Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 9: Complex analysis: general applications

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1. The Poisson integral formula

- We consider the closed unit disk, $U : |z| \leq 1$. Let $f(\theta)$ be a **given continuous function** for $0 \leq \theta \leq 2\pi$. We consider the problem of finding a solution to **Laplace's equation in** $0 \leq r < 1$. Thus we seek $u(r, \theta)$ satisfying:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \nabla^2 u \\ &= 0 \end{aligned}$$

  and the **boundary condition**, $u(1, \theta) = f(\theta)$. This is called the **Dirichlet problem** for the unit disk. We have already seen how knowing a function of a complex variable on the boundary of a closed curve we can calculate its interior values with **Cauchy's integral formula.** We are going to solve the Dirichlet problem for the unit circle using this result.

- Let $z = re^{i\theta}; r < 1; 0 \leq \theta < 2\pi$. Setting $\zeta = e^{i\phi}, d\zeta = i\zeta d\phi$. Now suppose $f(z)$ is holomorphic in U. Then, Cauchy's integral formula gives,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint \frac{f(\zeta)d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} d\phi \end{aligned} \tag{1}$$

1. Poisson formula: contd.

- Now, for any $|z| < 1$, the complex number, $z^* = \frac{1}{\bar{z}}$ must lie outside U. From **Cauchy's theorem**, the function $\frac{f(\zeta)}{\zeta - z^*}$ is analytic on and within the unit disk and we have the relation,

$$\begin{aligned} \frac{1}{2\pi i}\oint \frac{f(\zeta)d\zeta}{\zeta - z^*} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z^*}d\phi \\ &= 0 \end{aligned}$$

Since on the unit circle $\bar{\zeta} = \frac{1}{\zeta}$, we see that,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z^*}d\phi &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\bar{z}}{\bar{z} - \bar{\zeta}}d\phi \\ &= 0 \end{aligned} \quad (2)$$

Now subtracting/adding Eq.(2) from/to Eq.(1), we obtain,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[\frac{\zeta}{\zeta - z} \pm \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right]d\phi \quad (3)$$

2.1 The Poisson kernel

- Consider the $+$ sign first: clearly,

$$\left[\frac{\zeta}{\zeta - z} \pm \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right] = \frac{1 - r^2}{|\zeta - z|^2} \tag{4}$$

The function the right is purely **real and positive** and is a function of $r, (\theta - \phi)$. It is called the **Poisson Kernel** for the unit circle. It follows that,

$$\frac{1 - r^2}{|\zeta - z|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \tag{5}$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)(\frac{1 - r^2}{|\zeta - z|^2})d\phi \tag{6}$$

We may now separate real and imaginary parts in the last equation and letting, $f(z) = u(r, \theta) + iv(r, \theta)$, obtain the relations,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}d\phi \tag{7}$$

$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} v(\phi)\frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}d\phi \tag{8}$$

2.2 The Poisson formulae: discussion

- In these **Poisson formulae**, we see that the interior values of the conjugate harmonic functions $u(r,\theta), v(r,\theta)$ are expressed in terms of their respective values on the boundary. However, the remarkable property of analytic functions is that if one of these harmonic functions is known, the other can be calculated!
- To see this, we take the negative sign in Eq.(3), obtaining,

$$\frac{1-2\zeta\bar{z}+|z|^2}{|\zeta-z|^2} = 1+\frac{2i\mathrm{Im}(z\bar{\zeta})}{|\zeta-z|^2} \tag{9}$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[1+\frac{2i\mathrm{Im}(z\bar{\zeta})}{|\zeta-z|^2}\right]d\phi \tag{10}$$

From Cauchy's integral formula, Eq.(1), we get **Gauss' Mean Value Theorem**:

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi})d\phi \tag{11}$$

2.3 The Poisson integral: discussion

- Taking the imaginary part of Eq.(10), we obtain finally,

$$v(r,\theta) = v(0) + \frac{1}{\pi}\int_0^{2\pi} u(\phi)\frac{r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}d\phi \tag{12}$$

  This expresses $v(r,\theta)$ in terms of the boundary values of its harmonic conjugate, $u(\phi)$. The formulae may be combined to give:

$$f(z) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\frac{\zeta+z}{\zeta-z}d\phi \tag{13}$$

  We **assumed** that $f(z)$ is analytic even on the boundary.

