I F_{I_1 j I_2}. \qquad (2.47)$$

$\mathbf{D}_j$ does not preserve cyclic symmetry of tensors. However, $\mathbf{D}_j$ is a derivation of *conc*,

$$\mathbf{D}_j(FG) = (\mathbf{D}_j F)G + F(\mathbf{D}_j G). \qquad (2.48)$$

To see this, begin with the l.h.s. $[\mathbf{D}_j(FG)]_I = \delta^{I_1 I_2}_I (FG)_{I_1 j I_2}$,

$$[\mathbf{D}_j(FG)]_I = \delta^{I_1 I_2}_I \delta^{K_1 K_2}_{I_1 j I_2} F_{K_1} G_{K_2} = \delta^{L_1 L_2 L_3}_I F_{L_1 j L_2} G_{L_3} + \delta^{L_1 L_2 L_3}_I F_{L_1} G_{L_2 j L_3}. \qquad (2.49)$$

On the other hand,

$$[(\mathbf{D}_j F)G]_I = \delta^{I_1 I_2}_I (\mathbf{D}_j F)_{I_1} G_{I_2} = \delta^{I_1 I_2}_I \delta^{J_1 J_2}_{I_1} F_{J_1 j J_2} G_{I_2} = \delta^{L_1 L_2 L_3}_I F_{L_1 j L_2} G_{L_3}. \qquad (2.50)$$

Thus

$$[(\mathbf{D}_j F)G]_I + [F(\mathbf{D}_j G)]_I = \delta^{L_1 L_2 L_3}_I F_{L_1 j L_2} G_{L_3} + \delta^{L_1 L_2 L_3}_I F_{L_1} G_{L_2 j L_3} = [\mathbf{D}_j(FG)]_I. \qquad (2.51)$$

The commutator of derivations is a derivation irrespective of whether the product is commutative or not. This is analogous to the Lie bracket of vector fields being a vector field on a manifold. For example, merely using the fact that D_i is a derivation of $sh = \circ$, it is easy to show that

$$[D_i, D_j](F \circ G) = ([D_i, D_j]F) \circ G + F \circ ([D_i, D_j]G). \qquad (2.52)$$

It follows that iterated commutators of derivations (e.g. $[D_i, [D_j, D_k]]$) are also derivations. On the other hand, products of left annihilation operators are not derivations of the shuffle algebra. For e.g. $D_i D_j = D_{ij}$ is not a derivation of sh. This is analogous to the product of vector fields not being a vector field. Furthermore, left annihilation operators with a single index D_i do not form a Lie algebra by themselves. The commutator $[D_i, D_j] = D_{ij} - D_{ji}$ is not a linear combination of D_k's. However, by construction, the vector space spanned by the set of all iterated commutators of left annihilation operators $D_i, [D_i, D_j], [D_i, [D_j, D_k]], \ldots$ forms a Lie algebra, the Lie algebra of derivations of the shuffle product. This is the free Lie algebra. It is analogous to the Lie algebra of left invariant vector fields on a Lie group. Here, the role of the Lie group is played by the free group on Λ generators.

2.7 Derivation property of terms in Yang-Mills action

The action-dependent linear term $\mathcal{S}^i G(\xi)$ in the LE (2.27) is a sum of products of left annihilation operators $\mathcal{S}^i = \sum_{n \geq 0} (n+1) S^{j_1 \cdots j_n i} D_{j_n} \cdots D_{j_1}$. Suppose coupling tensors S^I are such that $\mathcal{S}^i$ is a linear sum of iterated commutators of left annihilation operators,

$$\mathcal{S}^i = C^{ij} D_j + C^{ijk}[D_j, D_k] + C^{ijkl}[[D_j, D_k], D_l] + \cdots \qquad (2.53)$$

[12] *Not* the cyclic gradient. The cyclic gradient is $\delta_i \xi^I = \delta^I_{I_1 i I_2} \xi^{I_2 I_1}$ and is *not* a derivation of concatenation.

Then $\mathcal{S}^i$ is a derivation of shuffle. Of what practical use is this property? The LE (2.27) are quadratically non-linear in *conc*, but involve left annihilation, which is a derivation of *sh*. In section 4 we introduce an approximation scheme where *conc* is expanded around *sh*. The main simplification for matrix models having the derivation property is that their LE can be turned into (an infinite system of) *linear* PDEs at 0^{th} order in this approximation. This is not the case for matrix models without the derivation property.

Among 1-matrix models, the only one with this property is the Gaussian $\operatorname{tr} S(A) = \frac{1}{2\alpha} \operatorname{tr} A^2$ for which $\mathcal{S} = \frac{1}{\alpha} D$. For $\Lambda = 1$, there is only one left annihilation operator, and all its iterated commutators vanish. Multi-matrix models provide non-trivial examples. It is remarkable that the gluonic terms in the Yang-Mills action (1.1) quadratic in momentum, linear in momentum and independent of momentum each separately has this derivation property[13]. These terms can be written as $\operatorname{tr} C^{ij} A_i A_j$, $\operatorname{tr} C^{ijk} A_i [A_j, A_k]$ and $\operatorname{tr} [A_i, A_j][A_k, A_l] g^{ik} g^{jl}$ for appropriate tensors C^{ij}, C^{ijk}, g^{ij}. Moreover, the zero momentum limits of the Gaussian, Chern-Simons and Yang-Mills matrix field theories all have this derivation property. They correspond to the simplest non-vanishing choices for the tensors $C^{ij}, C^{ijk}, C^{ijkl}$ in (2.53). In fact, this property also extends to the corresponding matrix field theories but we do not address that here.

Gaussian: The Gaussian multi-matrix model $\operatorname{tr} S(A) = \frac{1}{2} \operatorname{tr} C^{ij} A_i A_j$ has real-symmetric covariance $C^{ij} = C^{ji}$. $S^{ij} = \frac{1}{2} C^{ij}$ is cyclically symmetric and also satisfies $(S^{ij})^* = S^{ji}$ so that all correlations satisfy $G_I^* = G_{\bar{I}}$. We get $\mathcal{S}^i = 2 S^{ij} D_j = C^{ij} D_j$, which is a linear combination of left annihilation operators and therefore a derivation of *sh*. The LE are

$$C^{ij} D_j G(\xi) = G(\xi) \xi^i G(\xi). \tag{2.54}$$

Chern-Simons: For at least three matrices ($\Lambda \geq 3$), the CS type of matrix model has action $\frac{2i\kappa}{3} \operatorname{tr} C^{ijk} A_i [A_j, A_k]$ where C^{ijk} is any tensor which is anti-symmetric under interchange of any pair of indices. The part of C^{ijk} that is symmetric under interchange of a pair of indices does not contribute on account of antisymmetry of the commutator. The action can also be written as $\operatorname{tr} S(A) = \frac{2i\kappa}{3} \operatorname{tr} \tilde{C}^{ijk} A_i A_j A_k$ where $\tilde{C}^{ijk} = C^{ijk} - C^{ikj}$. The particular case of zero momentum 3d CS gauge theory results from the choice $\Lambda = 3$, $\tilde{C}^{ijk} = \epsilon^{ijk}$ (the Levi-Civita symbol), and integer-valued coupling constant $4\pi\kappa$. More importantly, terms in the Yang-Mills action (1.1) linear in momentum are of this form. Irrespective of its field theoretic origin, $S^{ijk} = (2i\kappa/3) \tilde{C}^{ijk}$ is cyclically symmetric since $\tilde{C}^{kij} = (-1)^2 \tilde{C}^{ijk}$. Moreover, $(S^{ijk})^* = S^{kji}$ so that $G_I^* = G_{\bar{I}}$. Now $\mathcal{S}^i$ is a linear combination of commutators of left annihilation operators:

$$\mathcal{S}^i = 2i\kappa \tilde{C}^{ijk} D_k D_j = i\kappa \{\tilde{C}^{ijk} D_k D_j - \tilde{C}^{ikj} D_k D_j\} = i\kappa \tilde{C}^{ijk} [D_k, D_j] \tag{2.55}$$

and therefore a derivation of *sh*. The 'Chern-Simons' loop equations are

$$i\kappa \tilde{C}^{ijk} [D_k, D_j] G(\xi) = G(\xi) \xi^i G(\xi). \tag{2.56}$$

[13] See [13] for the corresponding property after inclusion of ghosts.

Yang-Mills: For $\Lambda \geq 2$, the zero momentum limit of Yang-Mills theory has action ($\alpha = g^2$)

$$\text{tr } S(A) = -\frac{1}{4\alpha} \text{ tr } [A_i, A_j][A_k, A_l] g^{ik} g^{jl}, \tag{2.57}$$

where $g^{ij} = g^{ji}$ is the inverse metric, it is a real symmetric matrix. The action is rewritten as

$$\text{tr } S(A) = \frac{-1}{2\alpha} \text{ tr } (g^{ik}g^{jl} - g^{il}g^{jk}) A_{ijkl} = \frac{-1}{4\alpha} \text{ tr } [(2g^{ik}g^{jl} - g^{il}g^{jk} - g^{ij}g^{kl}) A_{ijkl}] \tag{2.58}$$

so that $S^{ijkl} = -\frac{1}{4\alpha}(2g^{ik}g^{jl} - g^{il}g^{jk} - g^{ij}g^{kl})$ is cyclically symmetric. Moreover, $S^{ijkl} = (S^{lkji})^* = S^{lkji}$ follows since g^{ij} is real symmetric. Then the differential operator $\mathcal{S}^i = (3+1) S^{ijkl} D_l D_k D_j$

$$\mathcal{S}^i = -\frac{1}{\alpha} g^{ik} g^{jl} (D_l D_k D_j - D_k D_l D_j + D_l D_k D_j - D_l D_j D_k) = -\frac{1}{\alpha} g^{ik} g^{jl} [D_j, [D_k, D_l]] \tag{2.59}$$

is a linear combination of iterated commutators of derivations and hence a derivation of the shuffle product. The Yang-Mills LE are thus

$$-\frac{1}{\alpha} g^{ik} g^{jl} [D_j, [D_k, D_l]] G(\xi) = G(\xi) \xi^i G(\xi). \tag{2.60}$$

On the other hand, *most matrix models do not have this derivation property.* For example, consider the popular [15] two matrix model $\text{tr } S(A_1, A_2) = \text{ tr } [A_1^4 + A_2^4 + 2A_1 A_2]$. Here, $\mathcal{S}^1 = 2D_2 + 4D_1^3$ and $\mathcal{S}^2 = 2D_1 + 4D_2^3$ are not linear combinations of iterated commutators of D_i and do not define derivations of the shuffle algebra.

3. Approximation method for one-matrix models

The LE of a 1-matrix model (2.9) with m^{th} order polynomial action

$$\sum_{l=1}^{m} l \ S_l D^{l-1} G(\xi) = G(\xi) \xi G(\xi). \tag{3.1}$$

can be written in terms of left annihilation[14] D. Concatenation, which appears on the r.h.s. is the usual product of calculus. But D satisfies the Leibnitz rule with respect to *sh*, not *conc*. So this is not a differential equation. We develop approximation methods to solve these LE either by expanding *conc* around *sh* or by expanding D around full annihilation $\mathbf{D}$ (2.47), which is a derivation of *conc*. Both these turn the LE into linear ODEs at each order of the expansion.

