International Journal of Modern Physics A
Vol. 21, No. 18 (2006) 3771–3808

$2+1$ ABELIAN "GAUGE THEORY" INSPIRED BY IDEAL HYDRODYNAMICS

GOVIND S. KRISHNASWAMI

Institute for Theoretical Physics & Spinoza Institute, Utrecht University,
Postbus 80.195, 3508 TD, Utrecht, The Netherlands
g.s.krishnaswami@phys.uu.nl

Received 11 October 2005

We study a possibly integrable model of Abelian gauge fields on a two-dimensional surface M, with volume form μ. It has the same phase-space as ideal hydrodynamics, a coadjoint orbit of the volume-preserving diffeomorphism group of M. Gauge field Poisson brackets differ from the Heisenberg algebra, but are reminiscent of Yang–Mills theory on a null surface. Enstrophy invariants are Casimirs of the Poisson algebra of gauge invariant observables. Some symplectic leaves of the Poisson manifold are identified. The Hamiltonian is a magnetic energy, similar to that of electrodynamics, and depends on a metric whose volume element is not a multiple of μ. The magnetic field evolves by a quadratically nonlinear "Euler" equation, which may also be regarded as describing geodesic flow on $\mathrm{SDiff}(M,\mu)$. Static solutions are obtained. For uniform μ, an infinite sequence of local conserved charges beginning with the Hamiltonian are found. The charges are shown to be in involution, suggesting integrability. Besides being a theory of a novel kind of ideal flow, this is a toy-model for Yang–Mills theory and matrix field theories, whose gauge-invariant phase-space is conjectured to be a coadjoint orbit of the diffeomorphism group of a noncommutative space.

Keywords: Gauge theory; coadjoint orbits; volume-preserving diffeomorphisms; Euler equation; integrability.

PACS numbers: 11.15.-q, 11.30.-j, 02.30.Ik, 47.15.Ki

1. Introduction and Summary

The classical theory of gauge fields[a] $A_1(x^1,x^2,t)$ and $A_2(x^1,x^2,t)$ we study in this paper, may be summarized in four equations. The Hamiltonian is a gauge-invariant magnetic energy,

$$H = \int \left(\frac{B}{\rho}\right)^2 \sigma\rho\, d^2x\,, \tag{1}$$

where $B = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field. ρ is a given volume element and g_{ij} a fixed metric on a two-dimensional surface M such that $\sigma = \rho^2/g$ is not a

[a]After gauge-fixing, there will be a single propagating field degree of freedom.

constant ($g = \det g_{ij}$). The Poisson bracket between gauge fields is a Lie algebra (independent of the metric)

$$\{A_i(x), A_j(y)\} = \delta^2(x-y)\left[A_j(y)\frac{\partial}{\partial y^i}\rho^{-1}(y) - A_i(x)\frac{\partial}{\partial x^j}\rho^{-1}(x)\right]. \tag{2}$$

Hamilton's equation for time evolution of gauge-invariant observables, $\dot{f} = \{H, f\}$ implies that the magnetic field evolves according to a nonlinear "Euler" equation

$$\dot{B} = \nabla(B/\rho) \times \nabla(B\sigma/\rho). \tag{3}$$

Viewed as a rigid body for the group of volume-preserving diffeomorphisms of M, the inverse of the inertia tensor is a twisted version of the Laplace operator

$$H = \frac{1}{2}\int A_i h^{il} A_l \rho \, d^2x \quad \text{with} \quad h^{il} = \frac{\varepsilon^{ij}\varepsilon^{kl}}{\rho}\left[\left(\partial_j \frac{\sigma}{\rho}\right)\partial_k + \frac{\sigma}{\rho}\partial_j\partial_k\right]. \tag{4}$$

This theory has the same phase-space and Poisson brackets as $2+1$ ideal (inviscid and volume-preserving) hydrodynamics, but a different Hamiltonian. It may be integrable, since we find an infinite number of conserved quantities $H_n = \int (B/\rho)^n \sigma\rho \, d^2x$ as well as an infinite number of Casimirs $I_n = \int (B/\rho)^n \rho d^2x$ for uniform ρ. It is remarkable that one can make this modification to $2+1$ ideal flow, which is sometimes studied as a toy-model for turbulence, to get a potentially integrable system. However, our original motivation for studying this model was different. We argue below that it is the simplest "gauge theory" that shares some quite deep, though unfamiliar, mathematical features of Yang–Mills theory.

The formulation of Yang–Mills theory in terms of gauge-invariant observables, and the development of methods for its solution are important and challenging problems of theoretical physics, since all the experimentally observed asymptotic states of the strong interactions are color-singlets. This problem has a long history stretching at least as far back as the work of Mandelstam.[1] Wilson loops are a natural choice for gauge-invariant variables, but they have trivial Poisson brackets on a spatial initial value surface, since the gauge field is canonically conjugate to the electric field on such a surface. More recently, it has been shown by Rajeev and Turgut,[2,3] that Wilson loops of $(3+1)$-dimensional Yang–Mills theory on a null initial value hypersurface satisfy a quadratic Poisson algebra with no need for electric field insertions. This is because the transverse components of the gauge field satisfy a nontrivial Poisson algebra among themselves, as opposed to the situation on a spatial surface. The Poisson algebra of Wilson loops is degenerate due to Mandelstam-like constraints. The gauge-invariant phase-space of Yang–Mills theory is conjectured to be a coadjoint orbit of this Poisson algebra. It is still a challenge to write the Hamiltonian in terms of these variables. However, this has been possible in dimensionally reduced versions[b] such as adjoint scalar field theories coupled to

[b]See also the work of Karabali, Nair and Kim who have made significant progress with a gauge-invariant Hamiltonian approach to $2+1$ Yang–Mills theory.[4]

quarks in $1+1$ dimensions, as shown by Lee and Rajeev.[5,6] In conjunction with 't Hooft's large N approximation,[7] viewed as an alternative classical limit, this is an approach to better understand the nonperturbative dynamics, especially of nonsupersymmetric gauge theories. In such an approach to $1+1$ QCD, the phase-space of gauge-invariant meson variables is an infinite Grassmannian, a coadjoint orbit of an infinite dimensional unitary group.[8] This allows one to understand baryons as well as mesons in the large N limit, going beyond the early work of 't Hooft.[9,8,10,11]

However, the groups and Lie algebras whose coadjoint orbits are relevant to matrix field theories and Yang–Mills theory are poorly understood noncommutative versions of diffeomorphism groups.[c] In the case of a multimatrix model, the group is, roughly speaking, an automorphism group of a tensor algebra. The Lie algebra is a Cuntz-type algebra which can be thought of as an algebra of vector fields on a noncommutative space.[12,5,6,13,14] However, it is still very challenging to find the proper mathematical framework for these theories and develop approximation methods to solve them even in the large N limit. To develop the necessary tools, it becomes worthwhile to practice on simpler theories whose gauge-invariant phase-space is the coadjoint orbit of a less formidable group. Here, we take a step in this direction by studying an Abelian gauge theory whose phase-space is a coadjoint orbit of the volume-preserving diffeomorphism group of a two-dimensional surface.

To put these remarks in perspective, recall the common classical formulation of Eulerian rigid body dynamics, ideal hydrodynamics, the KdV equation[15–17] and the large N limit of two-dimensional QCD.[8] The phase-space of each of these theories is a symplectic leaf of a degenerate Poisson manifold, which is the dual $\mathcal{G}^*$ of a Lie algebra. $\mathcal{G}^*$ always carries a natural Poisson structure. Symplectic leaves are coadjoint orbits of a group G acting on the dual of its Lie algebra $\mathcal{G}^*$. On any leaf, the symplectic structure is given by the Kirillov form. The appropriate groups in these examples are SO(3), the volume-preserving diffeomorphism group of the manifold upon which the fluid flows, and the central extensions of $\mathrm{Diff}(S^1)$ and of an infinite dimensional unitary group, respectively. The coadjoint orbits for the rigid body and 2D QCD are well-known symplectic manifolds: concentric spheres and the infinite dimensional Grassmannian manifold. The observables in each case are real-valued functions on $\mathcal{G}^*$. The Poisson algebra of observables is degenerate, i.e. has a center consisting of Casimirs. The symplectic leaves can also be characterized as the level sets of a complete set of Casimirs. In each case, the Hamiltonian is a quadratic function on the phase-space and classical time evolution is given by Hamilton's equations. Hamilton's equations are nonlinear despite a quadratic Hamiltonian, since the Poisson brackets of observables are more complicated than the Heisenberg algebra. In exceptional cases such as the rigid body and the KdV equation, these nonlinear equations are exactly integrable. In other cases, it is useful to develop

[c]This is *not* the structure group (sometimes called the gauge group) of the theory, which is still SU(N) or U(N). The gauge group plays little role in a gauge-invariant formulation of the theory.

approximation methods to solve them, that are adapted to the geometry of the phase-space.

Our earlier remarks indicate that it may be fruitful to regard Yang–Mills theory and matrix field theories as Hamiltonian dynamical systems along the lines of the more well-known ones listed in the last paragraph. As a toy-model in this direction, we seek a gauge theory where the gauge fields satisfy a closed Poisson algebra, without any need for electric fields. We want a theory whose phase-space is a coadjoint orbit of an ordinary diffeomorphism group, which is simpler than its noncommutative cousins. We would also like to understand in more detail the structure of the Poisson algebra of gauge-invariant observables, work out the equations of motion and try to solve them.

In this paper, we identify a classical theory of Abelian gauge fields in two spatial dimensions, different from Maxwell theory. In particular, it is *not* Lorentz covariant, indeed, time plays the same role as in Newtonian relativity. The theory is defined by a two-dimensional manifold M, a volume form μ and a metric g_{ij} whose volume element Ω_g is not a multiple of μ. The Hamiltonian is a gauge-invariant magnetic energy, much like that of Maxwell theory. Unlike in electrodynamics, the gauge field is a one-form on space, rather than on space–time. Thus, *even before any gauge fixing*, the gauge field has no time component. There is a magnetic field B, but no electric field, so to speak. After gauge fixing, there remains only one dynamical component of the gauge field. In this sense, the theory has the same number of degrees of freedom as $2+1$ electrodynamics. However, though the Hamiltonian is quadratic in the gauge fields, the classical theory is nonlinear due to the "non-canonical" Poisson algebra of gauge fields. Equations of motion are nonlinear and comparable to those of a $(2+1)$-dimensional non-Abelian gauge theory or ideal hydrodynamics.

The phase-space of the theory is a coadjoint orbit of the volume-preserving diffeomorphism group $\mathrm{SDiff}(M,\mu)$ of the spatial two-dimensional manifold. Roughly speaking, this means that $\mathrm{SDiff}(M,\mu)$ is a symmetry group of the Poisson algebra of observables. The gauge group (structure group) of the theory is U(1). The inspiration for this lies in ideal hydrodynamics. Indeed, even before the diffeomorphism group of a manifold appeared in general relativity, it was relevant as the configuration space of a fluid. The theory we study is not the same as, but is motivated by $(2+1)$-dimensional ideal hydrodynamics, regarded as a Hamiltonian system.[18,19,15,20–22] Though we arrived at it as a toy-model for Yang–Mills theory, it turns out to have a nice geometric and possibly even integrable structure. We find two infinite sequences of conserved charges. The first set are Casimirs, analogues of the enstrophy invariants of ideal hydrodynamics. In addition, we find another infinite set of conserved charges which are *not* Casimirs but are in involution. The theory we study here can also be regarded as a theory of geodesics of a right-invariant metric on the volume-preserving diffeomorphism group of a two-dimensional manifold. However, the right invariant metric on $\mathrm{SDiff}(M,\mu)$ implied by our Hamiltonian is different from that arising in ideal hydrodynamics (the L^2

metric leading to ideal Euler flow) as well as the H^1 metric leading to averaged Euler flow.[23]

Another way to view the current work is to recall that adding supersymmetry usually gives greater analytical control over gauge theories. But there may be other modifications of gauge theories that also lead to interesting toy-models or enhanced solvability. Our investigation concerns one such novel modification of gauge field Poisson brackets.

In Sec. 2 we introduce the space of Abelian gauge fields $A_i dx^i$ on a two-dimensional surface M as the dual of the Lie algebra $\mathrm{SVect}(M,\mu)$ of vector fields preserving a volume element $\mu = \rho\, d^2x$. This "duality" is known in hydrodynamics.[15] The differentials $df^i = \rho^{-1}\frac{\delta f}{\delta A_i}$ of differentiable gauge-invariant observables $f(A)$ are shown to be volume-preserving vector fields. In Sec. 3 we give the Poisson structure on gauge-invariant observables

$$\{f,g\} = \int d^2x \rho A_i \left[\rho^{-1}\frac{\delta f}{\delta A_j}\partial_j\left(\rho^{-1}\frac{\delta g}{\delta A_i}\right) - \rho^{-1}\frac{\delta g}{\delta A_j}\partial_j\left(\rho^{-1}\frac{\delta f}{\delta A_i}\right)\right] \tag{5}$$

turning the space of gauge fields into a Poisson manifold. The Poisson brackets of gauge fields are obtained explicitly (35) and compared with those of Yang–Mills theory on a spatial and null initial value hypersurface.

In Sec. 4 we give the coadjoint action of $\mathrm{SDiff}(M,\mu)$ and its Lie algebra $\mathrm{SVect}(M,\mu)$ on the Poisson manifold of gauge fields $\mathrm{SVect}(M,\mu)^*$, and show that the action is canonical, i.e. preserves the Poisson structure. The moment maps generate the coadjoint action. The symplectic leaves of the Poisson manifold are coadjoint orbits. The enstrophy invariants of hydrodynamics $I_n = \int_M (dA/\mu)^n \mu$ are an infinite sequence of Casimirs of the Poisson algebra. The coadjoint orbits of closed gauge field one-forms are shown to be finite dimensional. Single-point orbits for simply connected M are found. We argue that all other orbits are infinite dimensional and try to characterize their isotropy subalgebras as well as tangent spaces.

In Sec. 5 we first review the choice of Hamiltonian leading to ideal Eulerian hydrodynamics in $2+1$ dimensions. Then we propose a different gauge-invariant Hamiltonian depending on both μ and a metric g_{ij}, by analogy with the magnetic energy of Maxwell theory,

$$H = \frac{1}{2}\int_M \left(\frac{F\wedge *F}{\Omega_g}\right)\mu = \int \left(\frac{B}{\rho}\right)^2 \sigma\rho\, d^2x\,, \tag{6}$$

where $F = dA$ is the field strength, $*F$ is its Hodge dual, and Ω_g is the volume element of the metric g_{ij}. Here $\sigma = (\mu/\Omega_g)^2 = \rho^2/g$, $g = \det g_{ij}$ and ρ is the density associated to μ. H is shown to determine a nonnegative inner product on the dual of the Lie algebra $\mathrm{SVect}(M,\mu)^*$ and an inverse "inertia tensor" by analogy with the rigid body. If M is simply connected, the inverse inertia operator is nondegenerate and could be inverted to get an inner product on the Lie algebra $\mathrm{SVect}(M,\mu)$. This could be extended to the diffeomorphism group $\mathrm{SDiff}(M,\mu)$ by right translations.

Thus, the magnetic energy should define geodesic flow on $\mathrm{SDiff}(M,\mu)$ with respect to a right-invariant metric different from that coming from Eulerian hydrodynamics.

In Sec. 6 we find the equation of motion for the magnetic field $B = \varepsilon^{ij}\partial_i A_j$, $\dot{B} = \nabla(B/\rho) \times \nabla(B\rho/g)$. This simple quadratically nonlinear evolution equation is strikingly similar to the Euler equation of a rigid body $\dot{L} = L \times \Omega, L = I\Omega$. It can be regarded as the "Euler equation" for the group $\mathrm{SDiff}(M,\mu)$ with Hamiltonian given above. Remarkably, for a uniform measure μ we find an infinite sequence H_n of conserved charges in involution, which are not Casimirs. The Hamiltonian is $\frac{H_2}{2}$,

$$H_n = \int_M \left(\frac{dA}{\mu}\right)^n \sigma\mu\,, \quad n = 1, 2, 3, \ldots\,. \tag{7}$$

In Sec. 7 we find some static solutions of the equations of motion. We show that for circularly symmetric ρ and g, every circularly symmetric magnetic field is a static solution. We generalize this to the nonsymmetric case as well. We also find a one-parameter family of static solutions that are local extrema of energy even with respect to variations that are not restricted to the symplectic leaf on which the extremum lies. Some ideas for further study are given in Sec. 8.

