Mathematical Physics I, 2008
Assignment 4

1. In a cartesan coordinate system, the position vectors are given by

$$\vec{x} = \sum_{i=1}^{3} x_i \hat{x}_i \quad (1)$$

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij} \quad (2)$$

The components of any vector $\vec{A}$ in this coordinate system is given by A_i, where,

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{x}_i \quad (3)$$

A new coordinate system is defined by $(\Omega_1(x_1, x_2, x_3), \Omega_2(x_1, x_2, x_3), \Omega_3(x_1, x_2, x_3))$ and the basis in this coordinate system defined as,

$$\vec{e}_i \equiv \frac{\partial}{\partial \Omega_i} \vec{x} \quad (4)$$

The components of a vector $\vec{A}$ in this system are $\mathcal{A}_i$, where

$$\vec{A} = \sum_{i=1}^{3} \mathcal{A}_i \vec{e}_i \quad (5)$$

Compute $\vec{e}_i$ and $\mathcal{A}_i, \; i = 1, 2, 3$ when

(a)

$$\begin{aligned} \Omega_1 &= x_1 \; cos\theta + x_2 \; sin\theta \\ \Omega_2 &= -x_1 \; sin\theta + x_2 \; cos\theta \\ \Omega_3 &= x_3 \end{aligned} \quad (6)$$

(Rotated coordinates)

(b)

$$\begin{aligned} \Omega_1 &= x_1^2 - x_2^2 \\ \Omega_2 &= x_1 x_2 \\ \Omega_3 &= x_3 \end{aligned} \quad (7)$$

(A conformal transformation)

(c)

$$
\begin{aligned}
\Omega_1 &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\
\Omega_2 &= cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \\
\Omega_3 &= tan^{-1}\left(\frac{x_2}{x_1}\right) \qquad (8)
\end{aligned}
$$

(Spherical coordinates)

2. If $\vec{A}(\vec{x})$ is a vector field, express its divergence, $\vec{\nabla} \cdot \vec{A}$ in terms of Ω_i and $\mathcal{A}_i$ for all the three coordinate systems in question 1.

3. If $\vec{B} = \sum_{i=1}^{3} \mathcal{B}_i \vec{e}_i = \vec{\nabla} \times \vec{A}$, express $\mathcal{B}_i$ in terms of Ω_i and $\mathcal{A}_i$ for all the three coordinate systems in question 1.

4. If there was a magnetic monopole, its magnetic field in spherical coordinates would be given by,

$$\vec{B} = \frac{g}{r^2} \vec{e}_r \qquad (9)$$

   (a) Show that the divergence of $\vec{B}$ is zero everywhere except the origin.

   (b) Find a vector field $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$ (the vector potential).

   (c) The vector potential $\vec{A}$ will always be singular on a line starting from the origin. Locate this singular line for your solution.

5. Write down a vector field, $\vec{\lambda}(\vec{x})$ such that the line integral

$$\Phi = \oint_C \vec{\lambda} \cdot \vec{dx} \qquad (10)$$

over any closed curve, C which lies on the $x - y$ plane, gives the area of the region in the plane enclosed by the curve.

6. Write down a vector field, $\vec{\beta}(\vec{x})$ such that the line integral

$$\Phi = \oint_C \vec{\beta} \cdot \vec{dx} \qquad (11)$$

over any closed curve, C, which lies on the surface of the unit sphere, gives the area of the region on the unit sphere enclosed by the curve (the solid angle).