3.1 Shuffle, concatenation and their derivations

We give the 1-matrix versions of *conc*, *sh* and their derivations by specialization from sections 2.4 and 2.5. Then we define q-deformed products and derivations that we use to

[14] The 1-matrix left annihilation operator, $D\xi^n = \xi^{n-1}$ is *not* the same as the usual derivative of calculus.

solve the LE approximately. Suppose $F(\xi) = \sum_{n\geq 0} F_n \xi^n$ etc. *Conc* $= *_1$ is the usual product of calculus[15],

$$\xi^p *_1 \xi^q = \xi^{p+q} \quad \text{or} \quad (F *_1 G)_n = \sum_{r=0}^{n} F_r G_{n-r} \tag{3.2}$$

while shuffle $= *_0$ (previously denoted $\circ$) is,

$$\xi^p *_0 \xi^q = \binom{p+q}{p} \xi^{p+q}, \quad \text{or} \quad (F *_0 G)_n = \sum_{r=0}^{n} \binom{n}{r} F_r G_{n-r} \tag{3.3}$$

For example $\xi *_0 \xi = 2\xi^2$. Both are commutative. The notation anticipates $*_q$ that interpolates between *sh* ($q = 0$) and *conc* ($q = 1$). We also define 1-matrix analogs of left and full annihilation and name them in anticipation of q-annihilation D_q. Left annihilation $D_0 \xi^n = \xi^{n-1}$ is the 1-matrix version of D_i defined in (2.25). D_0 is a derivation of shuffle

$$(D_0 F)_n = F_{n+1}, \quad (D_0(F *_0 G))_n = ((D_0 F) *_0 G)_n + (F *_0 (D_0 G))_n. \tag{3.4}$$

Full annihilation $D_1 \xi^n = n \xi^{n-1}$ is the same as the usual derivative of calculus. It is the 1-matrix version of $\mathbf{D}_i$ defined in (2.47). D_1 is a derivation of *conc*,

$$[D_1 F]_n = (n+1) F_{n+1}, \quad (D_1(F *_1 G))_n = ((D_1 F) *_1 G)_n + (F *_1 (D_1 G))_n. \tag{3.5}$$

This follows from the easily verified formula

$$(n+1) \sum_{r=0}^{n+1} F_r G_{n+1-r} = \sum_{r=0}^{n} (r+1) F_{r+1} G_{n-r} + \sum_{r=0}^{n} (n-r+1) F_r G_{n-r+1}. \tag{3.6}$$

3.2 q-Deformed product

The q-product interpolates between *conc* ($q = 1$) and *sh* ($q = 0$)[16]

$$(F *_q G)_n = \sum_{r=0}^{n} \binom{n}{r}_{1-q} F_r G_{n-r}. \tag{3.7}$$

It is associative and commutative for $0 \leq q \leq 1$. The q-binomial coefficients or Gauss binomials $\binom{n}{r}_q$ are polynomials in q with non-negative coefficients. They reduce to unity for $q = 0$ and to the usual binomial coefficients when $q = 1$. To obtain their properties let $yx = qxy$. Then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}_q x^{n-r} y^r. \tag{3.8}$$

The first three Gauss binomials are

$$\binom{n}{0}_q = 1, \qquad \binom{n}{1}_q = 1 + q + q^2 + \cdots + q^{n-1},$$

[15] We also denote *conc* $= *_1$ by juxtaposition.

[16] The quantity $1 - q$ often occurs in formulae, so we call it $p = 1 - q$.

$$\binom{n}{2}_q = \begin{cases} (1+q^2+q^4+\cdots+q^{n-2})(1+q+q^2+\cdots q^{n-2}), & \text{if } n \text{ is even;} \\ (1+q^2+q^4+\cdots+q^{n-3})(1+q+q^2+\cdots+q^{n-1}), & \text{if } n \text{ is odd.} \end{cases} \tag{3.9}$$

The q-Pascal relation is got by multiplying $(x+y)^{n-1}$ by $(x+y)$ either from the right or left:

$$\begin{aligned} \binom{n}{r}_q &= q^r \binom{n-1}{r}_q + \binom{n-1}{r-1}_q \\ \binom{n}{r}_q &= \binom{n-1}{r}_q + q^{n-r}\binom{n-1}{r-1}_q . \end{aligned} \tag{3.10}$$

Substituting the first in the second gives

$$\binom{n}{r}_q = \frac{1-q^n}{1-q^{n-r}}\binom{n-1}{r}_q \quad \text{for} \quad 0 \le r < n. \tag{3.11}$$

Iterating, we get

$$\binom{n}{r}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}. \tag{3.12}$$

This can also be written as

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} \quad \text{where} \quad [n]_q! = [1]_q[2]_q\cdots[n]_q \quad \text{and} \quad [n]_q = \frac{1-q^n}{1-q}. \tag{3.13}$$

The symmetry $\binom{n}{r}_q = \binom{n}{n-r}_q$ is now manifest, which guarantees commutativity of the q-product (3.7). Some examples of the q-product are

$$\begin{aligned} (F *_q G)_0 &= F_0G_0; \quad (F *_q G)_1 = F_1G_0 + F_0G_1; \\ (F *_q G)_2 &= F_0G_2 + (1+p)F_1G_1 + F_2G_0; \\ (F *_q G)_3 &= F_0G_3 + (1+p+p^2)(F_1G_2+F_2G_1) + F_3G_0; \\ (F *_q G)_4 &= F_0G_4 + (1+p+p^2+p^3)(F_1G_3+F_3G_1) \\ &\quad +(1+p+2p^2+p^3+p^4)F_2G_2 + F_4G_0. \end{aligned} \tag{3.14}$$

We expand the q-binomials around the ordinary binomial coefficients ($q = 1$) in a Taylor series

$$\binom{n}{r}_q = \binom{n}{r}_1 \left\{1 - \frac{r(n-r)}{2}p + \mathcal{O}(p^2)\right\}. \tag{3.15}$$

Thus $*_q$ may be expanded around shuffle $*_0$

$$\begin{aligned} (F *_q G)_n &= (F *_0 G)_n - \frac{q}{2}\sum_{r=0}^{n}\binom{n}{r}_1 rF_r(n-r)G_{n-r} + \cdots \\ &= (F *_0 G)_n - \frac{q}{2}\sum_{r=0}^{n}\binom{n}{r}_1 [\xi *_0 D_0F(\xi)]_r[\xi *_0 D_0G(\xi)]_{n-r} + \cdots \\ (F *_q G)(\xi) &= (F *_0 G)(\xi) - \frac{q}{2}\,\xi *_0 (D_0F)(\xi) *_0 \xi *_0 (D_0G)(\xi) + \cdots . \end{aligned} \tag{3.16}$$

Taking $q = 1$, and using commutativity of $*_0$, we get an expansion for *conc* in terms of *sh*

$$(F *_1 G)(\xi) = (F *_0 G)(\xi) - \frac{1}{2}\,\xi *_0 \xi *_0 (D_0F)(\xi) *_0 (D_0G)(\xi) + \cdots . \tag{3.17}$$

3.3 q-Deformed annihilation operator

Recall from section 3.1 that left annihilation $[D_0 F]_n = F_{n+1}$ and full annihilation $[D_1 F]_n = F_{n+1}$. More generally, let

$$(D_q F)_n = [n+1]_q F_{n+1} = \left[\frac{q^{n+1}-1}{q-1}\right] F_{n+1} = \left[1+q+q^2+\cdots+q^n\right] F_{n+1}. \quad (3.18)$$

D_q reduces to left and full annihilation for $q = 0$ and $q = 1$. However, D_q is not a derivation of $*_q$ for $0 < q < 1$. Fortunately, we don't seem to need that. More importantly, we expand D_q around D_1 in powers of $p = 1-q$. Denoting *conc* reciprocal by usual division of calculus,

$$D_q F(\xi) = \frac{F(q\xi) - F(\xi)}{(q-1)\xi} = \sum_{k=1}^{\infty} (-p\xi)^{k-1} \frac{1}{k!} D_1^k F(\xi) \quad (3.19)$$

3.4 Gaussian one matrix model

Now we apply this formalism to the simplest of matrix models, the Gaussian 1-matrix model. We pick it as it is the only 1-matrix model with the derivation property. We show how expanding *conc* around *sh* and expanding D_0 around D_1, are used along with the derivation property to turn the non-linear LE into linear ODEs at each order in our approximation schemes. The resulting gluon correlations are compared with the exact solution.

From (2.9), the LE for the Gaussian 1 matrix model with action $S = \frac{1}{2\alpha}\,\mathrm{tr}\,A^2$ are

$$D_0 Z(\xi) = \alpha\xi Z(\xi) *_1 \xi *_1 Z(\xi) \quad \text{or} \quad G_{n+1} = \alpha \sum_{r+1+s=n,\ r,s\geq 0} G_r G_s, \quad n = 0,1,2,\ldots \quad (3.20)$$

with the boundary condition $G_0 = 1$. When the product is not specified, it is taken to be the concatenation product $*_1$. In this section, we call the generating function of moments $Z(\xi) = \sum_n G_n \xi^n$. This is because we will expand $Z(\xi)$ in powers of q, and the coefficients $Z_k(\xi)$ are not to be confused with the moments G_n, which are coefficients in an expansion in powers of ξ. Of course, q is a bookkeeping device which is eventually set to 1.

3.4.1 Exact solution

The loop equation for the Gaussian (3.20) may be solved since it is a quadratic equation

$$\frac{Z(\xi) - Z(0)}{\xi} = \alpha\, Z^2(\xi)\xi \quad \Rightarrow \quad \alpha\xi^2 Z^2 - Z + 1 = 0. \quad (3.21)$$

The solution is

$$Z(\xi) \;=\; \frac{1-\sqrt{1-4\alpha\xi^2}}{2\alpha\xi^2} \;=\; \sum \Gamma_{2n}\xi^{2n} \quad (3.22)$$

where Γ_n are the moments. Define Catalan numbers C_n by

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \quad \text{with} \quad C_n = \frac{(2n)!}{n!(n+1)!} \sim \frac{4^n}{\sqrt{n^3\pi}} \text{ as } n \to \infty. \quad (3.23)$$

Then the non-vanishing moments of the Gaussian 1-matrix model are

$$\Gamma_{2n} = C_n \alpha^n \sim \frac{(4\alpha)^n}{\sqrt{n^3\pi}} \quad \text{as} \quad n \to \infty. \quad (3.24)$$

3.4.2 Approximate solution by deforming the product

In (3.20), D_0 is a derivation of $sh = *_0$, not of $conc = *_1$. So it is not a differential equation. But we can expand $*_1$ in a series in powers of $q(=1)$ around $*_0$. Expanding $Z(\xi)$ also in a power series in q, turns the loop equation into a sequence of differential equations in the shuffle algebra. At order q^0, we get a nonlinear ODE for $Z_0(\xi)$. Beyond that, we get a linear inhomogeneous ODE for $Z_k(\xi)$ in terms of $Z_{k-1}(\xi)$. In the end, q is set to 1. Let us illustrate this at $\mathcal{O}(q^0)$ and $\mathcal{O}(q^1)$. From section 3.2, the expansion of $*_q$ around $*_0 = sh$ is

$$(F *_q G)(\xi) = (F *_0 G)(\xi) - \frac{q}{2}\, \xi *_0 \xi *_0 (D_0 F)(\xi) *_0 (D_0 G)(\xi) + \cdots . \qquad (3.25)$$

Moreover $D_0\xi = 1$, so keeping only terms to $O(q)$,

$$\begin{aligned}(Z *_q \xi) *_q Z &= (Z *_0 \xi - \frac{q}{2}\xi *_0 D_0 Z *_0 \xi *_0 D_0\xi) *_q Z \qquad (3.26)\\ &= Z *_0 \xi *_0 Z - \frac{q}{2}\xi *_0 \xi *_0 D_0(Z *_0 \xi) *_0 D_0 Z - \frac{q}{2}\xi *_0 \xi *_0 D_0 Z *_0 Z\\ &= \xi *_0 Z *_0 Z - \frac{q}{2}\Big[2\xi *_0 \xi *_0 Z *_0 D_0 Z + \xi *_0 \xi *_0 \xi *_0 D_0 Z *_0 D_0 Z\Big].\end{aligned}$$

So the LE are

$$D_0 Z = \alpha\left[\xi *_0 Z *_0 Z - \frac{q}{2}\Big\{2\xi *_0 \xi *_0 Z *_0 D_0 Z + \xi *_0 \xi *_0 \xi *_0 D_0 Z *_0 D_0 Z\Big\} + \mathcal{O}(q^2)\right]. \quad (3.27)$$

Suppose $Z(\xi) = Z_0(\xi) + qZ_1(\xi) + q^2 Z_2(\xi) + \cdots$. Comparing coefficients of q^0 and q^1 we get

$$\begin{aligned}\frac{1}{\alpha} D_0 Z_0 &= \xi *_0 Z_0 *_0 Z_0 \qquad (3.28)\\ \frac{1}{\alpha} D_0 Z_1 &= 2\xi *_0 Z_0 *_0 Z_1 - \frac{1}{2}\Big(2\xi *_0 \xi *_0 Z_0 *_0 D_0 Z_0 + \xi *_0 \xi *_0 \xi *_0 D_0 Z_0 *_0 D_0 Z_0\Big).\end{aligned}$$

So we have a non-linear ODE for $Z_0(\xi)$, and linear in-homogeneous ODEs for $Z_k, k \geq 1$. The boundary condition $Z(0) = 1$ becomes $Z_0(0) = 1,\ Z_k(0) = 0,\ k \geq 1$.