Volume-preserving diffeomorphisms and gauge theories have appeared together previously in the literature (see, for example, Refs. 24 and 25). Our investigation seems quite different, since $\mathrm{SDiff}(M,\mu)$ is *not* the gauge group of our theory but rather a symmetry of the Poisson algebra.

2. Volume-Preserving Vector Fields to Gauge-Invariant Observables

2.1. *Lie algebra of volume-preserving vector fields*

Let M be a surface with local coordinates x^i, to be thought of as the space on which a fluid flows. A vector field on M is regarded as the velocity field of a fluid at a particular time. The space of all vector fields on M forms a Lie algebra $\mathrm{Vect}(M)$ with Lie bracket

$$[u,v]^i = u^j\partial_j v^i - v^j\partial_j u^i\,. \tag{8}$$

$\mathrm{Vect}(M)$ is the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$. Conservation of the mass of the fluid during its flow implies the continuity equation for its density $\rho(x,t)$:

$$\frac{\partial\rho(x,t)}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0\,. \tag{9}$$

We are interested in flows where the density at any point of space does not depend on time. Using the continuity equation, this becomes $\nabla\cdot(\rho\mathbf{u}) = 0$. We call such a flow volume-preserving. Geometrically, we are considering a flow that generates diffeomorphisms of M that preserve a given volume form[d] $\mathcal{L}_u\mu = 0$. The density is

[d]A volume form must be nondegenerate. In two dimensions it is the same as an area form or a symplectic form.

constant along integral curves of u. To see the equivalence of this with the continuity equation for a volume-preserving flow, recall that

$$\mathcal{L}_u\mu = (di_u + i_u d)\mu = d(i_u\mu)\,, \tag{10}$$

where i_u is the contraction with u. Here $d\mu = 0$ since μ is a volume form. In local coordinates $\mu = \frac{1}{2}\mu_{ij}\, dx^i \wedge dx^j$ where $\mu_{ij} = -\mu_{ji} \equiv \epsilon_{ij}\rho$ and ϵ_{ij} is antisymmetric with $\epsilon_{12} = 1$. So $\mu = \rho(x)d^2x$ where $dx^1 \wedge dx^2 \equiv d^2x$. Then $i_u\mu = \frac{1}{2}\mu_{ij}(u^i\, dx^j - dx^i\, u^j) = \mu_{ij}u^i\, dx^j$, so that

$$\mathcal{L}_u\mu = d(i_u\mu) = \partial_k(\mu_{ij}u^i)dx^k \wedge dx^j = \partial_i(\rho u^i)dx^1 \wedge dx^2\,. \tag{11}$$

Thus $\mathcal{L}_u\mu = 0$ becomes $\nabla \cdot (\rho\mathbf{u}) = 0$. We will use the terms volume-preserving and area preserving interchangeably since M is a two-dimensional surface. Some of what we say has a generalization to higher (especially even) dimensional M.

The properties $\mathcal{L}_{\alpha u+\beta v} = \alpha\mathcal{L}_u + \beta\mathcal{L}_v$ and $\mathcal{L}_{[u,v]} = \mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u$ ensure that the space of volume-preserving vector fields $\mathcal{G} = \mathrm{SVect}(M,\mu)$ forms a Lie subalgebra of $\mathrm{Vect}(M)$. It is the Lie algebra of the group of volume-preserving diffeomorphisms $G = \mathrm{SDiff}(M,\mu)$.

Volume-preserving flow is a special case of incompressible flow, which occurs when the fluid speed is small compared to the speed of sound.[e] In particular, shock waves cannot form in incompressible flow since shock waves involve supersonic flow. Under ordinary conditions, air flow in the atmosphere is incompressible. Vertical currents mix regions of high and low density so they are incompressible but not volume-preserving. Horizontal air currents in the atmosphere are approximately volume-preserving.

If M is simply connected, the volume-preserving condition $\partial_i(\rho u^i) = 0$ may be solved in terms of a *stream function* ψ satisfying $i_u\mu = d\psi$. ψ is a scalar function on M that serves as a "potential" for the velocity field. In local coordinates

$$i_u\mu = d\psi \Rightarrow \mu_{ij}u^i = \partial_j\psi\,. \tag{12}$$

Since μ is nondegenerate ($\rho \neq 0$), it can be inverted $\rho^{-1}\varepsilon^{ij}\mu_{jk} = -\delta^i_k$ where ε^{ij} is a constant antisymmetric tensor with $\varepsilon^{12} = 1$, $\varepsilon^{ij}\epsilon_{jk} = -\delta^i_k$. This does not require a metric on M. Then

$$u^i = \rho^{-1}\varepsilon^{ij}\partial_j\psi\,. \tag{13}$$

u determines ψ up to an additive constant, which can be fixed by a boundary condition. If M is simply connected, then $\mathrm{SVect}(M,\mu)$ may be identified with the space of stream functions. Suppose two volume-preserving vector fields u, v have stream functions ψ_u and ψ_v,

$$u^i = \rho^{-1}\varepsilon^{ij}\partial_j\psi_u\,, \quad v^i = \rho^{-1}\varepsilon^{ij}\partial_j\psi_u\,. \tag{14}$$

[e]Some authors consider only the special case where density is a constant, $\nabla \cdot u = 0$. Note also that the same fluid may support both compressible and incompressible flow under different conditions, so our definitions refer to the flow and not just to the fluid.

Then their Lie bracket $[u, v]$ has stream function $\rho^{-1}\nabla\psi_v \times \nabla\psi_u$:

$$\begin{aligned}[u,v]^i &= \rho^{-1}\varepsilon^{il}\partial_l\left\{\rho^{-1}\varepsilon^{jk}(\partial_j\psi_v)(\partial_k\psi_u)\right\}, \\ \psi_{[u,v]} &= \rho^{-1}\varepsilon^{jk}(\partial_j\psi_v)(\partial_k\psi_u)\,.\end{aligned} \tag{15}$$

2.2. *Abelian gauge fields as the dual of* $\mathbf{SVect}(M, \mu)$

The Lie algebra $\mathrm{SVect}(M, \mu)$ is akin to the Lie algebra of angular velocities of a rigid body. The angular momenta are in the dual space to angular velocities, and satisfy the angular momentum Poisson algebra. As explained in App. A, the dual of any Lie algebra is a Poisson manifold. This is interesting because the observables of a classical dynamical system are real-valued functions on a Poisson manifold. The dual of the Lie algebra $\mathcal{G} = \mathrm{SVect}(M, \mu)$ is the space of Abelian gauge fields modulo gauge transformations,

$$\mathcal{G}^* = \mathrm{SVect}(M,\mu)^* = \Omega^1(M)/d\Omega^0(M)\,. \tag{16}$$

This fact is well known in hydrodynamics (see Ref. 15), though it is usually not thought of in terms of gauge fields. To see this duality, we define the pairing (A, u) between gauge fields and volume-preserving vector fields by integrating the scalar $A(u)$ with respect to μ:

$$(A,u) = \mu_u(A) = \int_M A(u)\mu = \int A_i u^i \rho\, d^2x\,. \tag{17}$$

The pairing $\mu_u(A)$ is also called the moment map. It is a gauge-invariant pairing. Under a gauge transformation $A \mapsto A' = A + d\Lambda$ for any scalar $\Lambda(x)$,

$$\mu_u(A') - \mu_u(A) = \int_M (\partial_i\Lambda)u^i\mu = -\int \Lambda\partial_i(\rho u^i)d^2x = 0\,, \tag{18}$$

since u is volume-preserving. We assume that gauge fields and gauge transformations Λ vanish on the boundary ∂M or at infinity. We make no such assumption about the vector fields.

Gauge Fixing: it is occasionally convenient to "gauge-fix," i.e. pick a coset representative for $\Omega^1(M)/d\Omega^0(M)$. Under a gauge transformation, $A_i' = A_i + \partial_i\Lambda$. We can pick Λ such that $A_1' = A_1 + \partial_i\Lambda = 0$, so that we are left with only one component of the gauge field A_2'. We can still make an x^1-independent "residual" gauge transformation, $A_2'' = A_2' + \partial_2\tilde{\Lambda}(x^2)$ to eliminate any additive term in A_2' depending on x^2 alone. Suppose we have gauge fixed on a particular spatial initial value surface at time $t = 0$. Unlike in Yang–Mills theory, the equations of motion of our theory are purely dynamical. They only evolve the gauge-fixed fields forward in time, and do not contain any further constraints. In effect, after gauge fixing, we will be left with one propagating field degree of freedom.

2.3. *Differentials of gauge invariant charges are volume-preserving vector fields*

We should regard $\mathrm{SVect}(M,\mu)^*$, the space of gauge fields modulo gauge transformations, as the Poisson manifold of some dynamical system. Real-valued functions on this space (i.e. gauge-invariant functions $f(A)$) are the observables. Given such an $f(A)$, we can define its differential $df(A)$:

$$(df(A))^i = \rho^{-1}(x)\frac{\delta f}{\delta A_i(x)} \equiv \rho^{-1}\delta^i f\,. \tag{19}$$

For each equivalence class of gauge fields $[A] = \{A|A \sim A + d\Lambda\}$, the differential[f] defines a vector field $df^i\partial_i$ on M. If $f(A)$ is nonlinear, the vector field $(df(A))^i(x)$ changes as $A \in \mathcal{G}^*$ changes. Suppose $f(A)$ is gauge-invariant and differentiable. Then we can show that its differential df^i is a volume-preserving vector field on M: $\partial_i(\rho df^i) = 0$. To see this, note that gauge invariance implies that the change in f under any gauge transformation $\delta A_i = \partial_i\Lambda$ must vanish:

$$\begin{aligned} 0 = \delta f &= \int \frac{\delta f}{\delta A_i(x)}\delta A_i(x)d^2x \\ &= \int \frac{\delta f}{\delta A_i(x)}\partial_i\Lambda d^2x \\ &= -\int \partial_i\left(\frac{\delta f}{\delta A_i(x)}\right)\Lambda(x)d^2x\,. \end{aligned} \tag{20}$$

Since $\Lambda(x)$ is arbitrary, it must follow that $\partial_i(\delta f/\delta A_i) = 0$. So the differential of a gauge-invariant function can be regarded as an element of the Lie algebra $\mathcal{G} = \mathrm{SVect}(M,\mu)$.

The simplest gauge-invariant observable is the field strength 2-form:

$$F = dA = \frac{1}{2}F_{ij}\,dx^i \wedge dx^j\,, \quad F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ij}B\,, \tag{21}$$

where $B = \varepsilon^{ij}\partial_i A_j = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field. The differential of F,

$$((dF)(A))^k = \rho^{-1}\frac{\delta F_{ij}(x)}{\delta A_k(y)} = \rho^{-1}(\delta^k_j\partial_i\delta(x-y) - \delta^k_i\partial_j\delta(x-y)) \tag{22}$$

is a volume-preserving vector field on M for each A:

$$\begin{aligned} \partial_k(\rho(dF)^k) &= \partial_k(\delta^k_j\partial_i\delta(x-y) - \delta^k_i\partial_j\delta(x-y)) \\ &= (\partial_i\partial_j - \partial_j\partial_i)\delta(x-y) = 0\,. \end{aligned} \tag{23}$$

[f] The differential $(df(A))^i$ is regarded as a vector field on M for each A and should not be confused with the closely related exterior derivative df, which is a one-form on $\mathcal{G}^*$. However, as we will see later (Sec. 4 and App. A), on any symplectic leaf of $\mathrm{SVect}(M,\mu)^*$ with symplectic form ω, the one-form df determines the canonical vector field V_f via $\omega(V_f,.) = df(.)$. V_f is a vector field on the leaf, and its relation to the differential is $V_f(A) = ad^*_{df}\,A$.

Similarly, the differential of the magnetic field,

$$((dB)(A))^k = \rho^{-1}\frac{\delta B(x)}{\delta A_k(y)} = \rho^{-1}\varepsilon^{ij}\partial_i\delta_j^k\delta(x-y) = \rho^{-1}\varepsilon^{ik}\partial_i\delta(x-y) \tag{24}$$

is volume-preserving $\partial_k(\rho(dB)^k) = \varepsilon^{ik}\partial_i\partial_k\delta(x-y) = 0$. We can regard the moment maps $\mu_u(A)$ as linear gauge-invariant observables. The differential of $\mu_u(A)$ is the volume-preserving vector field u, for all gauge fields A.

Other gauge-invariant observables $f(A)$ we will be interested in are "charges": integrals over M with respect to μ, of a local gauge-invariant scalar function $\mathcal{F}$. $\mathcal{F}$ can depend on A only through the field strength two-form $F = dA$. The analogue of the Chern–Simons three-form, vanishes identically since A is a one-form on space, not space–time. The quotient of dA and the nondegenerate volume two-form (dA/μ) is a scalar function on M. Then

$$f(A) = \int \mathcal{F}\left(\sigma, \left(\frac{dA}{\mu}\right), v^i\partial_i\left(\frac{dA}{\mu}\right), w^{ij}\partial_i\partial_j\left(\frac{dA}{\mu}\right), \dots\right)\mu\,, \tag{25}$$

where σ is a scalar function and v^i, w^{ij}, etc. are arbitrary but fixed contravariant tensor fields. We can get an explicit formula for the differential of such a gauge-invariant charge. Using $dA/\mu = B/\rho$ and

$$\frac{\partial B}{\partial A_i} = -\varepsilon^{ij}\partial_j\delta^2(x-y)\,; \quad \frac{\delta\partial_k(B/\rho)(x)}{\delta A_i(y)} = -\varepsilon^{ij}\partial_k\left(\frac{1}{\rho}\partial_j\delta^2(x-y)\right); \dots\,, \tag{26}$$

we get upon integrating by parts,

$$df(A)^i = \rho^{-1}\frac{\delta f}{\delta A_i} = \rho^{-1}\varepsilon^{ij}\partial_j\left[\left(\frac{\partial\mathcal{F}}{\partial(B/\rho)}\right) - \left(\frac{1}{\rho}\partial_k\left(\frac{\partial\mathcal{F}}{\partial\partial_k(B/\rho)}\right)\right) + \cdots\right]. \tag{27}$$

Due to the antisymmetry of ε^{ij}, it follows that df is volume-preserving $\partial_i(\rho df^i) = 0$. Moreover, if f is gauge-invariant and of the form assumed above, then its differential is also gauge-invariant.

Two families of gauge-invariant charges which play an important role in our theory are I_n and H_n defined below. Let

$$\begin{aligned} I_n(A) &= \int_M \left(\frac{dA}{\mu}\right)^n \mu = \int \left(\frac{B}{\rho}\right)^n \rho\, d^2x \\ &= \int_M \left(\frac{B}{\rho}\right)^{n-1} dA\,, \quad n = 1, 2, 3, \dots\,, \\ (dI_n)^i &= \frac{1}{\rho}\frac{\delta I_n}{\delta A_i} = \frac{1}{\rho}n\varepsilon^{ij}\partial_j\left(\left(\frac{B}{\rho}\right)^{n-1}\right). \end{aligned} \tag{28}$$

Their differentials are volume-preserving $\partial_i(\rho(dI_n(A))^i) = 0$ since ε^{ij} is antisymmetric. Note that $I_1 = \int_M dA = 0$. We assume B vanishes sufficiently fast at infinity and do not consider I_n for $n < 0$. Given a scalar function σ on M we can construct additional gauge-invariant charges. These are similar to the I_n, except that we multiply by σ before integrating over M:

$$H_n(A) = \int \left(\frac{dA}{\mu}\right)^n \sigma\mu = \int \left(\frac{B}{\rho}\right)^n \sigma\rho\, d^2x\,. \tag{29}$$

More generally, one can replace $(B/\rho)^n$ by an arbitrary function of B/ρ. The differential of H_n is volume-preserving:

$$(dH_n(A))^i = \frac{n\varepsilon^{ij}}{\rho}\partial_j\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right). \tag{30}$$

Gauge-invariant observables depending on the volume form μ and a metric g_{ij} on M will play an important role in determining the dynamics of our theory. They are given in Subsec. 5.2.

