RESEARCH STATEMENT

DISHANT M. PANCHOLI

1. Introduction

My research interests are in the geometry and topology of contact and symplectic manifolds. In particular, I am interested in the questions related to the existence of contact structures, symplectic cobordism classes of contact manifolds and classification of contact structures. My emphasis is on the study of higher dimensional manifold.

The study of contact and symplectic structures is at the forefront of mathematical research with countless applications and connections to various parts of mathematics. To name a few, Eliashberg's topological classification of Stein manifolds, Donaldson's symplectic embedding theorem of Kähler manifold – not necessarily of integral class – into a projective space, Mirror symmetry and its various applications to algebraic and complex geometry etc.

A contact structure on a manifold M is a nowhere integrable co-dimension 1 distribution. More precisely, it is a sub-bundle ξ of TM of co-rank 1 such that the 2-form ω defined on ξ by the condition $\omega(X,Y) = \{[X,Y]\}$ (where $[X,Y]$ denotes the Lie-bracket of X and Y and $\{\}$ denotes an equivalence class) with its values in TM/ξ is non-degenerate. A symplectic structure on a manifold is a non-degenerate closed 2-form. It clear form the definitions that the dimension of a manifold admitting a contact structure is always odd while the dimension of a manifold admitting a symplectic structure is always even. Given a symplectic manifold with boundary $(W, \partial W, \omega)$ a Liouville vector-field X on it is a vector-field satisfying $L_X(\omega) = \omega$. It is easy to see that if a symplectic manifold $(W, \partial W)$ admits a Liouville vector-field transverse to the boundary then boundary naturally has a contact structure given by the kernel of the contraction of ω by X, i.e. $i_X(\omega)$, restricted to boundary. In particular, for a Hamiltonian function H on W satisfying a suitable hypothesis the inverse image of a regular value is a contact manifold. This property leads to interesting relationships between these two structures which I will try to explain in a moment.

One of the main question in symplectic and contact topology is to construct and classify these structures on a smooth manifold. One possible approach, which has been very successful in many cases, is to consider a handle-body decomposition of the manifold. We know that for a connected manifold we can always assume that the manifold has only one 0-handle. A theorem due to Darboux – in some sense – says that 0-handle admits a unique symplectic structure. Hence given a smooth manifold one can start by putting the standard symplectic structure on the ball and try to understand – step by step – all possible extensions of this structure on the handles of higher index. It turns out that the approach is very fruitful when the boundary of the manifold obtained by taking union of handles of index (say)

k (k less than the dimension of manifold) admits a contact structure. Namely, the symplectic structure constructed by attaching handles is so that the neighborhood of the boundary admits an outward pointing Liouville vector-field. The reason being that the boundary having naturally defined contact structure – natural in the sense discussed earlier – allows one to carry forward handle attachment procedure in the symplectic category. See, for example, [3] and [11] for more precise explanations and applications.

In general, however, one does not get a contact structure on the boundary. However, one can naturally get a co-dimension 1 complex distribution (ξ, J). An odd dimensional manifold admitting a co-dimension 1- (almost) complex distribution J is known as an almost-contact manifold. In order to carry out surgery procedure describe earlier it is very useful to find out if one can perturb the boundary (hence the given Morse function providing handle decomposition) so that the boundary becomes contact. More concretely, in finding out a procedure which homotopes given almost-contact structure to a contact structure. This leads to many interesting questions about the existence and a possible classification of contact structures. This is the main focus of my research. Few central questions related to these topics are the following:

Question 1.1 (Chern, 1966)**.** *Find topological conditions under which a manifold admits a contact structure.*

A contact manifold is always almost-contact and deciding whether a manifold admits an almost-contact structure is fairly well understood so the more relevant question is:

Question 1.2. *Is every almost-contact structure on an odd dimensional manifold homotopic to a contact structure?*

Another important question from the classification point of view discussed earlier is clearly the following:

Question 1.3. *Classify contact structures on a manifold.*

One would like to point out that it is not possible to classify all possible symplectic or contact structure due to the reasons similar to that of impossibility of a complete classification of manifolds in higher dimensions (see for example [10]). However, on many important classes of manifold one can achieve this and this in many cases is what one needs for various useful applications. In next couple of sections I would like to explain the current status of the problems and my contributions towards its solutions.

2. Existence of contact structures

The first major breakthrough (in 1970's) related to question 1.2 is due to Lutz and Martinet. Martinet generalized techniques developed by Lutz – now known as Lutz twists – to showed that every homotopy class of hyperplane field (co-dimension 1 distribution) admits a contact structure. This, in particular, completely settles questions 1.1 and 1.2 for 3–manifolds as the orientability is necessary for an existence of contact structure on 3-manifold. However, the situation in higher dimensions is completely different.