Zeroth order $\mathcal{O}(q^0)$. Replace concatenation by shuffle product: The ODE for Z_0 can be linearized by passing to the shuffle reciprocal of $Z_0(\xi)$

$$Y(\xi) *_0 Z_0(\xi) = 1 \Rightarrow D_0 Z_0 = -Z_0 *_0 Z_0 *_0 D_0 Y. \qquad (3.29)$$

Y satisfies the inhomogeneous linear ODE $D_0 Y = -\alpha\xi$ with boundary condition $Y(0) = 1$. So

$$Y(\xi) = 1 - \frac{\alpha}{2}\, \xi *_1 \xi = 1 - \alpha\xi^2. \qquad (3.30)$$

Taking the shuffle reciprocal, we get (using $\xi^{*_0 n} = n!\ \xi^{*_1 n} = n!\ \xi^n$)

$$\begin{aligned}Z_0(\xi) &= (1 - \frac{\alpha}{2}\xi *_0 \xi)^{-1} = 1 + \frac{\alpha}{2}\xi *_0 \xi + (\frac{\alpha}{2})^2 \xi *_0 \xi *_0 \xi *_0 \xi + \cdots\\ &= \sum_{n=0}^{\infty} \frac{\alpha^n}{2^n}(2n)!\ \xi^{2n}. \qquad (3.31)\end{aligned}$$

So the generating function at order q^0 is

$$Z(\xi) = \sum_{n=0}^{\infty} \left(\frac{\alpha}{2}\right)^n (2n)!\ \xi^{2n} + \mathcal{O}(q). \tag{3.32}$$

And the non-vanishing moments in this approximation are $G_{2n} = \left(\frac{\alpha}{2}\right)^n (2n)! + \mathcal{O}(q)$. These are compared with the exact moments in the table below.

Moments	exact	$\mathcal{O}(q^0)$
G_2	α	α
G_4	$2\alpha^2$	$6\alpha^2$
G_6	$5\alpha^3$	$90\alpha^3$
G_8	$14\alpha^4$	$2520\alpha^4$
$G_{2n}, n \to \infty$	$\frac{(4\alpha)^n}{\sqrt{\pi n^3}}$	$(\frac{\alpha}{2})^n (2n)!$

(3.33)

Due to the $(2n)!$, the $\mathcal{O}(q^0)$ moments numerically exceed the exact moments. We have a crude zeroth order answer with the potential for calculating corrections. Of course, the gaussian is a trivial model to solve. The value of our method lies in its applicability to multi-matrix models for which no method of solution exists.

3.4.3 Approximate solution by deforming the left annihilation operator

Next, we expand D_0 around D_1 so that the loop equation (3.20) becomes a sequence of differential equations with respect to *conc*. This leads to a different approximation compared to section 3.4.2, where we used the deformed product. Here, the expansion parameter is $p = 1 - q$, which is eventually set to 1. Recall that the q-deformed annihilation operator is

$$D_q F(x) = \sum_{k=1}^{\infty} \frac{(-p\xi)^{k-1}}{k!} D_1^k F(\xi) = D_1 F(\xi) - \frac{p}{2}\xi D_1^2 F(\xi) + \frac{p^2}{6}\xi^2 D_1^3 F(\xi) + \mathcal{O}(p^3). \tag{3.34}$$

If we expand $Z(\xi)$ in powers of p, $Z(\xi) = \sum_n Z_n(\xi) p^n$, then

$$D_q Z(\xi) = \sum_{s=0}^{\infty} p^s \sum_{n=0}^{s} \frac{(-1)^n}{(n+1)!} D_1^{n+1} Z_{s-n}(\xi)\xi^n \tag{3.35}$$

and

$$Z(\xi) *_1 \xi *_1 Z(\xi) = \sum_{s=0}^{\infty} p^s \sum_{n=0}^{s} Z_n(\xi) *_1 \xi *_1 Z_{s-n}(\xi). \tag{3.36}$$

Comparing coefficients of p, we get a nonlinear ODE for $Z_0(\xi)$ and a sequence of 1^{st}-order linear ODEs for $Z_s(\xi)$ in terms of the lower order ones $Z_{s-1}(\xi), \ldots$:

$$\sum_{n=0}^{s} \frac{(-1)^n}{(n+1)!} D_1^{n+1} Z_{s-n}(\xi)\xi^n = \alpha \sum_{n=0}^{s} Z_n(\xi) *_1 \xi *_1 Z_{s-n}(\xi). \tag{3.37}$$

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The first couple of orders are (all products are concatenation products)

$$\begin{aligned} D_1 Z_0(\xi) &= \alpha Z_0(\xi)\xi Z_0(\xi) \\ D_1 Z_1(\xi) - \frac{1}{2} D_1^2 Z_0(\xi)\xi &= 2\alpha Z_0(\xi)\xi Z_1(\xi) \\ &\vdots \ . \end{aligned} \tag{3.38}$$

Zeroth order: At $\mathcal{O}(p^0)$ we have to solve the ODE $D_1 Z_0(\xi) = \alpha\ Z_0(\xi) *_1 \xi *_1 Z_0(\xi)$ with $Z_0(0) = 1$. The solution is the *conc* reciprocal

$$\begin{aligned} Z_0(\xi) &= \frac{1}{1 - \frac{1}{2}\alpha\xi *_1 \xi} = 1 + \frac{\alpha\,\xi^2}{2} + \frac{\alpha^2\,\xi^4}{4} + \frac{\alpha^3\,\xi^6}{8} + \frac{\alpha^4\,\xi^8}{16} + \frac{\alpha^5\,\xi^{10}}{32} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha}{2}\right)^n \xi^{2n}. \end{aligned} \tag{3.39}$$

The non-vanishing moments are thus

$$G_{2n} = \left(\frac{\alpha}{2}\right)^n + \mathcal{O}(p). \tag{3.40}$$

These are compared with exact moments $\Gamma_{2n} = C_n \alpha^n \sim \frac{(4\alpha)^n}{\sqrt{n^3\pi}}$, in table 3.47. We see that at leading order, deforming the annihilation operator underestimates the moments.

Next to lowest order $\mathcal{O}(p^1)$: At the next order in $p = 1 - q$ we have an inhomogeneous linear first order ODE for $Z_1(\xi)$

$$D_1 Z_1(\xi) - \frac{1}{2} D_1^2 Z_0(\xi)\xi = 2\alpha Z_0(\xi)\xi Z_1(\xi) \tag{3.41}$$

with boundary condition $Z_1(0) = 0$. Now $Y' + PY + Q = 0$ has solution

$$Y(\xi) = -I^{-1}(\xi) \int_0^\xi Q(\eta) I(\eta) d\eta \text{ where } I(\xi) = \exp \int_0^\xi P(\eta). \tag{3.42}$$

$Y = Z_1(\xi)$; $P = -2\alpha\xi Z_0(\xi)$; $Q = -\frac{1}{2}\xi Z_0''(\xi)$; $Z_0(\eta) = \frac{1}{1-\frac{1}{2}\alpha\eta^2}$; $I(\xi) = (1 - \frac{1}{2}\alpha\xi^2)^2$. Thus,

$$\begin{aligned} Z_1(\xi) &= -\frac{3\alpha\xi^2 + 8\log(1 - \frac{1}{2}\alpha\xi^2)}{4(1 - \frac{1}{2}\alpha\xi^2)^2} = \frac{\alpha\,\xi^2}{4} + \frac{\alpha^2\,\xi^4}{2} + \frac{25\,\alpha^3\,\xi^6}{48} + \frac{41\,\alpha^4\,\xi^8}{96} + \frac{99\,\alpha^5\,\xi^{10}}{320} + \cdots \\ &= \frac{1}{4}\alpha\xi^2 + \sum_{n\geq 2} \left(\frac{\alpha\xi^2}{2}\right)^n \left[\frac{n}{2} + 2\sum_{r=0}^{n-2} (\frac{r+1}{n-r})\right]. \end{aligned} \tag{3.43}$$

To get the asymptotic behavior of moments for large n, let $Z_1(\xi) = \sum \tilde{G}_{2n}\xi^{2n}$

$$\begin{aligned} \tilde{G}_2 = \frac{\alpha}{4}, \quad \tilde{G}_{2n} &= \left(\frac{\alpha}{2}\right)^n \left[\frac{n}{2} + 2\sum_{r=0}^{n-2} (\frac{r+1}{n-r})\right], \ n \geq 2 \\ \Rightarrow \quad \tilde{G}_{2n} &\sim \left(\frac{\alpha}{2}\right)^n \left[2n\log n - (\frac{7}{2} - 2\gamma)n + 2\log n + \mathcal{O}(n^0)\right], n \to \infty. \end{aligned} \tag{3.44}$$

Recall that $Z(\xi) = Z_0(\xi) + pZ_1(\xi) + \cdots$ and $Z_0(\xi) = \sum_n (\frac{\alpha}{2})^n \xi^{2n}$. Combining, at $\mathcal{O}(p)$ we have (after setting $p = 1$)

$$G_2 = \frac{3\alpha}{4} + \mathcal{O}(p^2); \quad G_{2n} = \left(\frac{\alpha}{2}\right)^n \left[1 + \frac{n}{2} + 2\sum_{r=0}^{n-2} (\frac{r+1}{n-r})\right] + \mathcal{O}(p^2), \; n \geq 2$$
$$G_{2n} \sim \left(\frac{\alpha}{2}\right)^n \left[2n \log n - (\frac{7}{2} - 2\gamma)n + 2\log n + \mathcal{O}(n^0)\right] + \mathcal{O}(p^2), n \to \infty. \tag{3.45}$$

This is to be compared with the exact moments

$$\Gamma_{2n} = \frac{(2n)!}{n!(n+1)!}\alpha^n \sim \frac{(4\alpha)^n}{\sqrt{\pi n^3}}, \quad n \to \infty. \tag{3.46}$$

Going to the next to leading order in p has improved the agreement with the exact correlations. For large n, the next to leading corrections to G_{2n} are bigger in magnitude than the 0^{th} order G_{2n}. The accompanying table summarizes the approximate correlations obtained by expanding the left annihilation around the full annihilation operator in powers of $p = 1 - q$.

Moments	exact	$\mathcal{O}(p^0)$	$\mathcal{O}(p)$
G_2	α	0.5α	0.75α
G_4	$2\alpha^2$	$0.25\alpha^2$	$0.75\alpha^2$
G_6	$5\alpha^3$	$0.125\alpha^3$	$0.646\alpha^3$
G_8	$14\alpha^4$	$0.0625\alpha^4$	$0.490\alpha^4$
$G_{2n}, n \to \infty$	$\frac{(4\alpha)^n}{\sqrt{\pi n^3}}$	$(\frac{\alpha}{2})^n$	$(\frac{\alpha}{2})^n (2n \log n)$

(3.47)

3.5 Non-Gaussian 1-matrix models

Recall that the 1-matrix loop equation (2.9) for a polynomial action $\operatorname{tr} S(A) = \operatorname{tr} \times \sum_{l=1}^m S_l A^l$ with $S_m \neq 0$ determines higher rank correlations $G_{m-1}, G_m, G_{m+1}, \ldots$ in terms of the lower rank ones $G_0 = 1, G_1, G_2, \ldots G_{m-2}$. Suppose we apply our approximation method here. At 0^{th} order we replace *conc* by *sh*. Since left annihilation $D\xi^n = \xi^{n-1}$ is a derivation of *sh*, the loop equation becomes a quadratically non-linear ODE in the commutative shuffle algebra

$$\sum_{l=1}^m l S_l D^{l-1} G(\xi) = G(\xi) \circ \xi \circ G(\xi). \tag{3.48}$$

However, for $m > 2$ (i.e. non-Gaussian models), the differential operator $\sum_{l=1}^m l S_l D^{l-1}$ is *not* a derivation of *sh* and our trick of passing to the shuffle reciprocal does not linearize this ODE. It can still be thought of as a set of recursion relations (use $\xi^s *_0 \xi^t = \binom{s+t}{s} \xi^{s+t}$)

$$\sum_{l=1}^m l \; S_l \; G_{r+l-1} = \sum_{\substack{s+t+1=r \\ s,t \geq 0}} \frac{r!}{s! \; t!} \; G_s \; G_t, \qquad \text{for} \quad r = 0, 1, 2, \ldots \tag{3.49}$$

which determine $G_{m-1}, G_m, G_{m+1}, \ldots$ in terms of $G_1, G_2, \ldots G_{m-2}$:

$$r = 0: \quad S_1 G_0 + 2S_2 G_1 + \cdots + m S_m G_{m-1} = 0$$

$$r=1: \quad S_1G_1+S_2G_2+\cdots+S_mG_m=1, \quad \text{e.t.c.} \tag{3.50}$$

Our approach *does not* lead to a significant simplification for non-Gaussian 1-matrix models. However, we observe that the passage to the limit $q=0$ (replacement of *conc* by *sh*) did not change the dimension of the space of solutions to the original loop equations.