There are other interesting gauge-invariant observables such as the circulation $C_\gamma(A) = \int_0^1 A_i \frac{d\gamma^i(s)}{ds} ds$ and its exponential, the Wilson loop. Since these observables are concentrated on one-dimensional curves on M, they may fail to be differentiable and their differentials exist only as distributional vector fields. They require a more careful analysis and are not considered in this paper. Henceforth, when we say observable, we will mean gauge-invariant observables that are differentiable, in which case, their differentials are guaranteed to be volume-preserving vector fields on M.

3. Poisson Brackets

3.1. *Definition of Poisson bracket on* $\mathcal{G}^* = \mathbf{SVect}(M,\mu)^*$

The Lie algebra structure of volume-preserving vector fields $\mathcal{G} = \mathrm{SVect}(M,\mu)$ can be used to define a Poisson structure on its dual (see App. A, Refs. 26–28 and 32). The dual space $\mathcal{G}^* = \mathrm{SVect}(M,\mu)^* = \Omega^1(M)/d\Omega^0(M)$ of gauge field one-forms then becomes a Poisson manifold. Observables are real-valued functions on it. The Poisson bracket (p.b.) between gauge-invariant observables $f(A)$ and $g(A)$ with volume-preserving differentials df and dg, is defined using the pairing (A,u) between $\mathcal{G}$ and $\mathcal{G}^*$ and the Lie algebra bracket $[df, dg]$:

$$\begin{aligned}
\{f,g\}(A) &\equiv (A,[df,dg]) = \int_M A_i[df,dg]^i\mu \\
&= \int d^2x\, \rho A_i[df^j\,\partial_j\, dg^i - dg^j\,\partial_j\, df^i] \\
&= \int d^2x\,\rho A_i\left[\rho^{-1}\frac{\delta f}{\delta A_j}\partial_j\left(\rho^{-1}\frac{\delta g}{\delta A_i}\right) - \rho^{-1}\frac{\delta g}{\delta A_j}\partial_j\left(\rho^{-1}\frac{\delta f}{\delta A_i}\right)\right] \\
&= \int A_i\big[(\delta^j f)\partial_j(\rho^{-1}\delta^i g) - (\delta^j g)\partial_j(\rho^{-1}\delta^i f)\big]\, d^2x\,,
\end{aligned} \tag{31}$$

where $\delta^i = \frac{\delta}{\delta A_i}$. The antisymmetry, linearity and Jacobi identity follow from the corresponding properties of the Lie bracket. The Leibnitz rule follows from the Leibnitz rule for differentials.

The p.b. (31) preserves the class of gauge-invariant functions. Suppose f, g are gauge-invariant. Then $df,\ dg \in \mathrm{SVect}(M,\mu)$. Recall (Subsec. 2.3) that if f and g are gauge-invariant, so are their differentials df^i and dg^i. It follows that $[df,dg]^i$ is also gauge-invariant. Now, under a gauge transformation $A' = A + d\Lambda$,

$$\begin{aligned}\{f,g\}(A') &= (A',[df,dg](A')) = (A',[df,dg](A))\\ &= \int A'_i([df,dg](A))^i\mu \Rightarrow \{f,g\}(A') - \{f,g\}(A)\\ &= -\int \Lambda\partial_i(([df,dg](A))^i\rho)d^2x = 0\,.\end{aligned} \tag{32}$$

Thus (31) is a gauge-invariant Poisson bracket.

A gauge-invariant observable $f(A)$ defines canonical transformations on the Poisson manifold $\mathcal{G}^*$ via the p.b. Suppose $g(A)$ is any observable, then its Lie derivative under the flow generated by f is $\mathcal{L}_{V_f}g(A) = \{f,g\}(A) = (A,[df,dg])$ (see Sec. 4).

Example. The p.b. of two moment maps $\mu_u(A)$ and $\mu_v(A)$ is

$$\{\mu_u,\mu_v\}(A) = \mu_{[u,v]}(A) = \int_M A_i(u^j\partial_j v^i - v^j\partial_j u^i)\mu\,. \tag{33}$$

If u and v are volume-preserving, so is $[u,v]$; therefore, if μ_u and μ_v are gauge-invariant functions of A, so is $\mu_{[u,v]}$. Moreover, the canonical transformation generated by the moment map μ_u is just the Lie algebra coadjoint action (see App. A and Sec. 4):

$$\mathcal{L}_{V_{\mu_u}}f(A) = \{\mu_u,f\}(A)\,,\quad \mathcal{L}_{V_{\mu_u}}A = ad^*_u\,A\,. \tag{34}$$

3.2. *Poisson brackets of gauge fields*

Equivalence classes of gauge fields are coordinates on our Poisson manifold $\mathcal{G}^* = \mathrm{SVect}(M,\mu)^*$. Thus, an explicit formula for the "fundamental" p.b. between components of the gauge field $A_i(x)$ is useful. This will also facilitate a comparison with electrodynamics. We will show that the p.b. between gauge field components is

$$\{A_i(x),A_j(y)\} = \delta^2(x-y)\left[A_j(y)\frac{\partial}{\partial y^i}\rho^{-1}(y) - A_i(x)\frac{\partial}{\partial x^j}\rho^{-1}(x)\right], \tag{35}$$

where the derivatives act on everything to their right. Though this formula for $\{A_i(x),A_j(y)\}$ looks a bit complicated, the right-hand side is linear in gauge fields. Thus, our Poisson algebra is actually a Lie algebra like the Lie algebra of angular momenta $\{L_i,L_j\} = \epsilon_{ijk}L_k$.

Recall that electrodynamics is based on the Heisenberg algebra between the spatial components of the gauge field and the spatial components of the electric field. In two spatial dimensions we have (before any gauge fixing)

$$\begin{aligned}\{A_i(\mathbf{x},t),E^j(\mathbf{y},t)\} &= \delta^2(\mathbf{x}-\mathbf{y})\delta_i^j\,,\\ \{A_i(\mathbf{x},t),A_j(\mathbf{y},t)\} &= 0\,,\\ \{E^i(\mathbf{x},t),E^j(\mathbf{y},t)\} &= 0\,.\end{aligned} \tag{36}$$

While in electrodynamics the electric field is canonically conjugate to the gauge field, this is *not* the case in our theory. The components of the gauge field in our theory obey p.b. relations with each other, without being canonically conjugate. This is not unusual. For instance, the components of angular momentum form a closed Poisson algebra though none of them is canonically conjugate to another. This is a generic feature of degenerate Poisson manifolds where canonically conjugate variables can only be chosen on individual symplectic leaves (the concentric spheres in the case of angular momenta).

Gauge fields obeying p.b. not involving the electric field are *not* alien to conventional Yang–Mills theory. For example, in a coordinate system where initial values of fields are specified on a null cone at past timelike infinity, the transverse components of gauge fields satisfy p.b. among themselves as shown by Rajeev and Turgut[2,3]

$$\{A^a_{ib}(z,R),A^c_{jd}(z',R')\} = \frac{1}{2}\,\delta^a_d\delta^c_b q_{ij}(z)\delta(z-z')\,\mathrm{sgn}(R-R')\,, \tag{37}$$

where z^i are transverse angular coordinates, q_{ij} is the round metric on S^2, R is a radial coordinate and a,b are color indices. In fact, our original motivation for studying the dynamical system in this paper was to find a toy-model that shared this feature with Yang–Mills theory.

Now we will establish (35) using (31) and the relation between the p.b. of functions and those between the "coordinates" $A_i(x)$:

$$\{f,g\}(A) = \int d^2x\,d^2y\{A_i(x),A_j(y)\}\frac{\delta f}{\delta A_i(x)}\,\frac{\delta g}{\delta A_j(y)}\,. \tag{38}$$

We rewrite $\{f,g\}(A)$ from (31) to make it look as this:

$$\begin{aligned}&\int d^2x\,A_j\left[\frac{\delta f}{\delta A_i}\partial_i\left(\frac{1}{\rho}\frac{\delta g}{\delta A_j}\right)-\frac{\delta g}{\delta A_i}\partial_i\left(\frac{1}{\rho}\frac{\delta f}{\delta A_j}\right)\right]\\ &\quad=\int d^2x\,d^2y\,\delta^2(x-y)A_j(y)\left[\frac{\delta f}{\delta A_i(x)}\frac{\partial}{\partial y^i}\left(\frac{1}{\rho(y)}\frac{\delta g}{\delta A_j(y)}\right)-f\leftrightarrow g\right]\\ &\quad=\int d^2x\,d^2y\,\delta^2(x-y)A_j(y)\frac{\partial}{\partial y^i}\left[\frac{1}{\rho(y)}\frac{\delta f}{\delta A_i(x)}\,\frac{\delta g}{\delta A_j(y)}-f\leftrightarrow g\right]\\ &\quad=\int d^2x\,d^2y\,\delta^2(x-y)\left[A_j(y)\frac{\partial}{\partial y^i}\rho^{-1}(y)-A_i(x)\frac{\partial}{\partial x^j}\rho^{-1}(x)\right]\frac{\delta f}{\delta A_i(x)}\frac{\delta g}{\delta A_j(y)}\,.\end{aligned} \tag{39}$$

Finally we read off the desired expression (35).

4. Structure of Poisson Algebra of Observables

In this section we give the canonical action of $\mathrm{SDiff}(M,\mu)$ (generated via p.b.) on the Poisson manifold $\mathcal{G}^* = \mathrm{SVect}(M,\mu)^*$ (see also App. A and Ref. 15). Symplectic leaves are coadjoint orbits of $\mathrm{SDiff}(M,\mu)$. The Poisson algebra of gauge-invariant observables is degenerate. The enstrophy invariants I_n of hydrodynamics are an infinite number of Casimirs. They are constant on the coadjoint orbits. Then we try to identify some of the simpler coadjoint orbits. We show that there is always at least one single-point orbit, that of the pure gauge configuration. If M has nonvanishing first cohomology, then we show that there are finite dimensional symplectic leaves lying inside $H^1(M)\setminus\{0\}$. This also shows that I_n could not be a complete set of coadjoint orbit invariants. If M is simply connected, we show that the only single-point orbits consist of the configurations for which dA/μ is constant. For simply connected M, we also argue that the orbit of $[A]\in\mathcal{G}^*$ for which dA/μ is not constant, is infinite dimensional. We identify the isotropy sub-algebra and tangent space of such an orbit and give an example where dA/μ is circularly symmetric. However, this analysis is far from complete. It would be useful to find a nice coordinate system on these orbits and get the symplectic structure in explicit form so as to study Hamiltonian reduction.

4.1. *Coadjoint action of* $\mathrm{SDiff}(M,\mu)$ *on Poisson algebra*

The Poisson manifold $\mathcal{G}^* = \Omega^1(M)/d\Omega^0(M) = \{A\in\Omega^1(M)|A\sim A+d\Lambda\}$ carries the coadjoint action of $\mathrm{SDiff}(M,\mu)$. For example, on a simply connected region M, $\mathcal{G}^*$ may be identified with the space of scalar functions $f=(dA/\mu)=(B/\rho)$ on M. The coadjoint action is the pull-back action of volume-preserving diffeomorphisms $\phi\in\mathrm{SDiff}(M,\mu)$ on functions $\phi^*f = f\circ\phi$. The action on equivalence classes of gauge fields is also the pull back $Ad^*_\phi[A]=[\phi^*A]$. For infinitesimal $\phi(t)=1+ut$ we get the Lie algebra coadjoint action of $u\in\mathrm{SVect}(M,\mu)$ on $\mathcal{G}^*$:

$$ad^*_u A = -\mathcal{L}_u A\,. \tag{40}$$

This action of $\mathrm{SDiff}(M,\mu)$ on $\mathcal{G}^*$ is canonical, i.e. there is a function on $\mathcal{G}^*$ (a gauge-invariant observable) that generates the coadjoint action via the p.b. The generating function is the moment map $\mu_u(A)$. To see this, suppose u is a volume-preserving vector field, then

$$\{\mu_u(A), A_j\} = -(\mathcal{L}_u A)_j\,. \tag{41}$$

To show this we begin with the right-hand side of (41),

$$\begin{aligned}\mathcal{L}_u A &= d(i_uA) + i_u(dA)\\ &= d(u^iA_i) + \frac{1}{2}i_u\{(\partial_jA_i - \partial_iA_j)dx^i\wedge dx^j\}\\ &= (\partial_jA_iu^i + A_i\partial_ju^i)dx^j + \frac{1}{2}(\partial_jA_i-\partial_iA_j)(u^j\,dx^i - dx^j\,u^i)\\ &= \{A_i\partial_ju^i + u^i\partial_iA_j\}dx^j\,.\end{aligned} \tag{42}$$

On the other hand, using (35), $\{\mu_u(A), A_k(z)\}$ is equal to

$$\int d^2x\, d^2y \{A_i(x), A_j(y)\} \frac{\delta \mu_u}{\delta A_i(x)} \frac{\delta A_k(z)}{\delta A_j(y)}$$

$$= \int d^2x\, d^2y\, \delta^2(x-y) \left[A_j(y) \frac{\partial}{\partial y^i} \rho^{-1}(y) - A_i(x) \frac{\partial}{\partial x^j} \rho^{-1}(x) \right] \rho(x) u^i(x) \delta^j_k \delta^2(z-y)$$

$$= \int d^2y\, A_k(y) \rho(y) u^i(y) \frac{\partial}{\partial y^i} (\rho^{-1}(y) \delta^2(y-z)) - \int d^2x A_i(x) \delta^2(z-x) \frac{\partial u^i(x)}{\partial x^k}$$

$$= -\rho^{-1} \partial_i (A_k \rho u^i) - A_i \partial_k u^i = -u^i \partial_i A_k - A_i \partial_k u^i = -(\mathcal{L}_u A)_k \,. \tag{43}$$

We integrated by parts ($A = 0$ on ∂M) and used $\partial_i(\rho u^i) = 0$. Thus, we obtain (41).

4.2. *Center of Poisson algebra*

From Subsec. 4.1, the symplectic leaves of $\mathrm{SVect}(M, \mu)^*$ are homogeneous symplectic manifolds, identified with coadjoint orbits of $\mathrm{SDiff}(M, \mu)$. Is the Poisson algebra of functions on $\mathrm{SVect}(M, \mu)^*$ degenerate? What are the Casimirs that constitute its center? Casimirs are unchanged under canonical transformations. So they are constant on symplectic leaves. A complete set of such Casimirs would allow us to distinguish between distinct leaves. Since the symplectic leaves are coadjoint orbits of $\mathrm{SDiff}(M, \mu)$, it suffices to find observables that commute with the moment maps $\mu_u(A)$, which generate the coadjoint action.

We first observe that the center of the Poisson subalgebra of linear observables $\mu_u(A)$ is trivial. $\{\mu_u, \mu_v\}(A)$ vanishes for all A iff $[u, v] = 0$. But the center of $\mathrm{SVect}(M, \mu)$ is trivial. This is seen by writing u and v in terms of their stream functions ψ_u, ψ_v. Taking some simple choices for ψ_v in the condition $[u, v] = 0$ will imply that $\psi_u = 0$. Thus none of the $\mu_u(A)$ lie in the center of the Poisson algebra of gauge-invariant functions, since they do not even commute with each other. Thus we need to look elsewhere to find the Casimirs of our Poisson algebra. It turns out that the charges $I_n = \int (dA/\mu)^n \mu$ are central observables. The monomials $(dA/\mu)^n$ can be replaced by any scalar function of (dA/μ). We found that I_n are Casimirs by showing that they Poisson commute with the moment maps. However, this involves lengthy calculations. For example, in App. B we show that I_n commute with each other and in App. C we show that I_2 commutes with $\mu_u(A)$ for uniform μ.

