DIFFERENTIAL EQUATIONS

(1) Let V be a continuous vector-field defined on a domain U in $\mathbb{R}^n$. Let x_0 be a point in U. Is it true that the differential equation $\frac{dx}{dt} = V(x)$ admits a solution $\phi(t)$ satisfying initial condition $\phi(0) = x_0$ defined on some open interval containing 0. Justify your answer.

(2) Give at least two distinct examples of differential equations that do not admit unique solution but do admit a solution.

(3) Let V_r be a smooth vector field defined on a sphere of radius r that is always tangential to the sphere on which it is defined. Define a vector field on $\mathbb{R}^3$ be declaring $V(x) = r^2(1 - r^2)V_r(x)$. Show that for each t in $\mathbb{R}$ the time t flow associated to this vector field is defined.

(4) Draw the integral curves (flow -lines) corresponding to the following vector-fields defined on $\mathbb{R}^2$.
  - $V(x) = K.x$, K an integer.
  - $V(x, y) = (x, -y)$.

(5) Suppose we are given two vector fields V_1 and V_2 – defined on $\mathbb{R}^n$ – such that the vectors $V_1(x)$ and $V_2(x)$ are linearly independent for each x. Is is possible to find a diffeomorphism ϕ from a neighborhood V of 0 to a neighborhood U of zero such that $(D\phi_x(V_1(x)), D\phi_x(V_2(x)) = (1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0)$ for each x in V. Justify your answer.