4. Approximation method for multi-matrix models

Recall the multi-matrix LE (2.27) for the generating series of gluon correlations $\mathcal{S}^iG(\xi) = G(\xi)\xi^iG(\xi)$ where $\mathcal{S}^i = \sum_{n\geq 0}(n+1)S^{j_1\cdots j_n i}D_{j_n}\cdots D_{j_1}$. Products on the r.h.s. are *conc* products, but D_j are not derivations of *conc*. So the LE are not differential equations. By analogy with 1-matrix models, two ways around this mismatch come to mind. We could p-expand D_i around full annihilation $\mathbf{D}_i$, which *is* a derivation of *conc*. Or, we could q-expand *conc* around *sh*, with respect to which D_i is a derivation. Both these turn LE into quadratically non-linear PDEs at 0^{th} order in p or q. In the former approach these are PDEs on the non-commutative concatenation algebra, while in the latter case, they are PDEs on the commutative shuffle algebra. We focus on the second approach in section 4.1 due to its similarity with deformation quantization, and briefly consider the first approach in section 4.2. Beginnings of a formalism to go beyond zeroth order are in section 4.3.

4.1 Multi-matrix LE at $\mathcal{O}(q^0)$ and the shuffle reciprocal

At 0^{th} order in q, we replace *conc* by *sh*. Then the factorized LE (2.27) become[17]

$$\mathcal{S}^iG(\xi) = G(\xi)\circ\xi^i\circ G(\xi) \tag{4.1}$$

with the boundary condition $G(0)=1$. (4.1) is a quadratically non-linear PDE on the shuffle algebra. In general, the order of the PDE is one less than the degree of $S(A)$. If $\mathcal{S}^i$ is a derivation of *sh*, we can change variables so that (4.1) becomes a linear PDE for the shuffle reciprocal of $G(\xi)$ denoted $F(\xi)$, $F(\xi)\circ G(\xi)=1$. The shuffle reciprocal exists as a formal series since the constant term $G_0=1$ does not vanish. Moreover, since G_I are cyclic and shuffle product preserves cyclicity, F_I are also cyclic. Assuming S^i is a derivation of *sh*,

$$\mathcal{S}^i(F(\xi)\circ G(\xi)) = 0 \quad\Rightarrow\quad F\circ\mathcal{S}^iG = -\mathcal{S}^iF\circ G \quad\Rightarrow\quad \mathcal{S}^iG = -G\circ\mathcal{S}^iF\circ G. \tag{4.2}$$

Putting this in (4.1) we get $G\circ\mathcal{S}^iF\circ G = G\circ\xi^i\circ G$. Shuffle multiplying by $F\circ F$ reduces the LE to a system of inhomogeneous linear PDEs in the shuffle algebra

$$\mathcal{S}^iF(\xi) = -\xi^i. \tag{4.3}$$

We call these *shuffle reciprocal LE*. We seek cyclically symmetric solutions to them. The l.h.s. of (4.3) is the same as in the LE (2.27) with G replaced by its reciprocal F. The $-$ sign due to inversion has been written on the r.h.s. . The r.h.s. , however is much simpler than

[17]Here, we use $\circ$ for $*_0 = sh$ to avoid subscripts.

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in (2.27) since the quadratic factor in $G(\xi)$ has been eliminated. For the zero-momentum Gaussian, Chern-Simons and Yang-Mills matrix models we get (from section 2.7)

$$\begin{aligned} Gaussian \quad & C^{ij}\, D_j F(\xi) \;=\; -\,\xi^i \\ Chern-Simons \quad & i\kappa\, \epsilon^{ijk}[D_k, D_j]F(\xi) \;=\; -\,\xi^i \\ Yang-Mills \quad & -\frac{1}{\alpha}\, g^{ik} g^{jl}[D_j,[D_k,D_l]]F(\xi) \;=\; -\,\xi^i. \end{aligned} \tag{4.4}$$

Thus, we have used the derivation properties of these theories to effectively linearize the LE at order q^0. We still have to solve these linear PDEs on the ∞-dimensional vector space spanned by ξ^I. First we find a formula to recover $G(\xi)$ from its shuffle reciprocal $F(\xi)$.

$$(F \circ G)(\xi) = 1 \Rightarrow \sum_{I=J\sqcup K} F_J G_K = \delta^I_\emptyset. \tag{4.5}$$

We can solve these equations starting from $G_0 = F_0 = 1$. The first few equations are

$$\begin{gathered} F_i + G_i = 0, \qquad F_{ij} + F_i G_j + F_j G_i + G_{ij} = 0, \\ F_{ijk} + F_{ij}G_k + F_{ik}G_j \;+\; F_{jk}G_i + F_i G_{jk} + F_j G_{ik} + F_k G_{ij} + G_{ijk} = 0, \quad \ldots \end{gathered} \tag{4.6}$$

Since each successive equation involves the next higher rank G_I only linearly, we need only solve a linear equation at each step. Thus for $|I| > 0$,

$$G_I = -\sum_{I=J\sqcup K, K\neq I} F_J G_K \tag{4.7}$$

expresses higher rank G_I in terms of lower rank ones and the reciprocal F. Iterating,

$$G_I = \sum_{n=1}^{|I|} (-1)^n \sum_{\substack{I=I_1\sqcup I_2\sqcup\cdots\sqcup I_n \\ I_k \,\neq\, \emptyset \;\forall\; k}} F_{I_1}F_{I_2}\cdots F_{I_n} \quad \text{for} \;\; I \;\neq\; \emptyset. \tag{4.8}$$

$I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n \Leftrightarrow I_1, \ldots, I_n$ are complementary order-preserving subwords of I. For example, $G_i = -F_i$, $G_{ij} = -F_{ij} + 2F_iF_j$ and

$$\begin{aligned} G_{ijk} &= -F_{ijk} + 2(F_iF_{jk} + F_jF_{ik} + F_kF_{ij}) - 6F_iF_jF_k \qquad (4.9) \\ G_{ijkl} &= -F_{ijkl} + 2(F_iF_{jkl} + F_jF_{ikl} + F_kF_{ijl} + F_lF_{ijk} + F_{ij}F_{kl} + F_{ik}F_{jl} + F_{il}F_{jk}) \\ &\quad -6(F_iF_{jk}F_l + F_jF_{ik}F_lF_iF_{jl}F_k + F_jF_{il}F_k + F_kF_{ij}F_l + F_iF_{kl}F_j) + 24F_iF_jF_kF_l. \end{aligned}$$

This formula shows that the mapping to shuffle reciprocal (for series with non-vanishing constant term) is one-to-one. We don't lose any information in going from $G(\xi)$ to $F(\xi)$ and back. Once we solve (4.3), for $F(\xi)$ we may straightforwardly recover G_I using (4.8).

4.1.1 Solution of Gaussian multi-matrix model at zeroth order in q

Consider the Gaussian multi-matrix model $\;\operatorname{tr}\, S(A) = \frac{1}{2}\, \operatorname{tr}\, C^{ij}A_iA_j$ with symmetric covariance $C^{ij} = C^{ji}$. At 0^{th} order in q, the shuffle reciprocal LE (4.4) are

$$D_k F(\xi) = -C_{kj}\xi^j, \tag{4.10}$$

where $C_{kj} = C_{jk}$ is the matrix inverse of C^{ij}. We seek a solution of (4.10) of the general form

$$F(\xi) = 1 + F_{i_1}\xi^{i_1} + F_{i_1 i_2}\xi^{i_1 i_2} + \cdots + F_{i_1\cdots i_n}\xi^{i_1\cdots i_n} + \cdots, \tag{4.11}$$

where F_I are cyclically symmetric. $G_0 = 1$ fixes $F_0 = 1$. Substituting in (4.10) using $D_i\xi^{i_i\cdots i_n} = \delta_i^{i_1}\xi^{i_2\cdots i_n}$ we get

$$F_i + F_{ii_2}\xi^{i_2} + F_{ii_2 i_3}\xi^{i_2 i_3} + \cdots + F_{ii_2\cdots i_n}\xi^{i_2\cdots i_n} + \cdots = C_{ij}\xi^j. \tag{4.12}$$

Comparing coefficients of words ξ^I we read off the solution

$$F_i = 0, \quad F_{ij} = -C_{ij}, \quad F_{i_1\cdots i_n} = 0 \ \text{ for } \ n \geq 3. \tag{4.13}$$

The solution is a quadratic polynomial $F(\xi) = 1 - C_{ij}\xi^{ij}$. Using (4.8) we get

$$G_0 = 1, \ \ G_i = 0, \ \ G_{ij} = -F_{ij} = C_{ij}, \ \ G_{ijk} = 0, G_{ijkl} = 2\{C_{ij}C_{kl} + C_{ik}C_{jl} + C_{il}C_{jk}\}. \tag{4.14}$$

Thus, for the Gaussian multi-matrix model, the linear equations (4.10) for shuffle reciprocal, along with the boundary condition $F_0 = 1$ have a unique solution. Comparing with exact moments from the planar Wick theorem, $\Gamma_0 = 1$, $\Gamma_{ij} = C_{ij}$, $\Gamma_{ijkl} = C_{ij}C_{kl} + C_{il}C_{jk}, \ldots$, we see that the approximation is an over estimate (as we found in the 1-matrix example in section 3.4.2).

4.1.2 Chern-Simons matrix model at zeroth order in q

Consider the zero-momentum limit of 3d Chern-Simons(CS) gauge theory. This corresponds to the 3-matrix model with action tr $S(A) = 2i\kappa$ tr $A_1[A_2, A_3]$. Such an action also results from considering terms in (1.1) that are linear in momentum. The 0^{th} order CS loop equation (4.4) for the shuffle reciprocal $F(\xi)$ is

$$i\kappa\ \epsilon^{ijk}[D_k, D_j]F(\xi) = -\xi^i \quad \text{ or } \quad 2i\kappa\ \epsilon^{ijk}D_k D_j F(\xi) = -\xi^i. \tag{4.15}$$

We seek a solution to (4.15) among formal series $F(\xi) = F_I\xi^I$ with cyclic coefficients F_I satisfying $F_I^* = F_{\bar{I}}$. eq. (4.15) is an inhomogeneous 2^{nd} order linear PDE in an infinite dimensional space spanned by the words ξ^I. $F_0 = 1$ does not suffice to fix a solution. For example, F_i are undetermined, since ξ^i is annihilated by the l.h.s. . Inserting $F_I\xi^I$ into (4.15) gives

$$2i\kappa\ \epsilon^{ii_1 i_2}F_{i_1\cdots i_n}\xi^{i_3\cdots i_n} = -\xi^i. \tag{4.16}$$

The PDEs become linear equations for the coefficients F_I with $|I| \geq 2$,

$$\begin{aligned} &n = 2: \quad \epsilon^{ijk}F_{jk} = 0; \quad n = 3: \quad 2i\kappa\ \epsilon^{ijk}F_{jkl} = -\delta^i_l; \quad \text{and} \\ &n > 3: \quad \epsilon^{ii_1 i_2}F_{i_1\cdots i_n} = 0. \end{aligned} \tag{4.17}$$