However, we found that I_n are closely related to the enstrophy invariants of hydrodynamics, which are known to be conserved quantities in Eulerian hydrodynamics. There is a simple argument in Ref. 15 based on earlier work[21,20,22] that leads to the conclusion that I_n are constant on coadjoint orbits. The argument is that the action of $\mathrm{SDiff}(M, \mu)$ is merely to change coordinates in the integral defining I_n. Since this integral is independent of the choice of coordinates, I_n must be invariant under the coadjoint action. Though I_n for $n = 1, 2, 3, \ldots$ are all Casimirs, they are not a complete set of orbit invariants. Their level sets do not necessarily distinguish the coadjoint orbits (see Subsec. 4.3). For some remarks on additional invariants, see Sec. 9 of Ref. 15. Our explicit calculations of $\{I_n, \mu_u\}$ given in Apps. B and C suggested to us how I_n could be modified in order to get an independent infinite sequence of conserved quantities (not Casimir invariants) for our choice of Hamiltonian; see Subsec. 6.3.

4.3. *Finite dimensional symplectic leaves in $H^1(M)$*

The simplest symplectic leaf in the dual of the Lie algebra of the rotation group is the origin of angular momentum space $\{L_i = 0\}$. It consists of a single point. Of course, one can also characterize the orbit $\{L_i = 0\}$ as the zero set of the Casimir $L^2 = 0$. Can we get a similar explicit characterization of the simplest symplectic leaves of $\mathrm{SVect}(M, \mu)^*$?

Orbits of closed one forms and zero set of Casimirs I_n

The simplest symplectic leaf in $\mathrm{SVect}(M, \mu)^*$ should be the orbit of exact one-forms. Suppose $A = d\Lambda$ is an exact one-form. Then under the coadjoint action of $u \in \mathrm{SVect}(M, \mu)$ it goes to $A' = A + ad_u^* A$:

$$A'_i = A_i - (\mathcal{L}_u A)_i = \partial_i \Lambda - (\partial_j \Lambda)(\partial_i u^j) + u^j \partial_j \partial_i \Lambda = \partial_i(\Lambda - u^j \partial_j \Lambda)\,. \qquad (44)$$

We see that a pure gauge is mapped to a pure gauge under the Lie algebra coadjoint action. Thus the tangent space to the orbit of exact one-forms is trivial. So pure gauges form a single-point orbit.

After the pure gauges, the next simplest configurations are closed but inexact one-forms $dA = 0$, $A \neq d\Lambda$. Suppose M is a manifold with nonvanishing first cohomology. What is the orbit of an element of $H^1(M)$? In particular, is the orbit a finite dimensional manifold? Does the orbit lie within $H^1(M)$? The answers to both these questions is affirmative. Suppose A is an exact one-form, $F = dA = 0$. Under the Lie algebra coadjoint action, $A' = A + ad_u^* A = A - \mathcal{L}_u A$:

$$dA' = d(A - \mathcal{L}_u A) = -d(di_u + i_u d)A = -di_u\, dA = 0\,. \qquad (45)$$

The tangent space to the orbit of a closed one-form contains only closed one-forms. Since the space of closed one-forms on a two-dimensional manifold is finite dimensional, the tangent space to the orbit must be finite dimensional. Moreover, we have shown that the exact one-forms form a single point orbit by themselves. Thus, at the infinitesimal level, the orbit of a nontrivial element of $H^1(M)$ must lie within $H^1(M) \setminus \{0\}$. This view will be reinforced by considering the zero set of Casimirs:

$$I_n = \int \left(\frac{dA}{\mu}\right)^n \mu = 0\,. \qquad (46)$$

I_n are constant on coadjoint orbits. Thus, the orbits must be contained in their level sets. I_{2n} is the integral of a positive quantity and therefore vanishes iff $dA = 0$. If $dA = 0$ then I_{2n+1} is also zero. Thus the level set $I_n = 0$ is the space of closed one-forms on M. Thus, the coadjoint orbit of a closed one-form must lie in $H^1(M)$.

Since we already established that the pure gauges form a single-point orbit, this means I_n cannot be a complete set of Casimirs if $H^1(M)$ is nontrivial.

For example, if M is the plane, then $H^1(\mathbf{R}^2)$ consists only of the equivalence class of pure gauge configurations and the zero set of I_n contains only one single-point orbit $[A] = 0$. The same is true of the 2-sphere $\mathbf{S}^2$ which has trivial first cohomology. For the 2-torus $H^1(\mathbf{T}^2) \simeq \mathbf{R}^2$. For $\mathbf{T}^2$, there must be symplectic leaves which are submanifolds of $\mathbf{R}^2 \setminus \{0\}$. It is interesting to find the orbits of cohomologically nontrivial gauge fields more explicitly as well as the induced symplectic structure. However, we do not investigate this further since the Hamiltonian we pick (66) vanishes on closed one-forms. There is no interesting dynamics on the finite dimensional symplectic leaves we have described above. Therefore, we turn to the case where M is simply connected and try to characterize the orbits of gauge fields that are not closed.

4.4. *Symplectic leaves when M is simply connected*

In the case of the rigid body, symplectic leaves other than $L_i = 0$ are concentric spheres of nonzero radius, all two-dimensional symplectic manifolds. These leaves may also be characterized as nonzero level sets of L^2. By analogy, we would like to find the orbits in $\mathrm{SVect}(M,\mu)^*$ of one-forms that are not closed. They must lie within nonzero level sets of I_n. Can we say something more about them, such as whether they are finite dimensional? We address these questions below, assuming M is simply connected.

By Poincaré's lemma, all the information in a gauge field on a simply connected M can be stored in the two-form field strength, $F = dA$. Then, the dual of the Lie algebra $\mathrm{SVect}(M,\mu)^*$ may be identified with the space of scalar functions on M:

$$\mathrm{SVect}(M,\mu)^* = \left\{ f = \frac{dA}{\mu} \right\}. \tag{47}$$

The coadjoint action of $\phi \in \mathrm{SDiff}(M,\mu)$ is just the pull back

$$Ad^*_\phi f = \phi^* f = f \circ \phi \tag{48}$$

and the Lie algebra coadjoint action of $u \in \mathrm{SVect}(M,\mu)$ is

$$f \mapsto f + ad^*_u f = f - \mathcal{L}_u f = f - u^i \partial_i f\,. \tag{49}$$

Thus the coadjoint orbit of f and the tangent space to the orbit at f are

$$\begin{aligned} \mathcal{O}_f &= \{ f \circ \phi \,|\, \phi \in \mathrm{SDiff}(M,\mu) \}\,, \\ T_f\mathcal{O} &= \{ f - u^i \partial_i f \,|\, u \in \mathrm{SVect}(M,\mu) \}\,. \end{aligned} \tag{50}$$

The picture that emerges from our analysis below, is that there are two types of orbits when M is simply connected. Orbits of the first type contain only a single point, namely a constant function $f = dA/\mu = c$. In the case of the plane with the uniform measure, the only one-point orbit is $\{f = (dA/\mu) = 0\}$, consisting of the pure gauge configuration. The orbits of the second type are all infinite dimensional. They are the orbits of nonconstant functions f. In the case of the plane with uniform measure, we expect an infinite number of such orbits of the second type, each contained within a level set of $\{I_n, n = 1, 2, 3, \ldots\}$, though we do not rule out the existence of more than one such orbit in any one level set of the I_n.

Single point orbits

We show that constant functions $f = (dA/\mu) = c$ are the only single point coadjoint orbits in the dual of the Lie algebra $\mathrm{SVect}(M, \mu)^*$ when M is simply connected. They lie within the level sets $I_n = \mathrm{Vol}(M, \mu)c^n$ where $\mathrm{Vol}(M, \mu) = \int_M \mu$.

Suppose f is a constant. Then $\mathcal{L}_u f = 0$. So the constant functions form single point orbits. The case $f = 0$ corresponds to the closed and exact one-forms A which we already identified as a single point leaf if M is simply connected. Nonzero constant functions are not admissible elements of $\mathrm{SVect}(M, \mu)^*$ if $M = \mathbf{R}^2$ and μ is the uniform measure. But they are allowed if M is compact or if $\mu \to 0$ at infinity on noncompact M. Note that $I_n = \int f^n \mu = \mathrm{Vol}(M, \mu)c^n$ if $f = c$. So the constant functions lie in the level sets with $I_n = \mathrm{Vol}(M, \mu)c^n$.

We can go one step further and show that constant functions are the *only* single point orbits if M is simply connected. Suppose f is a single point orbit. Then $\mathcal{L}_u f$ must vanish for all volume-preserving vector fields u. Since M is simply connected, any such vector field can be written in terms of a stream function $u^i = \rho^{-1}\varepsilon^{ij}\partial_j\psi$. Then denoting derivatives by subscripts,

$$\begin{aligned} \mathcal{L}_u f &= \rho^{-1}\varepsilon^{ij}(\partial_j\psi)(\partial_i f) = \rho^{-1}(f_x\psi_y - f_y\psi_x)\,, \\ \mathcal{L}_u f &= 0 \Rightarrow f_x\psi_y = \psi_x f_y\,. \end{aligned} \tag{51}$$

This must be true for all stream functions ψ. Taking $\psi = x$ and $\psi = y$ successively tells us that f must be independent of both x and y and hence a constant. *We conclude that the only single point orbits in $\mathcal{G}^* = \mathrm{SVect}(M, \mu)^*$ are the constant functions $f = (dA/\mu) = c$ when M is simply connected.*

Orbit and stabilizer of nonconstant element of $\mathrm{SVect}(M, \mu)^*$

The stabilizer algebra $\mathrm{Stab}(f)$ or isotropy subalgebra of a function f is the set of all volume-preserving vector fields u^i which leave it fixed under the coadjoint action. The coset space $\mathrm{SVect}(M, \mu)/\mathrm{Stab}(f)$ then has the same dimension as the tangent space to the orbit containing f. For u to be in $\mathrm{Stab}(f)$ we need $\mathcal{L}_u f = 0$. Since M is simply connected we can express u in terms of its stream function $u^i = \rho^{-1}\varepsilon^{ij}\partial_j\chi$. The condition $\mathcal{L}_u f = 0$ becomes

$$\varepsilon^{ij}(\partial_i f)(\partial_j\chi) = 0 \Rightarrow \chi_x f_y - \chi_y f_x = 0 \tag{52}$$

which in vector notation[g] says that $\nabla f \times \nabla \chi = 0$. Colloquially, the gradient of χ must be everywhere parallel to the gradient of f. $\chi = cf$ is clearly a solution for any real number c, so the isotropy subalgebra is at least one-dimensional, provided f is not a constant. The product of two solutions as well as real linear combinations of solutions are again solutions to this linear PDE. To better understand the general solution of this PDE, let us first consider the specific example of a function $f(r,\theta)$ that is circularly symmetric.

Example: *Stabilizer and orbit of circularly symmetric function*

For example, let $f = (dA/\mu)$ be a nonconstant function depending only on the radial coordinate $r = \sqrt{x^2+y^2}$. Let us call its orbit by the name $\mathcal{O}_f$. The gradient ∇f points radially, so to speak. The isotropy subalgebra of any such function f is the space of stream functions χ with $\nabla f \times \nabla \chi = 0$. The solutions are stream functions $\chi(r)$ that are independent of θ:

$$\mathrm{Stab}(f) = \{\chi(r,\theta)\,|\,\partial_\theta \chi = 0\}\,. \tag{53}$$

In this case, the isotropy subalgebra is infinite dimensional. The coset space

$$\mathrm{SVect}(M,\mu)/\mathrm{Stab}(f) = \{\psi(r,\theta)|\psi \sim \psi + \chi(r)\} \tag{54}$$

has the same dimension as the tangent space to the orbit $T_f\mathcal{O}_f$ at f. Though the stabilizer is infinite dimensional, the orbit of $f(r)$ is infinite dimensional as well. For example, the coset space can be parametrized using an infinite number Fourier coefficients

$$\psi(r,\theta) = \sum_{n=1}^{\infty} \psi_n^{(c)}(r)\cos n\theta + \sum_{n=1}^{\infty} \psi_n^{(s)}(r)\sin n\theta\,. \tag{55}$$

We omitted the θ independent additive term in order to quotient out by $\mathrm{Stab}(f)$.

To summarize, the isotropy subalgebra of a nonconstant circularly symmetric function consists of all stream functions that are circularly symmetric. Moreover, the tangent space to the orbit may be identified with the coset space in which two stream functions are identified if they differ by one depending on r alone.

Nonconstant functions have infinite dimensional orbits

Using the circularly symmetric example as a guide, we can characterize the isotropy subalgebra of a general nonconstant function. The condition for χ to be in the isotropy subalgebra is $f_x\chi_y - f_y\chi_x = 0$. We will describe the set of all solutions χ to this linear PDE in a region of M where df is never the zero two-form. Such a region is guaranteed to exist by an analogue of the inverse function theorem, since we assumed that f was not a constant. Our answer is that the isotropy subalgebra

[g]We emphasize that this equation is independent of any metric on M.

of f consists of all stream functions χ which are constant along level sets of f. The level sets of f are necessarily one-dimensional curves in such a region.

To see this, we argue as follows. In a region where $f(x, y)$ has no local extrema, the level sets of f are one-dimensional. These level curves foliate the region. Pick a coordinate $\Theta(x, y)$ along the level curves. Then $\partial_\Theta f = f_\Theta = 0$. Also pick a coordinate R "transversal" to the level curves of f. What this means is that ∂_R and ∂_Θ should be linearly independent at each point (R, Θ). There is a lot of arbitrariness in the choice of these coordinates, and one certainly does not require any metric to define them. The volume form μ is nondegenerate, so the volume enclosed by the miniparallelogram determined by the tangent vectors ∂_R and ∂_Θ is nonzero. Denoting partial derivatives by subscripts, we get

$$\begin{aligned}\mu(\partial_R, \partial_\Theta) &= \rho dx \wedge dy(\partial_R, \partial_\Theta) \\ &= \frac{1}{2}\rho(dx \otimes dy - dy \otimes dx)\left(\frac{1}{R_x}\partial_x + \frac{1}{R_y}\partial_y, \frac{1}{\Theta_x}\partial_x + \frac{1}{\Theta_y}\partial_y\right) \\ &= \frac{1}{2}\rho\left(\frac{1}{R_x\Theta_y} - \frac{1}{R_y\Theta_x}\right) \\ &= \frac{1}{2}\rho\left(\frac{R_y\Theta_x - R_x\Theta_y}{R_x\Theta_y R_y\Theta_x}\right) \neq 0\,. \end{aligned} \tag{56}$$

Since ρ is nonvanishing, this means

$$R_y\Theta_x - R_x\Theta_y \neq 0\,. \tag{57}$$

The condition for χ to lie in the isotropy subalgebra becomes (using $f_\Theta = 0$)

$$\begin{aligned} f_x\chi_y - f_y\chi_x &= (f_R R_x + f_\Theta\Theta_x)(\chi_R R_y + \chi_\Theta\Theta_y) - x \leftrightarrow y \\ &= f_R R_x(\chi_R R_y + \chi_\Theta\Theta_y) - f_R R_y(\chi_R R_x + \chi_\Theta\Theta_x) \\ &= f_R\chi_\Theta(R_x\Theta_y - R_y\Theta_x)\,. \end{aligned} \tag{58}$$

We have already shown that $R_x\Theta_y - R_y\Theta_x \neq 0$. Moreover, f is nonconstant and $f_\Theta = 0$ so we must have $f_R \neq 0$. We conclude that $\chi_\Theta = 0$. In other words, the stream functions χ in the isotropy subalgebra are constant on level sets of f:

$$\mathrm{Stab}(f) = \{\text{Stream functions } \chi \text{ constant on level sets of } f\}\,. \tag{59}$$

It is clear that the space of such stream functions is closed under linear combinations and also under multiplication, as we observed earlier. The tangent space to the orbit at f is then identified with the coset space:

$$\begin{aligned} T_f\mathcal{O}_f &= \mathrm{SVect}(M, \mu)/\mathrm{Stab}(f) \\ &= \{\psi\,|\,\psi \sim \psi + \chi,\, \chi \text{ constant on level sets of} f\}\,. \end{aligned} \tag{60}$$

Both the stabilizer and the orbit are infinite dimensional, provided f is not constant, for the same reason as given in the example where f was circularly symmetric. *Thus, nonconstant $f = dA/\mu$ have infinite dimensional orbits.*

5. Hamiltonian

Having specified observables (gauge-invariant functions), their p.b. and phase-space (coadjoint orbit of $\mathrm{SDiff}(M,\mu)$), we need to pick a gauge-invariant Hamiltonian.

