Mathematical Physics I, 2008
Assignment 5

1. What are the dimensions of the linear vector spaces which consist of all possible linear combinations of the four vectors $|n\rangle,\ n = 1, 2, ..4$ when,

   (a)
   $$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   (b)
   $$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} \frac{-1}{3} \\ \frac{1}{\sqrt{3}} \\ \frac{\sqrt{5}}{3} \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{\sqrt{3}} \\ \frac{\sqrt{-5}}{3} \\ 0 \end{pmatrix} \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   (c)
   $$|1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad |2\rangle = \frac{1}{2}\begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix} \quad |3\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad |4\rangle = \frac{1}{2}\begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}$$

2. The Pauli spin matrices are defined as,
   $$\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Prove that,

   (a) If the commutator of two matrices A and B is defined as,
   $$[A, B] \equiv AB - BA$$
   Then,
   $$[\sigma^a, \sigma^b] = 2i \sum_{c=1}^{3} \epsilon^{abc} \sigma^c$$

(b) If the anti-commutator of two matrices A and B is defined as,

$$\{A, B\} \equiv AB + BA$$

Then,

$$\{\sigma^a, \sigma^b\} = 2\delta_{ab}$$

(c) If the trace of a matrix A is defined as,

$$\text{tr } A \equiv \sum_{i=1}^{2} A_{ii}$$

and

$$\vec{v} \cdot \vec{\sigma} \equiv \sum_{a=1}^{3} v_i \sigma_i$$

Then

$$\frac{1}{2}\text{tr}\,(\vec{u} \cdot \vec{\sigma}\ \vec{v} \cdot \vec{\sigma}) = \vec{u} \cdot \vec{v}$$

(d)

$$\frac{1}{4i}\text{tr}\,(\vec{u} \cdot \vec{\sigma}\ \vec{v} \cdot \vec{\sigma}\ \vec{w} \cdot \vec{\sigma}) = \vec{u} \cdot \vec{v} \times \vec{w}$$

(e) If,

$$e^A \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Then,

$$e^{i\vec{v}\cdot\vec{\sigma}} = \cos|\vec{v}| + i\frac{\vec{v}}{|\vec{v}|}\sin|\vec{v}|$$

3. Given an $N \times N$ matrix A with components A_{ij}, the determinant of A is defined as,

$$|A| = \sum_P \epsilon_P \prod_{i=1}^{N} A_{iP(i)}$$

where, P denotes a permutation of the integers $1, 2, ..., N$, defined by $i \to P(i)$. $\epsilon_P \equiv 1$ for even permutations and $\epsilon_P \equiv -1$ for odd permutations.

The co-factor of the (ij)th element is defined as follows. For every (ij), define a $N-1 \times N-1$ matrix $M^{(ij)}$ which is obtained from A by deleting the ith row and jth column. The cofactor is then defined as,

$$c_{ij} = (-1)^{(i+j)}|M^{(ij)}|$$

Prove that

$$|A| = \sum_{i=1}^{N} a_{1i} c_{1i}$$

4. Prove that for $N \times N$ matrices:

   (a)
   $$|AB| = |A||B|$$

   (b)
   $$|\lambda A| = \lambda^N |A|$$
   where λ is a complex number.

   (c)
   $$|A| = |A^T|$$

   (d)
   $$|A| = |A^\dagger|^*$$

   (e)
   $$|A^{-1}| = |A|^{-1}$$

   (f) Prove that the determinant of a matrix is unchanged if a row (or column) is added to another row (or column)

   (g) Prove that the determinant of a matrix is zero if two rows (or columns) are equal.

5. Prove that

   (a)
   $$\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = (x_2 - x_1)$$

   (b)
   $$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

(c) If,

$$A_{ij} = (x_j)^{i-1},\ i, j = 1, 2, ..., N$$

Then,

$$|A| = \prod_{i=j+1}^{N} \prod_{j=1}^{N} (x_i - x_j)$$

This is called the Vandermonde determinant.

6. If the secular equation of a $N \times N$ hermitian matrix A is,

$$\sum_{n=1}^{N} a_n \lambda^n = 0$$

Then prove that,

$$\sum_{n=1}^{N} a_n A^n = 0$$

This statement that that a matrix satisfies its secular equation is true in general and not just for hermitian matrices. It is known as the Cayley-Hamilton theorem.

7. Compute the eigenvalues and eigenvectors of

$$h = \vec{k} \cdot \vec{\sigma}$$

where the notation is same as in question 2c and

$$\begin{aligned} k_1 &= k \sin\theta \cos\phi \\ k_2 &= k \sin\theta \sin\phi \\ k_3 &= k \cos\theta \end{aligned}$$

8. If A is an $N \times N$ matrix and B an $M \times M$ matrix. Then $C = A \otimes B$ is an $NM \times NM$ matrix defined as,

$$C_{ik,jl} \equiv A_{ij} B_{kl}$$

Prove that,

(a)

$$\mathrm{Tr} A \otimes B = \mathrm{tr} A\ \mathrm{tr} B$$

(b)

$$|A \otimes B| = |A||B|$$

(c)

$$A \otimes B\ C \otimes D = AC \otimes BD$$

9. Find the eigenvalues and eigenvectors of the following matrices, formed out of direct products of the three Pauli spin matrices defined in question 2

   (a)

   $$h = \sigma^3 \otimes \sigma^3$$

   (b)

   $$h = \sigma^3 \otimes \sigma^1$$

   (c)

   $$h = \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3$$

10. Find the eigenvalues and eigenvectors of the following matrices:

   (a)

   $$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   (b)

   $$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   (c)

   $$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

   (d)

   $$\begin{aligned} U_{ij} &= \delta_{i+1,j} \quad i < N \\ U_{Nj} &= \delta 1 j \end{aligned}$$