Being a system of inhomogeneous linear equations, the general solution is the sum of a particular solution and the general solution of the corresponding homogeneous system. A particular solution with minimal number of non-vanishing F_I is

$$F_0 = 1, \quad F_{123} = F_{231} = F_{312} = F^*_{321} = F^*_{213} = F^*_{132} = \frac{i}{4\kappa} \quad \text{and}$$

$$F_I = 0 \ \ \forall \ \ \text{other} \ \ I. \tag{4.18}$$

To see this we need only consider $n = 3$, where despite appearances, after accounting for cyclic symmetry, there are only a pair of independent equations, the real and imaginary parts of

$$F_{321} - F_{123} = \frac{1}{2i\kappa}. \tag{4.19}$$

By hermiticity, $F^*_{321} = F_{123}$ or $\Re F_{321} = \Re F_{123}$ and $\Im F_{321} = -\Im F_{123}$. Since κ is real, the real part of the above equation is an identity, so the real part $\Re F_{123} = \Re F_{321}$ is left undetermined, and we can set it to zero. Its imaginary part gives $\Im F_{123} = \frac{1}{4\kappa}$, which is the advertised particular solution. For this particular solution the gluon green functions at order q^0 can be non-trivial only if their rank is divisible by 3. For example, $G_0 = 1$,

$$\begin{aligned} G_i &= -F_i = 0, \ G_{ij} = 0, \quad G_{123} = G_{231} = G_{312} = G^*_{321} = G^*_{132} = G^*_{213} = \frac{1}{4i\kappa}, \\ G_{ijk} &= 0 \text{ otherwise }, \ G_{113322} = -8G_{132}F_{132} = -\frac{1}{2\kappa^2}, \quad G_{112233} = 0, \quad \text{etc.} \end{aligned} \tag{4.20}$$

Let us now consider the general solution to the inhomogeneous linear equations (4.17). It is straightforward to see that they have infinitely many solutions, since the corresponding homogeneous equations $\epsilon^{ii_1i_2}F_{i_1i_2\cdots i_n} = 0, \quad n \geq 2$ do. Indeed, any tensor $F_{i_1i_2i_3\cdots i_n}$ that is symmetric under interchange of a pair of adjacent indices is a solution to the homogeneous equations. By cyclic symmetry, the two indices can be chosen as i_1 and i_2. Then such an $F_{i_1i_2i_3\cdots i_n}$ is annihilated due to antisymmetry of $\epsilon^{ii_1i_2}$. Even after imposing hermiticity and cyclic symmetry, this will leave an infinite number of homogeneous solutions, for example any totally symmetric real tensor F_I is automatically cyclically symmetric and satisfies $F^*_I = F_{\bar{I}}$. To get an idea of how many solutions there are among tensors of a fixed rank, consider each rank individually since the equations do not mix tensors of different rank. For $n = 1$, we do not have any LE, but hermiticity implies that F_1, F_2, F_3 are three arbitrary real quantities. For $n = 2$, $\epsilon^{ijk}F_{jk} = 0$ does not impose any condition on F_{11}, F_{22}, F_{33}, which are real by hermiticity, and says that F_{12}, F_{23} and F_{31} are symmetric tensors, which must again be real. For $n = 3$, as we saw earlier, $2i\kappa \ \epsilon^{ijk}F_{jkl} = -\delta^i_l$ is just the single condition $\Im F_{123} = \frac{1}{4\kappa}$. After accounting for cyclicity and hermiticity, there are 11 independent components of F_{ijk}. The 10 undetermined components can be taken as the real numbers

$$F_{111}, F_{222}, F_{333}, \Re F_{123}, F_{122}, F_{233}, F_{311}, F_{133}, F_{211}, F_{322}. \tag{4.21}$$

For $n = 4$, accounting for cyclic F_I, there are only 9 conditions

$$\begin{aligned} F_{1123} = F_{1132} = F_{1213}, \quad F_{2231} = F_{2213} = F_{2321}, \quad F_{3312} = F_{3321} = F_{3132}, \\ F_{1212} = F_{1122}, \quad F_{2323} = F_{2233}, \quad F_{3131} = F_{3311}. \end{aligned} \tag{4.22}$$

But there are $c(n = 4, \Lambda = 3) = 24$ independent cyclic symmetric fourth rank tensors (see appendix A). Thus we have a large space of homogeneous solutions among fourth rank tensors. A similar situation continues for $n > 4$. The LE at order q^0 (4.15), though linear and easy to solve, have infinitely many solutions. As explained in section 2.2, this is true of the original LE and is not an artifact of our approximation scheme. It remains to see if the additional equations obtained in 2.3 fix this shortcoming.

4.1.3 Yang-Mills multi-matrix model at zeroth order in q

Consider the Yang-Mills matrix model with action $\operatorname{tr} S(A) = -\frac{1}{4\alpha} \operatorname{tr} [A_i, A_j][A_k, A_l] g^{ik} g^{jl}$. The LE for the shuffle reciprocal $F(\xi)$ of the moment generating series $G(\xi)$ at zeroth order in q are

$$g^{ik} g^{jl} [D_j, [D_k, D_l]] F(\xi) = \alpha\ \xi^i \quad \text{for} \quad 1 \le i \le \Lambda. \tag{4.23}$$

Interesting special cases are $\Lambda = 4, 2$ which correspond to the zero momentum limit of 4 and 2 dimensional large-N Yang-Mills theory. For $\Lambda = 2$ and a flat Euclidean metric $g^{ij} = \delta^{ij}$, the matrix model action is $\operatorname{tr} S(A) = -\frac{1}{2\alpha} \operatorname{tr} [A_1, A_2]^2$. (4.23) are a system of Λ inhomogeneous 3$^{\text{rd}}$ order linear PDEs for $F(\xi) = F^I \xi_I$ which is normalized to $F_0 = 1$. F_I must be cyclically symmetric. We need additional conditions to fix a solution since any quadratic polynomial is annihilated by the l.h.s., so F_i and F_{ij} are not fixed by (4.23). Let us assume the metric $g^{ij} = \delta^{ij}$ and not make a distinction between lower and upper indices, with repeated indices being summed. Then (4.23) becomes (using the short-hand $D_{ijk} = D_i D_j D_k$)

$$(2D_{jij} - D_{jji} - D_{ijj}) F(\xi) = \alpha\ \xi_i \Rightarrow (2F_{jiji_4 \cdots i_n} - F_{ijji_4 \cdots i_n} - F_{jjii_4 \cdots i_n}) \xi^{i_4 \cdots i_n} = \alpha\ \xi_i. \tag{4.24}$$

Comparing coefficients we get these conditions

$$\begin{aligned}
&n = 3 \Rightarrow 2F_{jij} - F_{ijj} - F_{jji} = 0 \ \ \forall \ \ i \\
&n = 4 \Rightarrow 2F_{jijk} - F_{ijjk} - F_{jjik} = \alpha\ \delta_{ik} \ \ \forall \ \ i, k \\
&n \ge 5 \Rightarrow 2F_{jiji_4 \cdots i_n} - F_{ijji_4 \cdots i_n} - F_{jjii_4 \cdots i_n} = 0 \ \ \forall \ \ i, i_4 \cdots i_n.
\end{aligned} \tag{4.25}$$

The condition for $n = 3$ is an identity for cyclically symmetric tensors, so we drop it. These are infinitely many linear equations for the tensors F_I. A major simplification is that the equations do not mix tensors of different ranks, i.e. the matrix defining the system is block diagonal with all blocks finite dimensional. Let us specialize to the simplest non-trivial case of the $\Lambda = 2$ matrix model. We will show that a particular (cyclically symmetric) solution is

$$F_0 = 1, \quad F_{1122} = F_{2112} = F_{2211} = F_{1221} = -\frac{\alpha}{2} \ \text{ and the remaining } \ F_I = 0. \tag{4.26}$$

The only non-trivial part of this particular solution involves the rank $n = 4$ tensors. The equations for the rest are homogeneous and they can be set to zero. For $n = 4$ we need to find a solution to $2F_{jijk} - F_{ijjk} - F_{jjik} = \alpha\ \delta_{ik}$. These look like four equations,

$$\begin{aligned}
&2F_{2121} - F_{1221} - F_{2211} = \alpha, \ \ 2F_{2122} - F_{1222} - F_{2212} = 0, \\
&2F_{1212} - F_{2112} - F_{1122} = \alpha, \ \ 2F_{1211} - F_{2111} - F_{1121} = 0.
\end{aligned} \tag{4.27}$$

But there is only one independent non-trivial condition after accounting for cyclic symmetry

$$F_{1122} - F_{1212} = -\frac{\alpha}{2}. \tag{4.28}$$

Thus we see that $F_0 = 1, F_{1122}$ and cyclic permutations $= -\alpha/2$ and all other $F_I = 0$ is a particular solution. The gluon green functions at order q^0 are obtained via the shuffle reciprocal (4.8) which imply that non-vanishing correlations have rank divisible by 4, for example,

$$G_0 = 1, \quad G_i = G_{ij} = G_{ijk} = 0, \quad G_{1122} \text{ and cyclic } = \frac{\alpha}{2}, \text{ and other } G_{ijkl} = 0, \quad \text{etc.} \tag{4.29}$$

Now comes the harder question of the general solution of the homogeneous linear system [18]

$$\begin{aligned} n = 4 &\Rightarrow 2F_{jijk} - F_{ijjk} - F_{jjik} = 0 \;\; \forall \;\; i, k \\ n \geq 5 &\Rightarrow 2F_{jiji_4\cdots i_n} - F_{ijji_4\cdots i_n} - F_{jjii_4\cdots i_n} = 0 \;\; \forall \;\; i, i_4 \cdots i_n \end{aligned} \tag{4.30}$$

For $n = 4$, as we saw before, there is only one non-trivial equation $F_{1122} = F_{1212}$. But there are $c(n = 4, \Lambda = 2) = 6$ independent cyclically symmetric rank 4 tensors (see appendix A) which can be taken as $F_{2222}, \;\; F_{1222}, \;\; F_{1122}, \;\; F_{1212}, \;\; F_{1112}, \;\; F_{1111}$. Hermiticity $F_I = F^*_{\bar{I}}$ implies that all of them are real since reversal of order of indices can be achieved by cyclic permutations in each case. Thus the general solution for rank four tensors assigns 5 arbitrary real parameters to $F_{2222}, \;\; F_{1222}, \;\; F_{1122} = F_{1212}, \;\; F_{1112}$ and F_{1111}.

For $n = 5$, once the dust settles, there are only two non-trivial equations

$$F_{11122} = F_{11212} \quad \text{and} \quad F_{11222} = F_{12122} \tag{4.31}$$

Taking account of cyclic symmetry, there are $c(5, 2) = 8$ independent rank 5 tensors, which can be taken as $F_{22222}, F_{12222}, F_{11222}, F_{12122}, F_{11122}, F_{11212}, F_{11112}$ and F_{11111}. In general these are complex, but hermiticity and cyclicity imply they are all real. Thus we have two linear constraints on 8 real parameters and therefore a six real-dimensional space of solutions to the shuffle reciprocal LE for rank 5 tensors:

$$F_{22222}, \quad F_{12222}, \quad F_{11222} = F_{12122}, \quad F_{11122} = F_{11212}, \quad F_{11112}, \quad F_{11111} \tag{4.32}$$

are freely specifiable real quantities.

This abundance of solutions continues to hold for $n \geq 6$. It is easy to see that the homogeneous linear equations $2F_{jijI} - F_{ijjI} - F_{jjiI} = 0$ have an infinite number of solutions. Observe that any tensor that is totally symmetric in any three adjacent indices[19] satisfies this equation. In particular, totally symmetric tensors are an infinite class of solutions. The underdetermined nature of the linear equations for $F(\xi)$ is not an artifact of our approximation scheme. It is already true of the full LE as shown in section 2.2. It remains to implement the additional conditions (2.22) to see if they select a solution.

[18] Recall that $n = 3$ was identically satisfied by cyclically symmetric tensors.

[19] By cyclic symmetry, those three indices can be taken as the first three.