5.1. *Hamiltonian leading to ideal hydrodynamics*

There is more than one interesting way to pick a Hamiltonian. The classic choice leading to ideal hydrodynamics, requires a positive metric g_{ij} on M which defines a positive, symmetric, inner product on $\mathcal{G} = \mathrm{SVect}(M,\mu)$:

$$\langle u, v\rangle_{\mathcal{G}} = \int_M g(u,v)\mu = \int_M g_{ij}u^i v^j \mu\,. \tag{61}$$

The Hamiltonian of Eulerian hydrodynamics is then

$$H(u) = \frac{1}{2}\langle u, u\rangle_{\mathcal{G}} = \frac{1}{2}\int g_{ij}u^i u^j \rho\, d^2x\,. \tag{62}$$

There is no *a priori* relation between the metric g and the volume form μ. $\Omega_g = \sqrt{\det g}\, dx^1 \wedge dx^2$ need not equal μ. If $\mu = \Omega_g$, the theory is particularly natural as well as nontrivial. The Hamiltonian defines an inertia operator I (generalization of the inertia tensor of a rigid body, see App. A and Ref. 15) from the Lie algebra $\mathcal{G} = \mathrm{SVect}(M,\mu)$ to its dual $\mathcal{G}^* = \Omega^1(M)/d\Omega^0(M)$. Suppose u, v are volume-preserving vector fields. Then the inertia operator is defined by the equation $(Iu, v) = \langle u, v\rangle_{\mathcal{G}}$. In other words,

$$\int (Iu)_j v^j \mu = \int g_{ij}u^i v^j \mu \Rightarrow \int ((Iu)_j - g_{ij}u^i)v^j \mu = 0\,. \tag{63}$$

Since this is true for an arbitrary volume-preserving v, $((Iu)_j - g_{ij}u^i)$ must be a total derivative: $(Iu)_j - g_{ij}u^i = \partial_j \Lambda$ for some scalar function Λ which vanishes on the boundary of M or at infinity. We see that the metric does not determine the one-form Iu uniquely, but rather up to an exact one-form. Thus, the image Iu of a volume-preserving vector field $u \in \mathcal{G}$ is an equivalence class in $\mathcal{G}^*$ = gauge fields modulo gauge transformations. A coset representative is given by the one-form $A_i = g_{ij}u^j$. The equation of motion is the well-known Euler equation

$$\frac{\partial u}{\partial t} = -\nabla_u u - \nabla p\,, \quad \mathcal{L}_u \mu = 0\,. \tag{64}$$

Here $\nabla_u u$ is the covariant derivative (with respect to g_{ij}) of u along itself. The two equations fix the pressure $p(x,t)$ up to an additive constant. The Euler equations of hydrodynamics have a geometric interpretation.[15] The configuration space of a volume-preserving fluid flowing on the manifold M is the group of volume-preserving diffeomorphisms $\mathrm{SDiff}(M,\mu)$. The inner product (61) on the Lie algebra of this group can be extended to a right-invariant metric on the whole group by right translations by group elements. Then, by the least action principle, the time evolution of the fluid is given by geodesics on $\mathrm{SDiff}(M,\mu)$ with respect to this metric.

5.2. *Hamiltonian as magnetic energy*

Now we propose a Hamiltonian different from that of Eulerian hydrodynamics. It is a magnetic energy, natural from the point of view of Yang–Mills theory. The result will still be a theory of geodesics on $\mathrm{SDiff}(M, \mu)$, but with respect to a different right-invariant metric on this group than that implied by ideal hydrodynamics. Suppose M is a two-dimensional surface with volume form μ and metric g_{ij}. The metric did not play any role so far since the phase-space and Poisson structure are independent of it. But to specify the dynamics, we need the metric. Any two-dimensional manifold is conformally flat, so the information in g_{ij} is encoded in its volume form $\Omega_g = \sqrt{g}\, dx^1 \wedge dx^2$, where $g = \det g_{ij}$. We *do not* assume that μ is equal to Ω_g. Indeed, if that is the case, the dynamics is trivial. By analogy with Yang–Mills theory, we pick the manifestly gauge-invariant magnetic energy

$$H = \frac{1}{2} \int \left(\frac{F \wedge *F}{\Omega_g} \right) \mu \tag{65}$$

as our Hamiltonian. It can be written in a variety of equivalent ways:

$$\begin{aligned} H &= \frac{1}{2} \int \left(\frac{F}{\Omega_g} \right)^2 \mu = \frac{1}{2} \int (*F)^2 \mu = \frac{1}{2} \int \left(\frac{F}{\mu} \right)^2 \sigma\mu \\ &= \frac{1}{4} \int F_{ij} F^{ij} \mu = \frac{1}{2} \int \left(\frac{B^2 \sigma}{\rho} \right) d^2x \,. \end{aligned} \tag{66}$$

We find the last formula most useful in calculations. Here σ is the positive scalar function on M given by the square of the quotient of the two volume forms:

$$\sigma = \left(\frac{\mu}{\Omega_g} \right)^2 = \frac{\rho^2}{g} \,. \tag{67}$$

We will see that for the dynamics to be nontrivial, σ must not be constant. The Hamiltonian in (66) is the simplest choice that is gauge-invariant, quadratic in gauge fields and involves two derivatives. It is easy to see the equivalence of the various formulae for the magnetic energy in (66). Let $\epsilon^{ij} = g^{ik} g^{jk} \epsilon_{kl}$, $\epsilon^{ij} = -\epsilon^{ji}$, $\epsilon^{12} = 1/g$. Moreover, $g\epsilon^{ij} = \varepsilon^{ij}$. The Hodge dual of F is the scalar

$$*F = *(dx^i \wedge dx^j) \frac{1}{2} F_{ij} = \frac{1}{2} \sqrt{g} \epsilon^{ij} F_{ij} = \frac{1}{2\sqrt{g}} \sum_{ij} F_{ij} \epsilon_{ij} = \frac{B}{\sqrt{g}} \,. \tag{68}$$

$F \wedge *F$ is the two-form $(B^2/\sqrt{g}) dx^1 \wedge dx^2$. $((F \wedge *F)/\Omega_g) = B^2/g$ is a scalar. Hence

$$\frac{(F \wedge *F)}{\Omega_g} = (*F)^2 = \left(\frac{F}{\Omega_g} \right)^2 = \frac{B^2}{g} = \frac{B^2 \sigma}{\rho^2} = \left(\frac{F}{\mu} \right)^2 \sigma \,. \tag{69}$$

Moreover, $F_{ij} F^{ij} = 2(F_{12} F^{12})$. But $F^{12} = g^{1i} g^{2j} F_{ij} = (g^{11} g^{22} - g^{12} g^{21}) F_{12} = g^{-1} F_{12} = B/g$. Thus $F_{ij} F^{ij} = 2B^2/g$. This shows the equivalence of all the forms of the Hamiltonian given in (66).

The Hamiltonian (66) involves only the spatial components of the field strength since there is no time component for the gauge field in our theory. So there is no analogue of electric energy. Moreover, our theory is not relativistically covariant unlike electrodynamics. The dynamics determined by (66) is *not* equivalent to Eulerian hydrodynamics, though both theories share the same phase-space. To see this, note that if $\Omega_g = \mu$, then (66) reduces to the Casimir I_2 and has trivial dynamics while the Hamiltonian (62) continues to have nontrivial dynamics.

The differential of the Hamiltonian (66) is ($\frac{\partial B(y)}{\partial A_i(x)} = -\varepsilon^{ij}\partial_j \delta^2(x-y)$):

$$(dH(A))^i = \frac{1}{\rho}\frac{\delta H}{\delta A_i(x)} = \frac{1}{\rho}\varepsilon^{ij}\partial_j\left(\frac{B\sigma}{\rho}\right). \tag{70}$$

For each A, $dH(A)$ is volume-preserving $\partial_i(\rho dH^i) = \varepsilon^{ij}\partial_i\partial_j(B\sigma/\rho) = 0$. H generates a 1-parameter (time) family of diffeomorphisms of a coadjoint orbit of $\mathrm{SDiff}(M,\mu)$, which serves as the phase-space. The Hamiltonian vector field V_H is a vector field on a coadjoint orbit $\mathcal{O}$. At each tangent space $T_{[A(x)]}\mathcal{O}$ to an orbit, $V_H([A])$ is given by the coadjoint action of the Lie algebra element $dH(A)$, $V_H = ad^*_{dH(A)}$. The change in an observable f under such an infinitesimal canonical transformation is the Lie derivative with respect to V_H:

$$\frac{df(A)}{dt} = -\mathcal{L}_{V_H} f = ad^*_{dH}\, f = \{H, f\} = (A, [dH, df]) = \int A_i [dH, df]^i \mu\,. \tag{71}$$

Explicit equations of motion in local coordinates are given in Sec. 6.

5.3. *Inertia operator and inner product on $\mathcal{G}^*$ from Hamiltonian*

Recall (Subsec. 5.1) that the Hamiltonian of ideal hydrodynamics (62) defines an inner product on the Lie algebra of volume-preserving vector fields via the inertia operator. Here, the magnetic energy defines a positive inner product $\langle\cdot\,,\cdot\rangle_{\mathcal{G}^*}$ on the dual of the Lie algebra, $\mathcal{G}^* = \Omega^1(M)/d\Omega^0(M)$, via an inverse inertia operator h, obtained below. This inner product is degenerate if M has nonvanishing first cohomology. Away from degeneracies, one can in principle invert h to obtain an inertia operator and an inner product $\langle\cdot\,,\cdot\rangle_{\mathcal{G}}$ on the Lie algebra $\mathcal{G} = \mathrm{SVect}(M,\mu)$. It should be possible to extend $\langle\cdot\,,\cdot\rangle_{\mathcal{G}}$ to a metric on the group $G = \mathrm{SDiff}(M,\mu)$ by means of right translations. The equations of motion in $\mathcal{G}^* = \mathrm{SVect}(M,\mu)^*$ determined by H should be the projection of the geodesic flow on $G = \mathrm{SDiff}(M,\mu)$ with respect to this right-invariant metric. In this sense, the dynamics determined by the magnetic energy describes geodesics on $\mathrm{SDiff}(M,\mu)$. Eulerian hydrodynamics also describes geodesics on $\mathrm{SDiff}(M,\mu)$, but with respect to a different right-invariant metric.[15] Here, we obtain the inverse of the inertia operator and the inner product on $\mathrm{SVect}(M,\mu)^*$ explicitly. Let us begin by writing the Hamiltonian as a quadratic form on gauge fields. The result is

$$H = \frac{1}{2}\int A_i h^{il} A_l \rho\, d^2x \quad \text{with} \quad h^{il} = \frac{\varepsilon^{ij}\varepsilon^{kl}}{\rho}\left[\left(\partial_j \frac{\sigma}{\rho}\right)\partial_k + \frac{\sigma}{\rho}\partial_j\partial_k\right]. \tag{72}$$

h^{ij} plays the role of inverse inertia operator, mapping equivalence classes of gauge fields to volume-preserving vector fields by raising the index $A_j \mapsto h^{ij}A_j = (dH(A))^i$. To get (72), note that H may be written in terms of its differential,

$$H = \frac{1}{2}(A, dH(A)) = \frac{1}{2}\mu_{dH}(A) = \frac{1}{2}\int A_i (dH(A))^i \rho \, d^2x\,. \tag{73}$$

This is easily checked by integrating by parts assuming $B \to 0$ on ∂M:

$$\begin{aligned} H &= \frac{1}{2}\int \left(\frac{B^2\sigma}{\rho}\right) d^2x = \frac{\varepsilon^{ij}}{2}\int \left(\frac{B\sigma}{\rho}\right)\partial_i A_j \, d^2x \\ &= \frac{\varepsilon^{ij}}{2}\int \partial_j \left(\frac{B\sigma}{\rho}\right) A_i \, d^2x = \frac{1}{2}\int A_i \, dH^i \, \rho d^2x\,. \end{aligned} \tag{74}$$

As a consequence,

$$H = \frac{1}{2}\int A_i (hA)^i \mu \quad \text{where} \quad (hA)^i = h^{ij}A_j = dH^i = \frac{\varepsilon^{ij}}{\rho}\partial_j\left(\frac{B\sigma}{\rho}\right). \tag{75}$$

Writing this out we get,

$$(hA)^i = \frac{\varepsilon^{ij}\varepsilon^{kl}}{\rho}\partial_j\left(\frac{\sigma}{\rho}\partial_k A_l\right) = \frac{\varepsilon^{ij}\varepsilon^{kl}}{\rho}\left[\left(\partial_j\left(\frac{\sigma}{\rho}\right)\right)\partial_k + \left(\frac{\sigma}{\rho}\right)\partial_j\partial_k\right] A_l\,. \tag{76}$$

From this we read off the inverse of the inertia operator (72). Through h, the magnetic energy naturally defines a symmetric positive inner product on the dual of the Lie algebra, $\mathcal{G}^* = \Omega^1(M)/d\Omega^0(M)$. This inner product may be written in the following equivalent ways:

$$\langle A, \tilde{A}\rangle_{\mathcal{G}^*} = (A, h\tilde{A}) = \mu_{h\tilde{A}}(A) = \int A_i h^{ij}\tilde{A}_j \rho \, d^2x = \int B\tilde{B}\left(\frac{\sigma}{\rho}\right) d^2x\,. \tag{77}$$

Here B, $\tilde{B}$ are the magnetic fields corresponding to A, $\tilde{A}$. Positivity of the inner product is ensured since $\sigma(x) = (\mu/\Omega_g)^2 = (\rho^2/g) \geq 0$ and the integrand is positive:

$$\langle A, A\rangle_{\mathcal{G}^*} = 2H(A) = \int \left(\frac{B^2\sigma}{\rho}\right) d^2x \geq 0\,. \tag{78}$$

Symmetry of the inner product is shown by establishing the last equality in (77):

$$\begin{aligned} \int A_i (hA)^i \rho \, d^2x &= \varepsilon^{ij}\int A_i \partial_j\left(\frac{\tilde{B}\sigma}{\rho}\right) d^2x \\ &= -\varepsilon^{ij}\int \partial_j A_i \left(\frac{\tilde{B}\sigma}{\rho}\right) d^2x \\ &= \int \left(\frac{B\tilde{B}\sigma}{\rho}\right) d^2x\,. \end{aligned} \tag{79}$$

Inner product (77) is degenerate precisely on gauge fields in the first cohomology of M, $H^1(M) = \{B = 0, A \neq d\Lambda\}$. If M is simply connected, (77) is a nondegenerate inner product on $\mathcal{G}^* = \mathrm{SVect}(M, \mu)^*$. Inverting it gives an inner product on $\mathcal{G}$ which may be extended by right translation to a right-invariant metric on $\mathrm{SDiff}(M, \mu)$.

6. Equations of Motion

Before working out the equations of motion in detail, it would be prudent to convince ourselves that unlike the Casimir I_2, the magnetic energy H does *not* lie in the center of the Poisson algebra for nonconstant σ. We show this in App. D by exhibiting a moment map that has nonvanishing p.b. with H.