Many partial result were obtained in higher dimensions due to Geiges and his collaborators. Most notably Geiges's proof [4][Chpt-8] that very simply connected orientable 5-manifold is contact and Geiges and Stipsicz's construction [7] of contact structure on orientable 5-manifold that occurs as a product of lower dimensional manifold. Despite these advances there was no complete answer known in 5-manifold. I in my joint work with R.Casals and F.Presas obtained the most general result for 5-manifold [1] . In particular, we established the following theorem:

Theorem 2.1 (Casals, Pancholi and Presas)**.** *Let M be a closed oriented 5–dimensional manifold. There exists a contact structure in every homotopy class of almost contact structure.*

In general, we show that:

Theorem 2.2 (Casals, Pancholi and Presas)**.** *Let M be closed 5–dimensional manifold. Given an almost contact structure ξ with the first Chern class $c_1(\xi)$, there exists a contact structure $\hat{\xi}$ such that $c_1(\xi) = c_1(\hat{\xi})$.*

It follows immediately from theorem 2.1 that an orientable 5-manifold admits a contact structure if and only if its third Steifel-Whitney class with integral coefficient vanish. This completely answers Chern's question 1.1 for orientable 5–manifolds. One crucial ingredient in the proof of theorem 2.1 is the operation of blow-up along a sub-manifold. M. Gromov in [6] defined the notion of symplectic blow-up and asked a few questions for a possible notion of contact blow-up. We answered this question in [2]. The blow-up – apart from being very useful in establishing theorems 2.2 and 2.1 – is one of the very few tools by which one constructs lots of new contact manifolds from a given contact manifold. It also allows one to perform certain surgery operations in the contact category.

3. Classifications of contact structures

Classification of contact structure has been well studied for 3-manifolds. To begin with a contact structure on 3-manifold is either *tight* or *overtwisted*. Tight structure typically occur as a boundary of symplectic manifolds having connected boundary. Such structures are known as fillable structures. The theory of J–holomorphic curve has been very successful in understanding these structures.

Overtwisted structure on other hand are very flexible. They satisfy Gromov's h-principle. To be precise, Eliashberg showed that there is a weak homotopy equivalence between the space $\mathbf{H}$ of all hyperplane field and the space $\mathbf{C_o}$ of all overtwisted contact hyperplanes. Here by weak homotopy equivalence between $\mathbf{H}$ and $\mathbf{C_o}$ one just means that their homotopy groups are identical. My main focus is to single out right flexibility condition for higher dimensional manifolds.

K. Niederkrüger [9] based on Gromov's idea [5] defined the notion of *plastikstufe* (also known as *embedded overtwisted family*. One of the main reason for defining this object was that a contact structure admitting plastikstufe can not be filled. Popularly, these structures are call PS-overtwisted structures. It was believed that such structures must satisfy some flexibility. I with John Etnyre [8] showed that at least at π_0 level these structures do have some flexibility. Namely, we established the following.

Theorem 3.1. *Every contact manifold admits a PS–overtwisted contact structure homotopic to the given contact structure.*

This theorem was established using generalizing one of the most fundamental concept due to Lutz – known as Lutz twists – in higher dimensions. For orientable five manifolds theorem 3.1 together with theorem 2.1 imply that there is a surjection from $\pi_0(\mathbf{H})$ to $\pi_0(\mathbf{C_o})$, where, $\mathbf{H}$ denotes the space of almost-contact hyperplane fields and the space $\mathbf{C_o}$ denotes the space of all PS-overtwisted hyperplane fields. It would be interesting to know if this is actually a bijection.

References

[1] Roger Casals, Dishant M. Pancholi and Francisco Presas *Almost Contact* 5*-folds are contact* arXiv:1203.2166v3 [math.SG], 2012

[2] Roger Casals, Dishant M. Pancholi and Francisco Presas *Contact blow-up* arXiv:1210.1687v1 [math.SG], 2012

[3] Kai Cieliebak and Yakov Eliashberg *From Stein to Weinstein and back* AMS Colloquium publications, Vol. 59

[4] Hansjörg Geiges *An Introduction to Contact Topology* Cambridge Studies in Advanced Mathematics, 2009

[5] Mikhail Gromov,
Pseudo holomorphic curves in symplectic manifolds. Inventiones Mathematicae vol. 82, 1985, pgs. 307-347.

[6] Mikhail Gromov
Partial differential relations Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol 9, Springer-Verlag.

[7] Hansjörg Geiges and András Stipsicz
Contact structures on product five-manifolds and fibre sums along circles Math. Ann. 348 (2010), 195-210

[8] John B. Etnyre and Dishant M. Pancholi *On generalizing Lutz twists.* Journal of London Math. Society. (84)–3, 2011, 670-688.

[9] Kluz Niederkrüger
The plastikstufe a generalization of the overtwisted disk to higher dimensions Algebraic and Geometric Topology 6 (2006) 24732508

[10] Paul Seidel, *A biased view of symplectic cohomology* arXiv:0704.2055v6 [math.SG]

[11] Alan Weinstein *Contact surgery and symplectic handlebodies* Hokkaido Mathematical Journal, 20 (2). 241-. (1991)