4.2 LE with deformed left annihilator and the concatenation reciprocal

We also have the option of approximating the LE (2.27) by replacing left annihilation D_i by full annihilation $\mathbf{D}_i$ at zeroth order in an expansion in powers of $p = 1 - q$. Since $\mathbf{D}_i$ is a derivation of *conc*, this again turns the LE into non-linear PDEs, but this time on the non-commutative free algebra. As before, it is possible to convert the non-linear PDEs into linear PDEs by passage to the concatenation reciprocal. Recall that $\mathcal{S}^i = \sum_n (n+1) S^{j_1 \cdots j_n i} D_{j_n} \cdots D_{j_1}$. When D_j is replaced by $\mathbf{D}_j$, we denote the resulting differential operator

$$\mathbf{S}^i = \sum_n (n+1) S^{j_1 \cdots j_n i} \mathbf{D}_{j_n} \cdots \mathbf{D}_{j_1}. \tag{4.33}$$

Moreover, assume couplings S^I are such that $\mathbf{S}^i$ is a linear combination of iterated commutators of $\mathbf{D}_j$ and therefore a derivation of *conc*. This is the case for the Gaussian, CS and YM matrix models or any linear combination thereof. At zeroth order in p, the LE become

$$\mathbf{S}^i G(\xi) = G(\xi) \xi^i G(\xi). \tag{4.34}$$

Now we'd like to use the same trick as before and turn this into a linear equation for the *conc* reciprocal of $G(\xi)$. Though *conc* is non-commutative, left and right concatenation reciprocals of $G(\xi)$ are both equal. Let $GR = 1$ and $LG = 1$. Multiplying the first equation by L from the left and using the second, we get $R = L$. So let $F(\xi)$ be the unique two-sided *conc* reciprocal[20] of $G(\xi)$. Assuming $\mathbf{S}^i$ is a derivation of *conc*, $\mathbf{S}^i(FG) = (\mathbf{S}^i F)G + F(\mathbf{S}^i G) = 0$. This turns (4.34) into a linear equation for the *conc* reciprocal

$$\mathbf{S}^i F(\xi) = -\xi^i. \tag{4.35}$$

Inserting $F(\xi) = F_I \xi^I$ into (4.35) gives linear equations for coefficients F_I. Once F_I are determined, we recover G_I at zeroth order in p using the following formula for *conc* reciprocal.

$$(FG)_I = \delta_{I,\emptyset} \;\Rightarrow\; G_0 = 1 \;\text{ and }\; \delta_I^{I_1 I_2} F_{I_1} G_{I_2} = 0 \;\Rightarrow\; G_I = -\sum_{\substack{I = I_1 I_2, \\ I_1 \neq \emptyset}} F_{I_1} G_{I_2} \;\text{ for }\; |I| > 0. \tag{4.36}$$

Iterating this, we solve for G_I

$$G_I = \sum_{n=1}^{|I|} (-1)^n \sum_{\substack{I = I_1 I_2 \cdots I_n \\ I_k \neq \emptyset \;\; \forall k}} F_{I_1} F_{I_2} \cdots F_{I_n} \quad \text{for} \quad I \neq \emptyset. \tag{4.37}$$

For example, the first few gluon correlations are

$$G_0 = 1; \;\; G_i = -F_i; \;\; G_{ij} = -F_{ij} + F_i F_j; \;\; G_{ijk} = -F_{ijk} + F_{ij} F_k + F_i F_{jk} - F_i F_j F_k; \tag{4.38}$$

[20] $F_0 = G_0 = 1$

Thus *conc* reciprocal is a $1-1$ map. However, unlike shuffle reciprocal, it does not preserve cyclicity. Though we do not lose any information in the passage from G to F, the cyclic property of G_I gets slightly garbled when expressed in terms of F_I. For example, cyclic symmetry of G_{ijk} implies the relation $F_{ij}F_k - F_{ijk} = F_j F_{ki} - F_{jki}$. Thus, we should look for solutions to (4.35) among F_I that lead to cyclically symmetric G_J's. This makes identifying the appropriate solutions of (4.35) potentially harder than for the corresponding shuffle reciprocal LE (4.3). There is another reason why the concatenation reciprocal LE (4.35) are a potentially harder infinite linear system to solve than their shuffle reciprocal counter part (4.1). Left annihilation acting on a monomial produces a monomial $[D_j F]_I = F_{jI}$. But due to its democratic nature, full annihilation produces a linear combination of monomials $[\mathbf{D}_j F]_I = \delta_I^{I_1 I_2} F_{I_1 j I_2}$. Thus the matrix defining the system of linear equations for F_I would be less sparse than before. Nevertheless, the moral is that replacing D_j by $\mathbf{D}_j$ at 0^{th} order in an expansion in p allows for an effective linearization of the LE provided the action has the derivation property.

4.3 Formalism for multi-matrix models beyond zeroth order

At $\mathcal{O}(q^0)$ our approximation amounted to replacement of non-commutative *conc* by commutative *sh* in the LE. This is like approximating the associative product of operators in quantum mechanics by a commutative product of functions on phase space. To go beyond this, we need a formula expressing *conc* as a series around *sh*, by analogy with the Moyal $*$-product formula

$$(\tilde{F} *_\hbar \tilde{G})(x,p) = \sum_{n=0}^{\infty} \left(\frac{-i\hbar}{2}\right)^n \frac{1}{n} \{\tilde{F}, \tilde{G}\}_{(n)} \; = \; \tilde{F}\tilde{G} - \frac{i\hbar}{2}\{\tilde{F},\tilde{G}\} + \cdots, \quad \text{where}$$
$$\{\tilde{F},\tilde{G}\}_{(n)} = \sum_{r=0}^{n} (-1)^r \tilde{F}^{j_1\cdots j_r}_{i_1\cdots i_{n-r}} \tilde{G}^{i_1\cdots i_{n-r}}_{j_1\cdots j_r} \quad \text{with} \quad \tilde{F}^i = \frac{\partial \tilde{F}}{\partial p_i}, \quad \tilde{F}_i = \frac{\partial \tilde{A}}{\partial x^i}, \quad \text{etc} \tag{4.39}$$

for the symbols of operators (here Weyl ordered) in quantum mechanics. The first non-trivial term in such a formula involves the classical Poisson bracket. So one strategy is to look for a natural Poisson bracket on the shuffle algebra. However, there are differences from the usual situation where Heisenberg equations are approximated by Hamilton's equations. While the Heisenberg equations of quantum mechanics involve commutators of the associative product, the LE directly involve the associative concatenation product and not its commutator. Another difference from the usual situation in deformation quantization is that we know the product at both $q = 0$ and $q = 1$ whereas one usually knows the product only at $\hbar = 0$. Once we have such a formula, then as we did for 1-matrix models (section 3.4.2), we would expand the generating series of gluon correlations $G(\xi) = \sum_{k=0}^{\infty} G^{(k)}(\xi) q^k$ in a power series in q and find equations for the $G^{(k)}(\xi)$ order by order in q, starting from the 0^{th} order equations for $G^{(0)}(\xi)$ of section 4.1. However, the situation for multi-matrix models is substantially more complicated than for the 1-matrix models of section 3. This is because *conc* is non-commutative while it was commutative in the single-matrix case.

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4.3.1 q-Deformed product and Poisson bracket on shuffle algebra

We exhibit a 1-parameter family of associative products $*_q$ that interpolate between commutative shuffle $*_0$ and concatenation $*_1$. It reduces to the q-product for a single generator introduced in (3.7) and is defined as $(F *_q G)(\xi) = [F *_q G]_I \xi^I$ where[21]

$$[F *_q G]_I \equiv \sum_{J \sqcup K = I} p^{\chi(I,J,K)} F_J G_K \qquad \text{and} \qquad p = 1 - q. \tag{4.40}$$

The (two-word) crossing number $\chi(I; J, K)$ of the ordered triple $\{I; J, K\}$ is the *minimum* number of transpositions of elements of J and K in order to transform JK into I when J and K are order-preserving sub-words of I. For example,

$$\chi(ijk; i, jk) = 0, \quad \chi(ijk; ik, j) = 1, \quad \chi(ijk; jk, i) = 2. \tag{4.41}$$

For $q = 1$ ($p = 0$), this formula reduces to *conc*. For, the only term that contributes is the one with $\chi(I; J, K) = 0$ i.e. no crossings, so $I = JK$. Then

$$(F *_1 G)_I = \delta_I^{JK} F_J G_K. \tag{4.42}$$

If $q = 0$ ($p = 1$), then $p^{\chi(I;J,K)} = 1$ independent of the crossing number and all terms contribute equally giving back shuffle

$$(F *_0 G)_I = \sum_{I = J \sqcup K} F_J G_K. \tag{4.43}$$

Examples: For $q \neq 1$, $*_q$ is non-commutative in general. The first few terms in the q-product of a pair of tensors are $(F *_q G)_0 = F_0 G_0$,

$$\begin{aligned}
(F *_q G)_i &= F_i G_0 + F_0 G_i, \quad (F *_q G)_{ij} = F_0 G_{ij} + F_i G_j + p F_j G_i + F_{ij} G_0, \\
(F *_q G)_{ijk} &= F_0 G_{ijk} + F_i G_{jk} + p F_j G_{ik} + p^2 F_k G_{ij} + F_{ij} G_k + p F_{ik} G_j + p^2 F_{jk} G_i + F_{ijk} G_0 \\
(F *_q G)_{ijkl} &= F_0 G_{ijkl} + (F_i G_{jkl} + p F_j G_{ikl} + p^2 F_k G_{ijl} + p^3 F_l G_{ijk}) \\
&\quad + (F_{ij} G_{kl} + p F_{ik} G_{jl} + p^2 F_{il} G_{jk} + p^2 F_{jk} G_{il} + p^3 F_{jl} G_{ik} + p^4 F_{kl} G_{ij}) \\
&\quad + (F_{ijk} G_l + p F_{ijl} G_k + p^2 F_{ikl} G_j + p^3 F_{jkl} G_i) + F_{ijkl} G_0.
\end{aligned} \tag{4.44}$$

Associativity: We show that the q-product is associative

$$((F *_q G) *_q H)_I = (F *_q (G *_q H))_I = \sum_{I = J \sqcup K \sqcup L} p^{\chi(I;J,K,L)} F_J G_K H_L. \tag{4.45}$$

We first checked explicitly that associativity holds for $|I| \leq 3$ by writing out all the terms, but it was very tedious to go further. Instead, we write

$$\begin{aligned}
((F * G) * H)_I &= \sum_{I = J \sqcup K} p^{\chi(I;J,K)} (F * G)_J G_K \\
&= \sum_{I = L \sqcup M \sqcup K} p^{\chi(I;L \sqcup M,K)} p^{\chi(L \sqcup M;L,M)} F_L G_M H_K
\end{aligned}$$

[21]To avoid too much clutter we will occasionally drop the subscript in $*_q$ and indicate it by $*$.

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$$= \sum_{I=J\sqcup K\sqcup L} p^{\chi(I;J\sqcup K,L)+\chi(J\sqcup K;J,K)} F_J G_K H_L$$

$$(F*(G*H))_I = \sum_{I=J\sqcup K\sqcup L} p^{\chi(I;J,K\sqcup L)+\chi(K\sqcup L;K,L)} F_J G_K H_L \qquad (4.46)$$

where $I = J \sqcup K \sqcup L$ is the condition that J, K, L are complementary order-preserving sub-words of I. Since F, G, H are arbitrary and so is p, associativity requires the equality of the sums of crossing numbers

$$\chi(I; J\sqcup K, L) + \chi(J\sqcup K; J, K) \text{ and } \chi(I; J, K\sqcup L) + \chi(K\sqcup L; K, L) \qquad (4.47)$$

for each I and any (fixed) choices of $J, K, L, J\sqcup K$ and $K\sqcup L$ satisfying $I = J\sqcup K\sqcup L$. In fact, these two sums of (two-word) crossing numbers are equal to the (three-word) crossing number $\chi(I; J, K, L)$ that has a simple meaning. $\chi(I; J, K, L)$ is the smallest number of transpositions needed to transform JKL into I where J, K, L are order-preserving sub-words of I. For example suppose $I = abcd$, $J = d, K = c, L = ab, J\sqcup K = cd$ and $K\sqcup L = abc$. Then $\chi(abcd; cd, ab) + \chi(cd; d, c) = 4 + 1 = 5$ while $\chi(abcd; d, abc) + \chi(abc; c, ab) = 3+2 = 5$. Similarly, if $I = abcd$, $J = b, K = ad, L = c, J\sqcup K = abd$ and $K\sqcup L = acd$. Then $\chi(abcd; abd, c) + \chi(abd; b, ad) = 1 + 1 = 2$ while $\chi(abcd; b, acd) + \chi(acd; ad, c) = 1 + 1 = 2$. Thus, associativity just says that there are two different ways of calculating the three-word crossing number $\chi(I; J, K, L)$ when $I = J\sqcup K\sqcup L$. This gives the simple formula (4.45) for the $*_q$ product of three series, which makes associativity manifest.

Reduction to one generator: When we reduce to a single generator in the above examples (4.44), the formulae agree with those obtained earlier (3.14) using the Gauss binomials. More generally, we can see from the definition of the Gauss binomials (3.8) that

$$\binom{|I|}{r}_q = \sum_{\substack{I=J\sqcup K\\ |J|=r}} q^{\chi(I;J,K)}. \qquad (4.48)$$

Thus, the above formula for the q-product reduces to the one for a single generator.