6.1. *Time evolution of magnetic field*

The time evolution of the magnetic field is given by the equation

$$\dot{B} = \{H, B\} = \varepsilon^{ij}\partial_i\left(\frac{B}{\rho}\right)\partial_j\left(\frac{B\sigma}{\rho}\right) = \nabla\left(\frac{B}{\rho}\right)\times\nabla\left(\frac{B\sigma}{\rho}\right), \tag{80}$$

where $\sigma = \rho^2/g$. It is manifestly gauge-invariant. Though the Hamiltonian is quadratic, (80) is nonlinear since the p.b. of gauge fields (35) is linear in gauge fields. The "interactions" are partly encoded in the Poisson structure and partly in the Hamiltonian, so to speak. Equation (80) is an analogue of the Euler equation $\dot{L} = L \times \Omega$, $L = I\Omega$ for the rigid body, (see App. A). Indeed, the equation of motion for the magnetic field can be regarded as the Euler equation for the group $\mathrm{SDiff}(M,\mu)$ with respect to the metric (77) defined by the Hamiltonian (66). The time evolution of the magnetic field can also be written as an equation of motion for the field strength $F = dA$ defined in (21):

$$\dot{F} = \frac{1}{2}\varepsilon^{ij}\partial_i\left(\frac{F}{\mu}\right)\partial_j\left(\frac{F\sigma}{\mu}\right)\epsilon_{kl}\, dx^k \wedge dx^l\,. \tag{81}$$

The formula (80) for $\dot{B}$ is obtained using the p.b. formula (31) using the relation $\frac{\delta B(z)}{\delta A_j(y)} = -\varepsilon^{jk}\frac{\partial\delta^2(z-y)}{\partial y^k}$, assuming $A = 0$ on ∂M. After some integration by parts,

$$\frac{dB}{dt} = \{H, B\} = -\varepsilon^{il}\varepsilon^{jk}\partial_k\left[\rho^{-1}(\partial_i A_j) + A_i(\partial_j\rho^{-1}) + \frac{1}{\rho}A_i\partial_j\right]\partial_l\left(\frac{B\sigma}{\rho}\right). \tag{82}$$

Expanding out derivatives of products, eliminating A in favor of $B = \varepsilon^{ij}\partial_i A_j$ and after a lot of cancelations, one arrives at the advertised evolution equation.

6.2. *Time evolution of moment maps*

We have already argued that the Casimirs $I_n = \int (dA/\mu)^n\mu$ are constant on symplectic leaves. Thus, they are conserved under time evolution $\frac{dI_n}{dt} = \{H, I_n\} = 0$, independent of the choice of Hamiltonian. As for the moment maps μ_u, we can show using (31) that

$$\frac{d\mu_u(A)}{dt} = \{H, \mu_u\} = \int d^2x\, A_i\left[\varepsilon^{jk}(\partial_j u^i) + \varepsilon^{ik}u^j\rho^{-1}(\partial_j\rho) - \varepsilon^{ik}u^j\partial_j\right]\partial_k\left(\frac{B\sigma}{\rho}\right). \tag{83}$$

For $\rho = 1$ this reduces to

$$\frac{d\mu_u(A)}{dt} = \int d^2x\, A_i[\varepsilon^{jk}(\partial_j u^i) - \varepsilon^{ik} u^j \partial_j]\partial_k(B\sigma)\,. \tag{84}$$

The right-hand side is gauge-invariant even though the gauge field appears explicitly.

6.3. *Infinitely many conserved charges in involution*

We find an infinite set of conserved charges:

$$\frac{dH_n}{dt} = \{H, H_n\} = 0\,, \quad n = 1, 2, 3, \ldots \tag{85}$$

for a uniform μ ($\rho = 1$) and g_{ij} an arbitrary metric.[h] We suspect that a similar result holds for nonuniform μ. The conserved quantities are

$$H_n = \int \left(\frac{dA}{\mu}\right)^n \sigma\mu = \int B^n \sigma\, d^2x\,, \tag{86}$$

where $\sigma = (\mu/\Omega_g)^2 = (1/g)$ and $g = \det g_{ij}$. Furthermore, we find that H_n are in involution $\{H_m, H_n\} = 0$. The Hamiltonian is $H = \frac{1}{2}H_2$. The presence of infinitely many conserved charges in involution suggests an integrable structure underlying the dynamics determined by the magnetic energy. The Hamiltonian of ideal hydrodynamics (62) is not known nor expected to have any conserved quantities besides the I_n and their close relatives, which are constant on coadjoint orbits. What is remarkable about H_n is that they are *not* constant on coadjoint orbits but still conserved quantities for the magnetic energy Hamiltonian (66). To establish this we investigate the time evolution of H_n:

$$\frac{dH_n}{dt} = \{H, H_n\} = \int A_i\left[\frac{\delta H}{\delta A_j}\,\partial_j\left(\rho^{-1}\frac{\delta H_n}{\delta A_i}\right) - \frac{\delta H_n}{\delta A_j}\partial_j\left(\rho^{-1}\frac{\delta H}{\delta A_i}\right)\right]d^2x\,. \tag{87}$$

Using

$$\frac{\delta H_n}{\delta A_i} = n\varepsilon^{ij}\partial_j\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right); \quad \frac{\delta H}{\delta A_i} = \varepsilon^{ij}\partial_j\left(\frac{\sigma B}{\rho}\right), \tag{88}$$

we get

$$\begin{aligned}\frac{dH_n}{dt} = n\varepsilon^{jk}\varepsilon^{il}\int A_i\Bigg[&\partial_k\left(\frac{\sigma B}{\rho}\right)\partial_j\left(\rho^{-1}\partial_l\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right)\right)\\ &- \partial_k\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right)\partial_j\left(\rho^{-1}\partial_l\left(\frac{\sigma B}{\rho}\right)\right)\Bigg]d^2x\,.\end{aligned} \tag{89}$$

For $\rho = 1$ this becomes

$$\frac{dH_n}{dt} = n\varepsilon^{jk}\varepsilon^{il}\int A_i[\partial_k(\sigma B)\partial_j\partial_l(\sigma B^{n-1}) - \partial_k(\sigma B^{n-1})\partial_j\partial_l(\sigma B)]d^2x\,. \tag{90}$$

[h] $\sigma = \rho^2/g$ must not grow too fast at ∞ and the magnetic field must vanish at ∞.

In the gauge $A_1 = 0$, $A_2 \equiv A$, $B = \partial_x A$, this may be written as

$$\frac{dH_n}{dt} = n \int dx\, dy\, A[\partial_x(B\sigma)\partial_{xy}(B^{n-1}\sigma) - \partial_x(B^{n-1}\sigma)\partial_{xy}(B\sigma) - \partial_y(B\sigma)\partial_x^2(B^{n-1}\sigma) + \partial_y(B^{n-1}\sigma)\partial_x^2(B\sigma)]\,. \tag{91}$$

We prove in App. E that $H_n, n = 1, 2, 3, \ldots$ are conserved by showing that the right-hand side of (91) vanishes. Having found an infinite number of conserved quantities we wanted to know whether their p.b. generates new conserved quantities. In our case, we discovered (again for $\rho = 1$), that the conserved quantities H_n are in involution, i.e. they mutually Poisson commute $\{H_m, H_n\} = 0$. The proof of this is somewhat lengthy and is relegated to App. E.

Intuitively, H_n are independent of each other since they are like average values of different powers of B. Furthermore, H_n are independent of the Casimirs I_m. I_m contain no information about the metric g_{ij} while H_n depend on the metric via σ. For example, we show in App. D that H_2 is not a Casimir. More generally, it would be nice to prove that I_n and H_m are functionally independent by showing that on every tangent space to an orbit, the cross product of their gradients is nonvanishing.

7. Static Solutions

7.1. *Zero energy configurations*

The magnetic energy is $H = \int_M (dA/\mu)^2 \sigma\mu = \int (B^2/\rho)\sigma\, d^2x$ where $\sigma = (\mu/\Omega_g)^2 = (\rho^2/g) \geq 0$. Thus, the energy is nonnegative, $H \geq 0$. Moreover, if A is closed, the energy automatically is a global minimum $dA = 0 \Rightarrow H = 0$. Moreover, since H is the integral of the square of dA, weighted by a positive function, closed gauge fields are the only configurations with zero energy. Any such closed gauge field is a static solution to the equations of motion (irrespective of μ and g_{ij})

$$\left.\frac{df}{dt}\right|_{dA=0} = \{H, f\}|_{dA=0} = 0\,. \tag{92}$$

As discussed in Subsec. 4.3, the closed gauge fields constitute one or more (according as M is simply connected or not) symplectic leaves of the Poisson manifold. They correspond to the zero set of Casimirs $I_n = \int (dA/\mu)^n \mu = 0$. Thus, the Hamiltonian vanishes on all the symplectic leaves with $I_n = 0$ and there is no interesting dynamics to speak of. If M is the plane, then the pure gauges are the only ones with zero energy. The leaf/leaves with $I_n = 0$ are the analogue of the $L^2 = 0$ point at the origin $L_i = 0$ of the angular momentum Poisson manifold. At that point, the energy $E = \sum_i L_i^2/2I_i$ of the rigid body vanishes as well.

It is interesting to find minima of energy on more interesting symplectic leaves. Initial conditions determine which symplectic leaf is the phase-space of the theory. The general problem of finding the minimum of energy on a given symplectic leaf (perhaps specified through values of invariants such as I_n) is potentially quite interesting and difficult. One would first have to find which gauge fields or magnetic field configurations satisfy the constraints and lie on the specified leaf. This is similar to the problem we solved in the large N limit of 2D QCD where we found the minimum of energy on the symplectic leaf with baryon number equal to one.[11,10] Here we do the opposite, find a few static solutions and then determine which orbit they lie on.

7.2. *Circularly symmetric static solutions*

Suppose $M = \mathbf{R}^2$. If both the volume forms μ and Ω_g are circularly symmetric, then any circularly symmetric initial magnetic field will remain unchanged with time. To see this, note that $\sigma = \rho^2/g$ depends only on the radial coordinate r. Suppose that at $t = 0$, $B(r, t = 0)$ depends only on r. The initial value problem for B given in (80) is

$$\dot{B} = \nabla\left(\frac{B}{\rho}\right) \times \nabla\left(\frac{B\sigma}{\rho}\right). \tag{93}$$

Due to circular symmetry, both the gradients point radially and their cross product vanishes. Thus $\dot{B} = 0$ and we have a static solution $B(r)$!

We already met the $B(r) = 0$ static solution before, it lies on a one point symplectic leaf of pure gauge configurations. However, the static solutions corresponding to nonconstant $B(r)$ lie on infinite dimensional symplectic leaves which we found previously (Subsec. 4.4). The values of Casimirs on the orbit containing a circularly symmetric static solution $B(r)$ are

$$I_n = 2\pi \int_0^\infty \left(\frac{B(r)}{\rho(r)}\right)^n \rho(r) r\, dr\,. \tag{94}$$

By a judicious choice[i] of $B(r)$ one should be able to find a static solution $B(r)$ for given $\rho(r)$ that lies on an orbit with practically any desired value for the invariants I_n. More generally, by an argument similar to the one given in Subsec. 4.4, we see that magnetic fields for which (B/ρ) and $(B\sigma/\rho)$ have common one-dimensional level sets, are static solutions of (80).

7.3. *Some other local extrema of energy*

For a gauge field to be an extremum of energy on a given symplectic leaf, the variation of energy in directions tangential to the leaf must vanish. There is no need for variations in other, let alone all, directions to vanish. However, though it is not necessary, if $[A]$ is such that all variations of the Hamiltonian $H(A)$ vanish, then, A must be a local extremum of energy. Such an extremum $\frac{\delta H}{\delta A_k} = 0$ is automatically a static solution to the equations of motion for any gauge-invariant observable f:

$$\frac{df}{dt} = \{H, f\} = \int d^2x\, d^2y \{A_i(x), A_j(y)\} \frac{\delta H}{\delta A_i(x)} \frac{\delta f}{\delta A_j(y)} = 0\,. \tag{95}$$

[i]Finding $B(r)$ for given I_n, $\rho(r)$ is similar to the Classical Moment Problem.

These extrema are given by solutions of

$$\frac{\delta H}{\delta A_k} = 2\varepsilon^{ij}\partial_j\left(\frac{B\sigma}{\rho}\right) = 0 \Rightarrow \partial_1\left(\frac{B\sigma}{\rho}\right) = 0 \quad \text{and} \quad \partial_2\left(\frac{B\sigma}{\rho}\right) = 0\,. \tag{96}$$

The only solutions are $B = (c\rho/\sigma) = (cg/\rho)$ where c is a constant and $g = \det g_{ij}$. These extrema lie on leaves where the Casimirs take the values

$$I_n = c^n \int \left(\frac{\Omega_g}{\mu}\right)^{2n} \mu = c^n \int \sigma^{-n}\rho\, d^2x\,. \tag{97}$$

We do not yet understand the physical meaning of these extrema of energy. They have a finite energy if $B = (c\rho/\sigma)$ vanishes at infinity sufficiently fast.

8. Discussion

A summary of the paper was given in Introduction (Sec. 1). Here, we mention a few directions for further study. We would like to know whether there is a deeper integrable structure that would explain the presence of an infinite number of conserved charges in involution for the uniform volume measure μ. The extension to an arbitrary volume form seems likely. Are there any time-dependent exact solutions of the nonlinear evolution equation? What is the Poisson algebra of loop observables and can the Hamiltonian be written in terms of them? Can this model be quantized and is there a non-Abelian extension? An extension to $3+1$ dimensions is possible, though the Poisson algebra has only one analytically known Casimir, the Hopf or link invariant. Is there a Lorentz covariant theory along these lines? Can the idea that gauge fields be thought of as dual to volume-preserving vector fields be exploited in any other context?

How is our gauge theory related to hydrodynamics and turbulence? Recently, Jackiw *et al.*[29] have studied perfect fluids and certain non-Abelian extensions. Our gauge theory shares the same phase-space as ideal hydrodynamics, but the two theories have different Hamiltonians. However, similar methods may be useful in the study of both theories. For example, Iyer and Rajeev[30] (see also Sec. 11.D of Ref. 15) have proposed a statistical approach to two-dimensional turbulence, based on a matrix regularization of the phase-space. It may be possible to use a similar regularization for our gauge theory. For an $N \times N$ matrix regularization to be integrable, it would appear that we need $\mathcal{O}(N^2)$ conserved quantities, while I_n and H_m furnish only $2N$ conserved quantities. It is unclear what this implies for the integrability of the continuum theory we have proposed in this paper. On the other hand, Polyakov has proposed a theory of turbulence in $2+1$ dimensions based on conformal invariance.[31] Since the group of conformal transformations and area preserving transformations are disjoint except for isometries, it appears unlikely that there is any direct relation of our work to Polyakov's.

Acknowledgments

This investigation was motivated by discussions with S. G. Rajeev. The author also benefitted from discussions with J. Ambjorn, G. Arutyunov, G. 't Hooft and J. Noldus and thanks G. Arutyunov, G. 't Hooft, L. F. Alday and S. G. Rajeev for comments on the manuscript.

Appendix A. Poisson Manifolds and Coadjoint Orbits

We collect a few facts about classical mechanics[32,15] that we use, to make the paper self-contained and fix notation.

A.1. *Poisson manifolds and symplectic leaves*

The basic playground of classical mechanics is a Poisson manifold. It is a manifold M with a product $\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ (the Poisson bracket (p.b.)) on the algebra of observables. $\{\cdot,\cdot\}$ is bilinear, skew symmetric and satisfies the Jacobi identity and Leibnitz rule. $\mathcal{F}(M)$ is a class of real-valued functions (say, $C^\infty(M)$). Any such function $f : M \to \mathbf{R}$ generates canonical transformations on M. A canonical transformation is a flow on M, associated to the canonical vector field V_f. The Lie derivative of any function $g(A)$ along the flow is given by the p.b. $\mathcal{L}_{V_f} g(A) = \{f, g\}(A)$, $A \in M$. Flow lines of the canonical transformation generated by f are integral curves of V_f.

Often, Poisson algebras of observables are degenerate. They have a center (Casimirs) which have zero p.b. with *all* observables. In such a situation, the Poisson manifold as a whole cannot serve as the phase-space of a physical system, since it would not be a symplectic manifold. Rather, it is the symplectic leaves of a Poisson manifold that can serve as phase-spaces. The symplectic leaf of a point $A \in M$ is the set of all points of M reachable from A along integral curves of canonical vector fields. On a symplectic leaf, the Poisson structure is nondegenerate and can be inverted to define a symplectic structure, a nondegenerate closed two-form ω. Indeed, if ξ, η are tangent vectors at A to a symplectic leaf, then the symplectic form at A is $\omega(\xi, \eta) = \{f, g\}(A)$ where f and g are *any* two functions whose canonical vector fields at A coincide with ξ and η; i.e. $\xi = V_f|_A$, $\eta = V_g|_A$. Moreover, on any symplectic leaf, $\omega(V_f, \cdot) = df(\cdot)$ where df is the exterior derivative of f.

The Hamiltonian vector field V_H is the canonical vector field of the Hamiltonian $H : M \to \mathbf{R}$. V_H generates time evolution $\frac{df}{dt} = \{H, f\} = \mathcal{L}_{V_H} f$. Irrespective of the Hamiltonian, time evolution always stays on the same symplectic leaf.

A.2. *Coadjoint orbits in the dual of a Lie algebra*

A natural example of a Poisson manifold, that occurs in many areas of physics, is the dual of a Lie algebra. The symplectic leaves of the dual of a Lie algebra are the coadjoint orbits of the group. To understand this, suppose G is a group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ the dual of the Lie algebra. Then we have a bilinear pairing between dual spaces $(A, u) \in \mathbf{R}$ for $A \in \mathcal{G}^*$ and $u \in \mathcal{G}$. Suppose f, g are two real-valued functions on $\mathcal{G}^*$. Then their p.b. is defined using the differential $df(A) \in \mathcal{G}$:

$$\{f, g\}(A) = (A, [df, dg]) \,. \tag{A.1}$$

$[df, dg]$ is the commutator in $\mathcal{G}$. This turns $\mathcal{G}^*$ into a Poisson manifold, which is often degenerate. The symplectic leaves are coadjoint orbits of the action of G on $\mathcal{G}^*$. To see this, first define the inner automorphism $A_g : G \to G$, $A_g h = ghg^{-1}$ which takes the group identity e to itself. The group adjoint representation $Ad_g : \mathcal{G} \to \mathcal{G}$ is the linearization of the inner automorphism at e: $Ad_g = A_{g\,*}|_e$ and is $Ad_g\, u = gug^{-1}$ for matrix groups. The group coadjoint representation $Ad_g^* : \mathcal{G}^* \to \mathcal{G}^*$ is defined as $(Ad_g^*\, A, u) = (A, Ad_g\, u)$. The group coadjoint orbit of $A \in \mathcal{G}^*$ is

$$\mathcal{O}_A = \{Ad_g^*\, A \,|\, g \in G\} \,. \tag{A.2}$$

The Lie algebra adjoint representation is $ad_u : \mathcal{G} \to \mathcal{G}$ where $ad_u = \frac{d}{dt}|_{t=0}\, Ad_{g(t)}$ for a curve $g(t)$ on the group with $g(0) = e$ and $\dot{g}(0) = u$ and takes the form $ad_u\, v = [u, v]$ for matrix groups. The Lie algebra coadjoint representation $ad_u^* : \mathcal{G}^* \to \mathcal{G}^*$ is defined by $(ad_u^*\, A, v) = (A, ad_u\, v)$. The Lie algebra coadjoint orbit of $A \in \mathcal{G}^*$ is the tangent space at A to the group coadjoint orbit of A:

$$\{ad_u\, A \,|\, u \in \mathcal{G}\} = T_A \mathcal{O}_A \,. \tag{A.3}$$

Thus, a tangent vector ξ to a coadjoint orbit at A may be written as $\xi = ad_u^*\, A$ for some (not necessarily unique) $u \in \mathcal{G}$.

There is a natural symplectic structure on coadjoint orbits, which turns them into homogeneous symplectic leaves of $\mathcal{G}^*$. The symplectic form (Kirillov form) ω acting on a pair of tangent vectors to the orbit $\mathcal{O}_A$ at A is given by $\omega(\xi, \eta) = (A, [u, v])$ where $u, v \in \mathcal{G}$ are *any* two Lie algebra elements such that $\xi = ad_u^*\, A$ and $\eta = ad_v^*\, A$. Thus, the coadjoint orbits are symplectic manifolds which foliate $\mathcal{G}^*$ in such a way as to recover the Poisson structure on the whole of $\mathcal{G}^*$. The different coadjoint orbits in $\mathcal{G}^*$ are not necessarily of the same dimension, but are always even dimensional if their dimension is finite.

The canonical vector field V_f of an observable $f : \mathcal{G}^* \to \mathbf{R}$ at a point $A \in \mathcal{G}^*$ is given by the Lie algebra coadjoint action of the differential $df(A)$, $V_f(A) = ad_{df}^*\, A$:

$$\mathcal{L}_{V_f}\, g(A) = \{f, g\}(A) = (A, [df, dg]) = (A, ad_{df}\, dg) = (ad_{df}^*\, A, dg) \,. \tag{A.4}$$

In particular, infinitesimal time evolution is just the coadjoint action of the Lie algebra element dH, the differential of the Hamiltonian. For example, if $\mu_u(A) = (A, u)$ is the *moment map* for $u \in \mathcal{G}$, then the canonical vector field $V_{\mu_u}(A) = ad_u^*\, A$. In other words, infinitesimal canonical transformations generated by moment maps are the same as Lie algebra coadjoint actions.

Observables in the center of the Poisson algebra (Casimirs) are constant on coadjoint orbits. They are invariant under the group and Lie algebra coadjoint actions. To show that an observable is a Casimir, it suffices to check that it commutes with the moment maps which generate the coadjoint action.

Example (Eulerian Rigid Body). Let Ω^i be the components of angular velocity of a rigid body in the corotating frame. The components of angular velocity lie in the Lie algebra $\mathcal{G}$ of the rotation group $G = \mathrm{SO}(3)$. The dual space to angular velocities consists of angular momenta L_i with the pairing (or moment map) $(L, \Omega) = L_i\Omega^i$. The space of angular momenta is the dual $\mathrm{SO}(3)^* = \mathbf{R}^3$. The latter carries a Poisson structure $\{L_i, L_j\} = \epsilon_{ijk}L_k$. Observables are real-valued functions of angular momentum $f(L)$ and satisfy the p.b.:

$$\{f, g\}(L) = \sum_{i,j}\{L_i, L_j\}\frac{\partial f}{\partial L_i}\frac{\partial g}{\partial L_j}\,. \tag{A.5}$$

The space of angular momenta $\mathbf{R}^3$ must be a degenerate Poisson manifold, since it is not even dimensional. Indeed, the symplectic leaves are concentric spheres centered at the point $L_i = 0$ as well as the point $L_i = 0$. These symplectic leaves are the coadjoint orbits of SO(3) acting on the space $\mathbf{R}^3$ of angular momenta. The symplectic form on a sphere of radius r is given by $r\sin\theta d\theta \wedge d\phi$. The Casimirs are functions of $L^2 = \sum_i L_i^2$ and are constant on the symplectic leaves, they are invariant under the coadjoint action of the rotation group on $\mathbf{R}^3$. The Hamiltonian is $H = \sum_i L_i^2/2I_i$ if the axes are chosen along the principle axes of inertia. I_i are the principle moments of inertia, the eigenvalues of the inertia operator $I_{ij} : \mathcal{G} \to \mathcal{G}^*$ which maps angular velocities to angular momenta $L_i = I_{ij}\Omega^j$. The equations of motion $\dot{L} = ad^*_{dH}\,L = ad^*_{I^{-1}L}\,L = ad^*_{\Omega}\,L = L\times\Omega$ are $\dot{L}_i = \{H, L_i\}$, $\dot{L}_1 = a_{23}L_2L_3$ and cyclic permutations thereof, where $a_{ij} = I_j^{-1} - I_i^{-1}$.

Appendix B. The Charges I_n are in Involution

We show that the charges I_n are in involution $\{I_m, I_n\} = 0$. To calculate

$$\{I_m, I_n\} = \int_M A_i\left[\frac{\delta I_m}{\delta A_j}\partial_j\left(\frac{1}{\rho}\frac{\delta I_n}{\delta A_i}\right) - \frac{\delta I_n}{\delta A_j}\partial_j\left(\frac{1}{\rho}\frac{\delta I_m}{\delta A_i}\right)\right]d^2x \tag{B.1}$$

we need

$$I_m = \int\left(\frac{B}{\rho}\right)^m \rho d^2x \Rightarrow \frac{\delta I_m}{\delta A_i} = m\varepsilon^{ij}\partial_j\left(\left(\frac{B}{\rho}\right)^{m-1}\right). \tag{B.2}$$

So the p.b. becomes

$$\{I_m, I_n\} = mn\varepsilon^{il}\varepsilon^{jk}\int_M d^2x\, A_i\left[\partial_k\left(\frac{B}{\rho}\right)^{m-1}\partial_j\left(\frac{1}{\rho}\partial_l\left(\frac{B}{\rho}\right)^{n-1}\right) - m \leftrightarrow n\right]. \tag{B.3}$$

The first term in square brackets $\partial_k(B/\rho)^{m-1}\partial_j(\frac{1}{\rho}\partial_l(B/\rho)^{n-1})$ can be written as

$$\begin{aligned}&(m-1)(n-1)\left(\frac{B}{\rho}\right)^{m-2}\partial_k\left(\frac{B}{\rho}\right)\partial_j\left\{\rho^{-1}\left(\frac{B}{\rho}\right)^{n-2}\partial_l\left(\frac{B}{\rho}\right)\right\}\\ &\quad = (m-1)(n-1)\left(\frac{B}{\rho}\right)^{m+n-4}\partial_j(\rho^{-1})\partial_k\left(\frac{B}{\rho}\right)\partial_l\left(\frac{B}{\rho}\right)\end{aligned}$$

$$+ (m-1)(n-1)(n-2)\left(\frac{B}{\rho}\right)^{m+n-5} \rho^{-1}\partial_k\left(\frac{B}{\rho}\right)\partial_j\left(\frac{B}{\rho}\right)\partial_l\left(\frac{B}{\rho}\right)$$
$$+ (m-1)(n-1)\left(\frac{B}{\rho}\right)^{m+n-4} \rho^{-1}\partial_k\left(\frac{B}{\rho}\right)\partial_j\partial_l\left(\frac{B}{\rho}\right). \tag{B.4}$$

The first and third terms are symmetric under $m \leftrightarrow n$ and therefore do not contribute to $\{I_m, I_n\}$. Therefore we get ($I_1 = 0$, so we can ignore $m, n = 1$):

$$\frac{\{I_m, I_n\}}{mn(m-1)(n-1)(n-m)} = \int d^2x \frac{(B/\rho)^{m+n-5}}{\rho}\left[\varepsilon^{il}\varepsilon^{jk}A_i\partial_l\left(\frac{B}{\rho}\right)\partial_j\left(\frac{B}{\rho}\right)\partial_k\left(\frac{B}{\rho}\right)\right].$$

Now the term in square brackets vanishes identically due to antisymmetry of ε^{jk}. We conclude that $\{I_m, I_n\} = 0$. Thus, $I_n, n = 1, 2, 3\ldots$ are an infinite number of charges in involution.

Appendix C. I_2 is a Casimir of the Poisson Algebra

To find the infinitesimal change of I_n under the coadjoint action, we calculate $\{I_n, \mu_u\}$. If this vanishes for all volume-preserving u, then I_n would be constant on symplectic leaves and hence a Casimir. Recall (28) that the differential of I_n is

$$(dI_n(A))^i = n\rho^{-1}\varepsilon^{ij}\partial_j\left(\frac{B}{\rho}\right)^{n-1} \tag{C.1}$$

and the differential of the moment map is $(d\mu_u)^i = u^i$. Their Lie bracket is

$$[u, dI_n]^i = u^j\partial_j\left(n\rho^{-1}\varepsilon^{ik}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\right) - n\rho^{-1}\varepsilon^{jk}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\partial_j u^i$$
$$= n\left[\varepsilon^{ik}u^j\partial_j\left(\rho^{-1}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\right) - \rho^{-1}\varepsilon^{jk}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\partial_j u^i\right]. \tag{C.2}$$

Thus $\{\mu_u, I_n\} = \int A_i[u, dI_n]^i\rho d^2x$ gives

$$\{\mu_u, I_n\} = n\int d^2x\, A_i\left[\varepsilon^{ik}\rho u^j\partial_j\left(\rho^{-1}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\right) - \varepsilon^{jk}\partial_k\left(\frac{B}{\rho}\right)^{n-1}\partial_j u^i\right]. \tag{C.3}$$

For $\rho = 1$ this becomes

$$\{\mu_u, I_n\} = n\int d^2x\, A_i\left[\varepsilon^{ik}u^j(\partial_k\partial_j B^{n-1}) - \varepsilon^{jk}(\partial_j u^i)(\partial_k B^{n-1})\right]. \tag{C.4}$$

Specializing to $n = 2$, and writing u in terms of its stream function, $u^i = \varepsilon^{il}\partial_l\psi$,

$$\{I_2, \mu_u\} = 2\int d^2x\left[\varepsilon^{il}\varepsilon^{jk}A_i(\partial_l\partial_j\psi)(\partial_k B) - \varepsilon^{ik}\varepsilon^{jl}A_i(\partial_k\partial_j B)(\partial_l\psi)\right]. \tag{C.5}$$

$\{I_2, \mu_u\}$ is gauge-invariant, so we calculate it in the gauge $A_1 = 0$, $B = \partial_1 A_2$ and denote $A_2 = A$, $x^1 = x$, $x^2 = y$ and derivatives by subscripts:

$$\frac{1}{2}\{I_2, \mu_u\} = \int dx\, dy\, A[\psi_y A_{xxx} - \psi_x A_{xxy} + \psi_{xy} A_{xx} - \psi_{xx} A_{xy}]\,. \tag{C.6}$$

The idea is to integrate by parts and show that this expression vanishes. Let us temporarily call the second factor of A by the name $\tilde{A}$. Write $\frac{1}{2}\{I_2, \mu_u\}$ as a sum of four terms $T_1 + T_2 + T_3 + T_4$:

$$\begin{aligned} T_1 &= \int dx\, dy\, A\psi_y \tilde{A}_{xxx}\,, \quad T_2 = -\int dx\, dy\, A\psi_x \tilde{A}_{xxy}\,, \\ T_3 &= \int dx\, dy\, A\psi_{xy} \tilde{A}_{xx}\,, \quad T_4 = -\int dx\, dy\, A\psi_{xx} \tilde{A}_{xy}\,. \end{aligned} \tag{C.7}$$

We will show that $T_1 + T_2 = -T_3 - T_4$. Integrating by parts till there are no derivatives on $\tilde{A}$,

$$\begin{aligned} T_1 &= -\int dx\, dy\, \tilde{A}(\psi_y A_{xxx} + 3A_{xx}\psi_{xy} + 3A_x \psi_{xxy} + A\psi_{xxxy})\,, \\ T_2 &= \int dx\, dy\, \tilde{A}(\psi_x A_{xxy} + 2A_{xy}\psi_{xx} \\ &\qquad + 2A_x \psi_{xxy} + A_{xx}\psi_{xy} + A_y \psi_{xxx} + A\psi_{xxxy})\,, \\ T_3 &= \int dx\, dy\, \tilde{A}(\psi_{xy} A_{xx} + 2A_x \psi_{xxy} + A\psi_{xxxy})\,, \\ T_4 &= -\int dx\, dy\, \tilde{A}(A_{xy}\psi_{xx} + A_y \psi_{xxx} + A_x \psi_{xxy} + A\psi_{xxxy})\,. \end{aligned} \tag{C.8}$$