Poisson Bracket: It may help to find a Poisson bracket on the shuffle algebra that serves as a first approximation to the q-commutator. The q-commutator is

$$([F,G]_q)_I \equiv (F *_q G - G *_q F)_I = \sum_{I=J\sqcup K} (1-q)^{\chi(I;J,K)} (F_J G_K - G_J F_K). \qquad (4.49)$$

For small q, $-\frac{1}{q}([F,G]_q)_I = \sum_{I=J\sqcup K} \chi(I; J, K)(F_J G_K - G_J F_K) + \mathcal{O}(q)$. So let us define the bracket $\{F, G\} = \{F, G\}_I \xi^I$ by

$$\{F,G\}_I = -\lim_{q\to 0} \frac{1}{q}([F,G]_q)_I = \sum_{I=J\sqcup K} \chi(I; J, K)(F_J G_K - G_J F_K). \qquad (4.50)$$

It is clearly bilinear and anti-symmetric. The first few examples with lowest $|I|$ are

$$\{F,G\}_0 = 0; \quad \{F,G\}_i = 0; \quad \{F,G\}_{ij} = F_j G_i - G_j F_i;$$

$$
\begin{aligned}
\{F,G\}_{ijk} &= F_jG_{ik} + 2F_kG_{ij} + F_{ik}G_j + 2F_{jk}G_i - (F \leftrightarrow G);\\
\{F,G\}_{ijkl} &= F_jG_{ikl} + 2F_kG_{ijl} + 3F_lG_{ijk} + F_{ik}G_{jl} + 2F_{il}G_{jk} + 2F_{jk}G_{il}\\
&\quad +3F_{jl}G_{ik} + 4F_{kl}G_{ij} + F_{ijl}G_k + 2F_{ikl}G_j + 3F_{jkl}G_i - (F \leftrightarrow G). \qquad (4.51)
\end{aligned}
$$

It satisfies the Jacobi identity since the q-product was associative.

$$
\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0. \tag{4.52}
$$

This can also be checked explicitly. For example, the first non-trivial case is

$$
\{\{F,G\},H\}_{ijk} = 2(F_iG_jH_k + F_kG_jH_i - F_jG_kH_i - F_jG_iH_k). \tag{4.53}
$$

Upon adding its cyclic permutations, the Jacobi identity is satisfied. Moreover, the Leibnitz rule (with respect to $sh = \circ = *_0$)

$$
\{F \circ G, H\} = F \circ \{G,H\} + \{F,H\} \circ G \tag{4.54}
$$

is also satisfied due to the corresponding identity for the q-commutator. Thus $\{\ldots\}$ is a Poisson bracket on the commutative shuffle algebra.

In order to be practically useful in going beyond the 0^{th} order solution of the LE, we need a q-expansion for $*_q$ around $*_0 = sh$ involving left annihilation D_j^0. For small q,

$$
\begin{aligned}
(F *_q G)_I &= \sum_{I=J\sqcup K} (1-q)^{\chi(I;J,K)} F_J G_K\\
&= \sum_{I=J\sqcup K} F_J G_K - q \sum_{I=J\sqcup K} \chi(I;J,K) F_J G_K + \mathcal{O}(q^2)\\
&= (F *_0 G)_I - q \sum_{I=J\sqcup K} \chi(I;J,K) F_J G_K + \mathcal{O}(q^2)\\
\Rightarrow \quad & \lim_{q\to 0} \frac{(F *_q G - F *_0 G)_I}{-q} = \sum_{I=J\sqcup K} \chi(I;J,K) F_J G_K. \qquad (4.55)
\end{aligned}
$$

For example,

$$
\begin{aligned}
\lim_{q\to 0} \frac{(F *_q G - F *_0 G)_{ij}}{-q} &= F_jG_i\\
\lim_{q\to 0} \frac{(F *_q G - F *_0 G)_{ijk}}{-q} &= F_jG_{ik} + 2F_kG_{ij} + F_{ik}G_j + 2F_{jk}G_i. \qquad (4.56)
\end{aligned}
$$

Our aim is to express this $\mathcal{O}(q)$ contribution to $F *_q G$ in terms of D_i^0 and $*_0$. But we are yet to find such a formula that generalizes (3.16) and hope further investigation will reveal it.

4.3.2 q-Deformed annihilation

There is one parameter family of annihilation operators D_j^q that interpolates between left annihilation D_j^0 and full annihilation D_j^1. For a single generator, it was defined in (3.18) as $(D_qG)_n = (1+q+q^2+\cdots+q^n)G_{n+1}$. By analogy we define $[D_j^qG]_I = \delta_I^{I_1I_2} q^{|I_1|} G_{I_1jI_2}$, i.e.

$$
[D_j^qG]_{i_1\cdots i_n} = G_{ji_1\cdots i_n} + qG_{i_1ji_2\cdots i_n} + q^2G_{i_1i_2ji_3\cdots i_n} + \cdots + q^nG_{i_1\cdots i_nj}. \tag{4.57}
$$

We pick up one more power of q as the annihilation operator travels through each index of the tensor from left to right. It is easily seen that

$$\lim_{q\to 0}\ [D_j^q G]_I = G_{jI} \quad \text{and} \quad \lim_{q\to 1}\ [D_j^q G]_I = \delta_I^{I_1 I_2} G_{I_1 j I_2} \tag{4.58}$$

reproduce left and full annihilation which are derivations of *sh* and *conc*. To make the LE (2.27) differential equations with respect to *conc*, we want to expand D_j^0 around D_j^1 in powers of $p = 1 - q$ and finally set $p = 1$. Recall that for 1-generator (3.19),

$$D_q G(\xi) = \sum_{k=1}^{\infty} \frac{1}{k!}(-p\xi)^{k-1} D_0^k G(\xi) = D_1 G(\xi) - \frac{p}{2}\xi D_1^2 G(\xi) + \frac{p^2}{6}\xi^2 D_1^3 G(\xi) + \mathcal{O}(p^3) \tag{4.59}$$

For several generators,

$$\begin{aligned}
&[D_j^q G]_{i_1\cdots i_n} = \\
&= \left[G_{ji_1\cdots i_n} + \cdots + G_{i_1\cdots i_n j}\right] - p\left[G_{i_1 j i_2\cdots i_n} + 2G_{i_1 i_2 j i_3\cdots i_n} + \cdots + nG_{i_1\cdots i_n j}\right] \\
&+p^2\left[G_{i_1 i_2 j i_3\cdots i_n} + 3G_{i_1 i_2 i_3 j i_4\cdots i_n} + \cdots + \frac{n(n-1)}{2} G_{i_1\cdots i_n j}\right] + \cdots + (-p)^n G_{i_1\cdots i_n j}
\end{aligned} \tag{4.60}$$

Drawing inspiration from (3.19) we would like to recognize the coefficients of powers of p as combinations of full annihilation and some multiplication operator acting on G. However, we have not yet succeeded in this.

5. Discussion

Despite their formidable reputation, the loop equations(LE) of a large-N multi-matrix model show much simplicity and structure when expressed in terms of gluon correlations G_I. Non-linearities are mild in the sense that in any equation, highest rank correlations appear linearly. So the LE are systems of inhomogeneous linear difference equations for correlations of a given rank with lower rank correlations appearing non-linearly as 'sources'. Solving these equations in the absence of additional structure would be tedious at best. But this is not possible because the LE are underdetermined in most interesting cases. We observed that there are additional equations involving the G_I that a naive passage from finite N Schwinger-Dyson equations to large-N LE misses. These equations have to do with changes of variables in matrix integrals that leave both action and measure invariant. However, we are yet to implement these additional constraints in detail to see whether they suffice to fix a unique solution to the LE. On the other hand, we saw that part of the difficulty in understanding the LE lies in the fact that they are not differential equations. Left annihilation does not satisfy the Leibnitz rule with respect to the concatenation product appearing in these equations. We proposed two schemes to remedy this situation by expanding either annihilation or product around one that is a derivation of the other. For the Gaussian, Chern-Simons and Yang-Mills models, it was possible to altogether eliminate the non-linearities of the LE and arrive at inhomogeneous linear PDEs at the zeroth order of these expansions. But the under-determinacy of the loop equations prevented us from picking a unique solution except in the case of the gaussian, where the two approximations were shown to give over and underestimates for correlations. This underscores the importance of better understanding the remaining constraints on G_I (section 2.3) as well as any other conditions that would ameliorate the under-determinacy of the LE. In [13] we hope to extend these algebraic and differential properties to matrix models with both gluon and ghost matrices, of the sort appearing in the gauge-fixed action of Yang-Mills theory.

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A. Cyclically symmetric tensors of rank n

What is the dimension $c(n,\Lambda)$ of the space of cyclically symmetric real tensors $G_{i_1\cdots i_n}$ of rank (=number of indices) n if the indices can take the values $1 \le i_k \le \Lambda$? The dimension of the space of all tensors of rank n is Λ^n. On the other hand, the space of symmetric rank n tensors, which is a subspace of cyclically symmetric tensors, is $\binom{\Lambda+n-1}{n}$ dimensional. Thus

$$\binom{\Lambda+n-1}{n} \;\le\; c(n,\Lambda) \;\le\; \Lambda^n \tag{A.1}$$

For a $\Lambda = 3$ matrix model, $\frac{1}{2}(n^2+3n+2) \le\; c(n,3) \;\le\; 3^n$. For a 2-matrix model, $n+1 \;\le\; c(n,2) \;\le\; 2^n$. The cyclic group of order n acts on rank n tensors $G_{i_1\cdots i_n}$ by cyclically permuting indices. $c(n,\Lambda)$ is the number of orbits. For example, if $\Lambda = 2$ and $n = 4$, the orbits are

$$\begin{aligned}&(G_{2222});\;\; (G_{1222}{=}G_{2122}{=}G_{2212}{=}G_{2221});\quad (G_{1122}{=}G_{2112}{=}G_{2211}{=}G_{1221});\\ &(G_{1212}{=}G_{2121});\;\; (G_{1112}{=}G_{2111}{=}G_{1211}{=}G_{1121});\quad (G_{1111})\end{aligned} \tag{A.2}$$

So $c(n=4,\Lambda=2) = 6$, significantly less than 2^4. The cardinality of different orbits are not necessarily equal. Some other examples are

$$\begin{aligned}&c(n,1) = 1;\;\; c(1,\Lambda) = \Lambda;\quad c(2,\Lambda) = \frac{1}{2}\Lambda(\Lambda+1);\quad c(3,2) = 4;\\ &c(3,3) = 11;\;\; c(4,2) = 6;\quad c(4,3) = 24;\quad c(5,2) = 8.\end{aligned} \tag{A.3}$$

It would be nice to have formula for $c(n,\Lambda)$, at least for the $\Lambda = 2$ matrix model.

Note on hermiticity condition: Actually, the tensors G_I are complex numbers, so the real-dimension of the space of cyclically symmetric tensors of rank n is $2\,c(n,\Lambda)$. However, the hermiticity condition $G_{i_1 i_2\cdots i_n} = G^*_{i_n\cdots i_2 i_1}$ halves this real-dimension to $c(n,\Lambda)$. If reversal of indices can be achieved by a cyclic permutation (e.g. $G_{1122} = G^*_{2211} = G_{2211}$) then the correlation is real. If $\bar{I}$ cannot be obtained from I via cyclic permutations, then hermiticity means that $\Re G_I = \Re G_{\bar{I}}$ and $\Im G_I = -\Im G_{\bar{I}}$. For example $\Re G_{1123} = \Re G_{3211}$ and $\Im G_{1123} = -\Im G_{3211}$. In either case, hermiticity halves the number of independent parameters in cyclically symmetric correlations of a given rank.

B. Concatenation, shuffle and their co-products

By V, let us denote the infinite-dimensional complex vector space spanned by the monomial words $\xi^{i_1\cdots i_n}$ in the Λ non-commuting sources ξ^i. A typical element is the formal series $G(\xi) = G_I \xi^I$. V is the basic arena for our algebraic study of the loop equations[22].