Using the fact that $\tilde{A} = A$ we get for T_1 and T_2:

$$\begin{aligned} 2 \times T_1 &= -\int dx\, dy\, \tilde{A}(3A_{xx}\psi_{xy} + 3A_x \psi_{xxy} + A\psi_{xxxy})\,, \\ 2 \times T_2 &= \int dx\, dy\, \tilde{A}(2A_{xy}\psi_{xx} + 2A_x \psi_{xxy} + A_{xx}\psi_{xy} + A_y \psi_{xxx} + A\psi_{xxxy})\,. \end{aligned} \tag{C.9}$$

T_3 and T_4 give us the identities

$$\begin{aligned} \int dx\, dy\, \tilde{A} A_x \psi_{xxy} &= -\int dx\, dy\, \tilde{A} A \psi_{xxxy}\,, \\ \int dx\, dy\, \tilde{A} A_y \psi_{xxx} &= \int dx\, dy\, \tilde{A} A_x \psi_{xxy}\,. \end{aligned} \tag{C.10}$$

Use these to simplify T_1 and T_2 by eliminating ψ_{xxxy} and ψ_{xxx} in favor of ψ_{xxy}:

$$\begin{aligned} 2 \times T_1 &= -\int dx\, dy\, \tilde{A}(3A_{xx}\psi_{xy} + A_x \psi_{xxy})\,, \\ 2 \times T_2 &= \textstyle\int dx\, dy\, \tilde{A}(2A_{xy}\psi_{xx} + A_{xx}\psi_{xy} + A_x \psi_{xxy})\,. \end{aligned} \tag{C.11}$$

Adding these and setting $A = \tilde{A}$,

$$T_1 + T_2 = \int dx\, dy\, A(A_{xy}\psi_{xx} - A_{xx}\psi_{xy})\,. \tag{C.12}$$

Meanwhile by definition,

$$T_3 + T_4 = \int dx\, dy\, A(A_{xx}\psi_{xy} - A_{xy}\psi_{xx})\,,$$

thus

$$\{I_2, \mu_u\} = 2(T_1 + T_2 + T_3 + T_4) = 0\,. \tag{C.13}$$

We conclude that I_2 lies in the center of the Poisson algebra for constant ρ.

Appendix D. $H = \frac{1}{2}\int (dA/\mu)^2\sigma\mu$ is *not* a Casimir

$I_2 = \int (dA/\mu)^2\mu$ turned out to be in the center of the Poisson algebra of gauge-invariant functions. Here we show that the magnetic energy $H = \frac{1}{2}\int (dA/\mu)^2\sigma\mu$ with a nonconstant $\sigma = \rho^2/g$, *does not* lie in the center, and therefore leads to nontrivial time evolution. We do this by giving an explicit example of a gauge-invariant function with which it has a nonvanishing p.b. Consider $\rho = 1$, then from (83),

$$\{H, \mu_u\} = \int d^2x\, A_i[\varepsilon^{jk}(\partial_j u^i)\partial_k(B\sigma) - \varepsilon^{ik}u^j(\partial_j\partial_k B\sigma)]\,. \tag{D.1}$$

In gauge $A_1 = 0$, $A_2 \equiv A$ and with $u^i = \varepsilon^{ij}\partial_j\psi$ we get ($x^1 = x, x^2 = y$):

$$\begin{aligned}\{H, \mu_u\} = \int dx\, dy\, A[&-(\partial_x^2\psi)\partial_y(B\sigma) + (\partial_{xy}\psi)\partial_x(B\sigma) \\ &+ (\partial_y\psi)\partial_x^2(B\sigma) - (\partial_x\psi)\partial_{xy}(B\sigma)]\,.\end{aligned} \tag{D.2}$$

Now for the simple choices $\psi = xy$, $A = \sigma = e^{-(x^2+y^2)/2}$ we have $B = \partial_x A = -xe^{-(x^2+y^2)/2}$. The p.b. can be calculated exactly to yield

$$\{H, \mu_u\} = \int dx\, dy\, e^{-3(x^2+y^2)/2}(-1 - 4x^4 - 2y^2 + 4x^2(2 + y^2)) = \frac{2\pi}{27}\,. \tag{D.3}$$

Thus H does not lie in the center of the Poisson algebra. We also checked that H transforms nontrivially under many other generators μ_u of the coadjoint action.

Appendix E. H_n are in Involution for Uniform Measure

Suppose μ is the uniform measure ($\rho = 1$). We prove that the charges

$$H_n = \int \left(\frac{dA}{\mu}\right)^n \sigma\mu = \int \left(\frac{B}{\rho}\right)^n \sigma\rho\, d^2x = \int B^n\sigma\, d^2x \tag{E.1}$$

are in involution

$$\{H_m, H_n\} = 0 \quad \text{for} \quad m,\ n = 0, 1, 2, 3, \ldots\,. \tag{E.2}$$

An immediate corollary is that H_n are conserved quantities since the Hamiltonian is $\frac{1}{2}H_2$. Here $B \to 0$ at ∞ and $\sigma = (\mu/\Omega_g)^2 = 1/g$ must be such that these integrals converge. The proof involves explicitly computing the p.b. and integrating by parts several times. Using

$$\frac{\delta H_n}{\delta A_i} = \delta^i H_n = n\varepsilon^{ij}\partial_j\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right) \tag{E.3}$$

we can express the p.b. as

$$\begin{aligned}\{H_m, H_n\} &= \int A_i[\delta^j H_m \partial_j(\rho^{-1}\delta^i H_n) - m \leftrightarrow n]d^2x \\ &= mn\varepsilon^{il}\varepsilon^{jk}\int A_i\left[\partial_k\left(\sigma\left(\frac{B}{\rho}\right)^{m-1}\right)\partial_j\left(\rho^{-1}\partial_l\left(\sigma\left(\frac{B}{\rho}\right)^{n-1}\right)\right) - m \leftrightarrow n\right]d^2x\,.\end{aligned} \tag{E.4}$$

For $\rho = 1$ this becomes

$$\frac{\{H_{m+1}, H_{n+1}\}}{(m+1)(n+1)} = \varepsilon^{il}\varepsilon^{jk}\int A_i[\partial_k(\sigma B^m)\partial_j\partial_l(\sigma B^n) - m \leftrightarrow n]d^2x\,. \tag{E.5}$$

Now we expand out the derivatives of products of B and σ and eliminate terms that vanish due to antisymmetry of ε^{jk} or antisymmetry in m and n. We get

$$\begin{aligned}\frac{\{H_m, H_n\}}{mn(n-m)} = \varepsilon^{il}\varepsilon^{jk}\int A_i B^{m+n-4}[&B(\partial_k\sigma)(\partial_l\sigma)(\partial_j B) + \sigma B(\partial_k\sigma)(\partial_j\partial_l B) \\ &+ (n+m-3)\sigma(\partial_k\sigma)(\partial_j B)(\partial_l B) + \sigma B(\partial_j B)(\partial_k\partial_l\sigma)]\,.\end{aligned} \tag{E.6}$$

Since this p.b. is gauge-invariant, calculate in the gauge $A_1 = 0$, call $A_2 = A$, $B = \partial_1 A$ and denote derivatives by subscripts $(x^1 = x, x^2 = y)$,

$$\begin{aligned}\{H_{m+1}, H_{n+1}\} = -\varepsilon^{jk}(m+1)(n+1)(n-m)&\int dx\,dy\,AB^{m+n-2} \\ &\times [B\sigma_x B_j\sigma_k + \sigma BB_{jx}\sigma_k + (n+m-1)\sigma B_x B_j\sigma_k + \sigma BB_j\sigma_{kx}]\,.\end{aligned} \tag{E.7}$$

After collecting terms, this becomes

$$\begin{aligned}\{H_m, H_n\} = mn(m-n)\int AB^{m+n-4}[&(B_x\sigma_y - B_y\sigma_x)(B\sigma_x + (n+m-3)B_x\sigma) \\ &+ \sigma B\{B_{xx}\sigma_y - B_{xy}\sigma_x + B_x\sigma_{xy} - B_y\sigma_{xx}\}]\,.\end{aligned} \tag{E.8}$$

To simplify it we define $k = m + n - 4$ and use identities such as $B^k B_x = (B^{k+1})_x/(k+1)$ and $\sigma\sigma_x = \frac{1}{2}(\sigma^2)_x$ to combine the factors of B and σ to write

$$\frac{1}{mn(m-n)}\{H_m, H_n\} = \sum_{i=1}^{8} T_i\,,$$

where

$$\begin{aligned}
T_1 &= \frac{1}{k+2}\int A(B^{k+2})_x\sigma_x\sigma_y\,, & T_2 &= -\frac{1}{k+2}\int A(B^{k+2})_y\sigma_x\sigma_x\,,\\
T_3 &= \frac{1}{2}\int A(B^{k+1})_x B_x(\sigma^2)_y\,, & T_4 &= -\frac{1}{2}\int A(B^{k+1})_y B_x(\sigma^2)_x\,,\\
T_5 &= \frac{1}{2}\int AB^{k+1}B_{xx}(\sigma^2)_y\,, & T_6 &= -\frac{1}{2}\int AB^{k+1}B_{xy}(\sigma^2)_x\,,\\
T_7 &= \frac{1}{k+2}\int A(B^{k+2})_x\sigma\sigma_{xy}\,, & T_8 &= -\frac{1}{k+2}\int A(B^{k+2})_y\sigma\sigma_{xx}\,.
\end{aligned} \tag{E.9}$$

Numerically we find $T_1+T_7+T_2+T_8 = -(T_3+T_5+T_4+T_6)$. To see this analytically, integrate by parts with the aim of eliminating A in favor of $B = \partial_x A$. Using similar identities as before,

$$\begin{aligned}
T_1 &= -T_7 - \frac{1}{2}\int \frac{(B^{k+3})_x(\sigma^2)_y}{k+3} - \frac{1}{2}\int \frac{A(B^{k+2})_{xx}(\sigma^2)_y}{k+2}\,,\\
T_2 &= -T_8 + \frac{1}{2}\int \frac{(B^{k+3})_y(\sigma^2)_x}{k+3} + \frac{1}{2}\int \frac{A(B^{k+2})_{xy}(\sigma^2)_x}{k+2}\,,\\
T_3 &= -T_5 - \frac{1}{2}\int \frac{(B^{k+3})_x(\sigma^2)_y}{k+3} - \frac{1}{2}\int \frac{A(B^{k+2})_x(\sigma^2)_{xy}}{k+2}\,,\\
T_4 &= -T_6 + \frac{1}{2}\int \frac{A_y(B^{k+2})_x(\sigma^2)_x}{k+2} + \frac{1}{2}\int \frac{A(B^{k+2})_x(\sigma^2)_{xy}}{k+2}\,.
\end{aligned} \tag{E.10}$$

Then

$$\begin{aligned}
T_3+T_5+T_4+T_6 &= -\frac{1}{2}\int \frac{(B^{k+3})_x(\sigma^2)_y}{k+3} + \frac{1}{2}\int \frac{A_y(B^{k+2})_x(\sigma^2)_x}{k+2}\,,\\
T_1+T_7+T_2+T_8 &= \frac{1}{2(k+2)}\int A\big\{(B^{k+2})_{xy}(\sigma^2)_x - (B^{k+2})_{xx}(\sigma^2)_y\big\}\,,\\
\{H_m, H_n\} &= mn(m-n)(U_1+U_2+U_3+U_4)\,,
\end{aligned} \tag{E.11}$$

where $T_3+T_5+T_4+T_6 \equiv U_1+U_2$ and $T_1+T_7+T_2+T_8 \equiv U_3+U_4$. Finally, integration by parts shows that

$$U_2 = -U_3 - \frac{1}{2(k+2)}\int A(B^{k+2})_x(\sigma^2)_{xy} \tag{E.12}$$

and that $U_2+U_3 = -U_4-U_1$. We conclude that $\sum_{i=1}^4 U_i = 0$ and therefore $\{H_m, H_n\} = 0$. Since the Hamiltonian is half the second charge, $H = \frac{1}{2}H_2$, we have shown that H_n for $n = 1, 2, 3, \dots$ are an infinite number of conserved quantities, which moreover, are in involution!

References

1. S. Mandelstam, *Phys. Rev.* **175**, 1580 (1968).
2. S. G. Rajeev, Two-dimensional meson theory, in *Proc. 1991 Summer School in High Energy Physics and Cosmology*, eds. E. Gava *et al.* (World Scientific, Singapore, 1992), p. 955.
3. S. G. Rajeev and O. T. Turgut, *Int. J. Mod. Phys. A* **10**, 2479 (1995).
4. D. Karabali, C. J. Kim and V. P. Nair, *Phys. Lett. B* **434**, 103 (1998).
5. C. W. H. Lee and S. G. Rajeev, *J. Math. Phys.* **39**, 5199 (1998).
6. C. W. H. Lee and S. G. Rajeev, *Nucl. Phys. B* **529**, 656 (1998).
7. G. 't Hooft, *Nucl. Phys. B* **72**, 461 (1974).
8. S. G. Rajeev, *Int. J. Mod. Phys. A* **9**, 5583 (1994).
9. G. 't Hooft, *Nucl. Phys. B* **75**, 461 (1974).
10. V. John, G. S. Krishnaswami and S. G. Rajeev, *Phys. Lett. B* **492**, 63 (2000).
11. G. S. Krishnaswami, Large-N limit as a classical limit: Baryon in two-dimensional QCD and multi-matrix models, Ph.D. thesis, University of Rochester (2004), hep-th/0409279.
12. S. G. Rajeev and O. T. Turgut, *J. Math. Phys.* **37**, 637 (1996).
13. L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Int. J. Mod. Phys. A* **17**, 2413 (2002).
14. A. Agarwal, L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Int. J. Mod. Phys. A* **18**, 917 (2003).
15. V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Applied Mathematical Sciences, Vol. 125 (Springer, 1998).
16. B. A. Khesin, *Not. Amer. Math. Soc.* **52**, 9 (2005).
17. V. Yu. Ovsienko and B. A. Khesin, *Funct. Anal. Appl.* **21**, 329 (1987).
18. V. I. Arnold, *Ann. Inst. Fourier* (*Grenoble*) **16**, 319 (1966).
19. V. I. Arnold, *Usp. Mat. Nauk* **24**, 225 (1969).
20. S. P. Novikov, *Russ. Math. Surv.* **37**, 1 (1982).
21. J. E. Marsden and A. Weinstein, *Physica D* **7**, 305 (1983).
22. B. A. Dubrovin, I. M. Krichever and S. P. Novikov, *Encyclopaedia of Mathematical Sciences*, Vol. 4 (Springer-Verlag, 1990), pp. 173–280.
23. D. D. Holm, J. E. Marsden and T. S. Ratiu, *Phys. Rev. Lett.* **349**, 4173 (1998).
24. E. I. Guendelman, E. Nissimov and S. Pacheva, *Phys. Lett. B* **360**, 57 (1995).
25. M. Kreuzer, *Class. Quantum Grav.* **7**, 1303 (1990).
26. A. A. Kirillov, *Russ. Math. Surv.* **17**, 57 (1962).
27. A. Weinstein, *J. Diff. Geom.* **18**, 523 (1983).
28. A. Weinstein, *J. Diff. Geom.* **22**, 255 (1985).
29. R. Jackiw, V. P. Nair, S. Y. Pi and A. P. Polychronakos, *J. Phys. A* **37**, R327 (2004).
30. S. V. Iyer and S. G. Rajeev, *Mod. Phys. Lett. A* **17**, 1539 (2002).
31. A. M. Polyakov, *Nucl. Phys. B* **396**, 367 (1993).
32. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd edn., Graduate Texts in Mathematics (Springer, 1989).