The concatenation product $conc : V \otimes V \to V$ denoted by juxtaposition, was defined in (2.23) $\xi^I \xi^J = \delta^{IJ}_K \xi^K = \xi^{IJ}$. It has the structure constants $c^{I,J}_K = \delta^{IJ}_K$. For $\Lambda > 1$, *conc* is non-commutative. The vector space V, along with the concatenation product is the free associative algebra $\mathcal{T}$ on the generators $\xi^1, \ldots, \xi^\Lambda$. It is the universal envelope of the free Lie algebra. The commutative shuffle product $sh : V \otimes V \to V$ was defined in (2.38). V, equipped with sh is the shuffle algebra. The shuffle product of monomials

$$\xi^I \circ \xi^J = s^{I,J}_K \xi^K = \sum_{I \sqcup J = K} \xi^K. \tag{B.1}$$

leads to the shuffle structure constants $s^{I,J}_K = |\{I \sqcup J = K\}|$.

There is a natural inner product $(.,.)$ on V, for which ξ^I form an orthonormal basis

$$(\xi^I, \xi^J) = \delta^{I,J} \quad \text{or} \quad (F_I \xi^I, G_J \xi^J) = F_I G_J \delta^{I,J} = \sum_I F_I G_I. \tag{B.2}$$

The Kronecker symbol $\delta^{I,J} = 1$ if $I = J$ and 0 otherwise. We can use the 'metric' $\delta^{I,J}$ and its inverse $\delta_{I,J}$ to raise and lower indices. The inner product allows us to define co-products $V \to V \otimes V$. We call them co-concatenation $\Delta = sh^\dagger$ and co-shuffle $\Delta' = conc^\dagger$. They are adjoints of sh and *conc* respectively. For three formal series F, G, H, we define Δ and Δ' by

$$(F \otimes G, \Delta(H)) = (F \circ G, H) \quad \text{and} \quad (F \otimes G, \Delta'(H)) = (FG, H). \tag{B.3}$$

We define the structure constants of co-concatenation and co-shuffle as

$$\Delta(\xi^K) = s^K_{L,M} \xi^L \otimes \xi^M \quad \text{and} \quad \Delta'(\xi^K) = c^K_{L,M} \xi^L \otimes \xi^M. \tag{B.4}$$

We use the same letter c to denote the structure constants of *conc* and $\Delta' = conc^\dagger$ because they are related by raising and lowering indices using the metric $\delta^{I,J}$. The same goes for the letter s for the structure constants of sh and $\Delta = sh^\dagger$. The expressions for these are

$$c^I_{J,K} = c^{L,M}_N \delta^{I,N} \delta_{J,L} \delta_{K,M} = \delta^I_{JK} \quad \text{and}$$

[22]A superior approach that makes cyclic symmetry of G_I manifest might be to consider the quotient by the relation $\xi^I \sim \xi^J$ if I is a cyclic permutation of J. Then a basis for V would consist of words ξ^I where I labels orbits of the cyclic group action. In this paper we just allow all words ξ^I and impose the condition that G_I be cyclically symmetric, by hand, so to speak.

$$s^I_{J,K} = s^{L,M}_N \delta^{I,N} \delta_{J,L} \delta_{K,M} = s^{J,K}_I = |\{I = J \sqcup K\}|. \tag{B.5}$$

To obtain the co-shuffle structure constants $c^I_{J,K}$, we use the definition of adjoint to get

$$\begin{aligned} &\langle \xi^I \xi^J, \xi^K \rangle = \langle \xi^I \otimes \xi^J, \Delta'(\xi^K) \rangle \quad \Rightarrow \quad \delta^{IJ,K} = c^K_{L,M} \langle \xi^I \otimes \xi^J, \xi^L \otimes \xi^M \rangle = c^K_{L,M} \delta^{I,L} \delta^{J,M} \\ &\Rightarrow \quad c^K_{N,P} = \delta^{IJ,K} \delta_{I,N} \delta_{J,P} = \delta^K_{NP}. \end{aligned} \tag{B.6}$$

We use a similar procedure for the co-concatenation structure constants $s^I_{J,K}$

$$\begin{aligned} &\langle \xi^I \circ \xi^J, \xi^K \rangle = \langle \xi^I \otimes \xi^J, \Delta(\xi^K) \rangle \quad \Rightarrow \quad s^{I,J}_L \langle \xi^L, \xi^K \rangle = s^K_{L,M} \langle \xi^I \otimes \xi^J, \xi^L \otimes \xi^M \rangle \\ &\Rightarrow \quad s^{I,J}_L \delta^{L,K} = s^K_{L,M} \delta^{I,L} \delta^{J,M} \quad \Rightarrow \quad s^K_{P,Q} = s^{I,J}_L \delta^{L,K} \delta_{I,P} \delta_{J,Q} = s^{P,Q}_K. \end{aligned} \tag{B.7}$$

On formal series, co-shuffle $\Delta' = conc^\dagger$ acts as

$$\Delta' F = [\Delta' F]_{I,J}\ \xi^I \otimes \xi^J = F_{IJ} \xi^I \otimes \xi^J. \tag{B.8}$$

In particular, $\Delta'(\xi^I) = \delta^I_{JK} \xi^J \otimes \xi^K$ and $\Delta'(\xi^i) = (\xi^i \otimes 1 + 1 \otimes \xi^i)$. On formal series, co-concatenation $\Delta = sh^\dagger$ acts according to

$$\Delta F = [\Delta F]_{J,K} \xi^J \otimes \xi^K \quad \text{where} \quad [\Delta F]_{J,K} = \sum_{I = J \sqcup K} F_I. \tag{B.9}$$

In particular, $\Delta(\xi^I) = \sum_{I = J \sqcup K} \xi^J \otimes \xi^K$ and $\Delta(\xi^i) = \xi^i \otimes 1 + 1 \otimes \xi^i$.

C. Bialgebra structures on $V = Span(\xi^I)$

V has two bialgebra (algebra + compatible coalgebra) structures. In one, the product (sh) is commutative while the co-product (adjoint of $conc$) is non-co-commutative. In the dual bialgebra, the product ($conc$) is non-commutative while the co-product (adjoint of sh) is co-commutative.

To establish that shuffle and co-shuffle[23] combine to define a bialgebra on V, we show that co-shuffle $\Delta' = conc^\dagger$ is a homomorphism of the shuffle product

$$\Delta'(F \circ G) = \Delta'(F) \circ \Delta'(G). \tag{C.1}$$

Note that the l.h.s. is

$$\Delta'(F \circ G) = \sum_{L \sqcup M = K} F_L G_M \Delta'(\xi^K) = \sum_{L \sqcup M = IJ} F_L G_M \xi^I \otimes \xi^J. \tag{C.2}$$

While the r.h.s. is

$$\begin{aligned} \Delta'(F) \circ \Delta'(G) &= F_I G_J \Delta'(\xi^I) \circ \Delta'(\xi^J) = F_I G_J \delta^I_{KL} \delta^J_{MN} (\xi^K \otimes \xi^L) \circ (\xi^M \otimes \xi^N) \\ &= F_{KL} G_{MN} (\xi^K \circ \xi^M) \otimes (\xi^L \circ \xi^N) = \sum_{K \sqcup M = I, L \sqcup N = J} F_{KL} G_{MN} \xi^I \otimes \xi^J. \end{aligned} \tag{C.3}$$

[23]This justifies the name co-shuffle for the adjoint of *conc*.

Comparing coefficients, $\Delta' = conc^\dagger$ is a homomorphism of the shuffle product if

$$\sum_{J_1 \sqcup J_2 = I_1 I_2} F_{J_1} G_{J_2} = \sum_{L_1 \sqcup M_1 = I_1, L_2 \sqcup M_2 = I_2} F_{L_1 L_2} G_{M_1 M_2} \quad \forall \; I_1, \; I_2. \tag{C.4}$$

To prove this, observe that J_1 may be uniquely decomposed as $J_1 = L_1 L_2$ with $L_1 \subset I_1$ and $L_2 \subset I_2$ and similarly for J_2, $J_2 = M_1 M_2$ with $M_1 \subset I_1$ and $M_2 \subset I_2$. Then we observe that every riffle-shuffle $J_1 \sqcup J_2 = I_1 I_2$ arises from a unique pair of riffle-shuffles $L_1 \sqcup M_1 = I_1$ and $L_2 \sqcup M_2 = I_2$. This establishes that co-shuffle Δ is a homomorphism of sh.

A similar argument shows that $\Delta = sh^\dagger$ is a homomorphism of *conc*: $\Delta(FG) = \Delta(F)\Delta(G)$.

$$\Delta(F)\Delta(G) = \sum_{J = I_1 I_3, K = I_2 I_4} (\Delta F)_{I_1, I_2} (\Delta G)_{I_3, I_4} = \sum_{J = I_1 I_3, K = I_2 I_4} F_{I_1 \sqcup I_2} G_{I_3 \sqcup I_4}. \tag{C.5}$$

On the other hand, the l.h.s. gives

$$[\Delta(FG)]_{J,K} = \sum_{L = J \sqcup K} (FG)_L = \sum_{L_1 L_2 = J \sqcup K} F_{L_1} G_{L_2} = \sum_{J = I_1 I_3, K = I_2 I_4} F_{I_1 \sqcup I_2} G_{I_3 \sqcup I_4}. \tag{C.6}$$

In the last equality, we used the unique decomposition $J = I_1 I_3, K = I_2 I_4$ where $I_1, I_2 \subset L_1$ and $I_3, I_4 \subset L_2$ as before. Thus we have shown that $\Delta = sh^\dagger$ is a homomorphism of *conc*.

The unit element for *conc* is 1, $F1 = 1F = F$. The co-unit for co-concatenation is $\epsilon : V \to \mathbf{C}$. It picks out the constant term in a formal series $\epsilon(F_I \xi^I) = F_\emptyset \equiv F_0$. Just like co-concatenation, the co-unit is a homomorphism of *conc*

$$\epsilon(FG) = (FG)_0 = F_0 G_0 = \epsilon(F)\epsilon(G). \tag{C.7}$$

The unit element for shuffle too is 1, $(F \circ 1)_I = \sum_{I = J \sqcup K} F_J \delta^0_K = F_I$. The co-unit for co-shuffle is again $\epsilon : V \to \mathbf{C}$. The co-unit ϵ is a homomorphism of the shuffle product

$$\epsilon(F \circ G) = (F \circ G)_0 = \sum_{J \sqcup K = \emptyset} F_J G_K = F_0 G_0 = \epsilon(F) \circ \epsilon(G). \tag{C.8}$$

To summarize, $(conc, sh^\dagger = \Delta = co{-}conc, 1, \epsilon)$ defines a non-commutative but co-commutative bialgebra (algebra plus compatible co-algebra) structure on $V = span(\xi^I)$. Similarly, $(sh, conc^\dagger = \Delta' = co{-}sh, 1, \epsilon)$ defines a commutative but non-co-commutative bialgebra structure on V. These two bialgebras are not independent. Structure constants of the product and co-product of one can be obtained from those of the other using the inner product $\delta^{I,J}$ on V.

Remark: In addition to being a bialgebra $\mathcal{T} = (conc, \Delta, 1, \epsilon)$, is the universal envelope of the free Lie algebra. So it is a *Lie* algebra with the Lie product $[\xi^I, \xi^J] = \xi^{IJ} - \xi^{JI}$. Does Δ define a *Lie* bialgebra [36] with respect to the commutator? No! On the one hand, $\Delta : V \to V \otimes V$ is not skew-symmetric. Rather, its image lies within $Sym(V \otimes V)$.

$$\Delta(\xi^I) = \sum_{I = J \sqcup K} \xi^J \otimes \xi^K = \sum_{I = J \sqcup K} \xi^K \otimes \xi^J = (\tau\Delta)(\xi^I), \tag{C.9}$$

where $\tau(a \otimes b) = b \otimes a$. Here we used the fact that if J and K are order preserving complementary subwords of I, then so are K and J. Furthermore, Δ is not a 1-cocycle for the free associative algebra. In order to be a 1-cocycle, it must satisfy

$$\Delta[F, G] = (ad_F \otimes 1 + 1 \otimes ad_F)\Delta(G) - F \leftrightarrow G \tag{C.10}$$

for any $F, G \in \mathcal{T}$. However, taking $F = \xi^i$ and $G = \xi^j$ gives

$$l.h.s. \ = \Delta[\xi^i, \xi^j] = \xi^{ij} \otimes 1 + 1 \otimes \xi^{ij} - \xi^{ji} \otimes 1 - 1 \otimes \xi^{ji} \tag{C.11}$$

and $r.h.s. \ = 2 \times l.h.s. \ \neq l.h.s.$. There may be some other skew-symmetric 1-cocycle $\tilde{\Delta} : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ which defines a Lie bialgebra structure on the universal envelope of the free Lie algebra.

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