- It can be shown that for **any** continuous $u(\phi)$ on the unit circle, the Poisson integral, Eq.(7) defines a **harmonic function** $u(r,\theta)$ within the unit disk which tends to $u(\phi)$ as $z = re^{i\theta} \to e^{i\phi}$.

3.1 The Maximum-modulus Theorem

- We next discuss a remarkable property of analytic functions, which is also shared by harmonic functions.
- **Theorem 9.1:** If $f(z)$ is holomorphic inside and on a simple closed curve C, then the maximum value of $|f(z)|$ must occur on C, unless $f(z)$ is a constant.
- **Proof:** We know from Cauchy's integral formula and the fact that if $f(z)$ is holomorphic, all positive integral powers of it are too that for any **interior point** z:

$$(f(z))^n = \frac{1}{2\pi i}\oint_C \frac{(f(\zeta))^n}{\zeta - z} d\zeta, \qquad n = 1, 2, ..$$

Taking absolute values and using the fact that as a continuous function on C, $|f(\zeta)|$ must take its maximum value, M (say) on C, we obtain the obvious inequalities:

$$\begin{aligned} |f(z)|^n &\leq \frac{M^n}{2\pi}\oint_C \frac{1}{|z - \zeta|}|d\zeta| \\ &\leq \frac{LM^n}{2\pi d} \end{aligned}$$

where L_C is the length of C and d is the minimum distance of z from C.Taking the nth root on both sides and taking the limit as $n \to \infty$ we get the result, $|f(z)| \leq M$. Obviously, the equality holds only if $f(z)$ is a constant.

2 Maximum-modulus principle: corollarie

- **Theorem 9.2: ("Minimum-modulus theorem")** If $f(z)$ is holomorphic inside and on a simple closed curve C, and $f(z) \neq 0$ in its interior, the **minimum value** of $|f(z)|$ must occur on C.
- **Proof:** Since $1/f(z)$ vanishes nowhere in the interior, it is holomorphic in the interior. From the Maximum-modulus principle, $|1/f(z)|$ cannot have its maximum value in the interior, which means $|f(z)|$ cannot reach its minimum inside the boundary. It must therefore attain its minimum (since it is continuous, it **will** have a minimum!) **on the boundary.**
- **Theorem 9.3: ("Max-min principle for harmonic functions")** If $u(x, y)$ is a real, non-constant **harmonic function** on and within a closed curve C, it cannot have a maximum or minimum at an interior point (x, y).
- **Proof:** If (x, y) is an interior point where $u(x, y)$ has a maximum (or minimum), a **necessary condition is:** $u_x = u_y = 0$. We know that since $u(x, y)$ is a harmonic function, $u_{xx} + u_{yy} = 0$. If $(x + \delta x, y + \delta y)$ is any neigbouring point, we must have from Taylor's theorem (in real variables), $u \simeq u_{xx}(\delta x)^2 + 2u_{xy}\delta x \delta y + u_{yy}(\delta y)^2$. We see that since $u_{xx} \neq 0; u_{xx} = -u_{yy}$, the point (x, y) cannot be a maximum or a minimum, but must be a "saddle point". This proves the theorem, which has interesting applications in physics and engineering.

4.1 Evaluation of integrals-1

- Residue calculus provides a very powerful tool for the evaluation of definite integrals. We will look at a number of typical examples of the techniques involved.

- **Example 1:** Let $f(z)$ be holomorphic on and within the unit circle. Let $Q(z)$ be any polynomial with degree $m \geq n$, having isolated first order zeros, $a_1, .., a_n$ within the unit circle and no zeros on the unit circle. Then the following formula holds:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\exp i\phi)}{Q(\exp i\phi)} \exp i\phi d\phi \quad = \quad \Sigma_{i=1}^n \frac{f(a_i)}{Q'(a_i)} \tag{14}$$

  Proof: Consider the function, $f(z)/Q(z)$. It is analytic and single-valued on and within the unit circle and has simple poles at a_i. From the properties of polynomials, we can write, $Q(z) \equiv (z-a_1)(z-a_2)..(z-a_n)q(z)$, where $q(z)$ is a polynomial having **no** zeros within or on the unit circle. Plainly, the residue of the integrand at $z = a_i$ is given by, $\mathrm{Lim}_{z\to a_i}(z-a_i)f(z)/Q(z) = \frac{f(a_i)}{Q'(a_i)}$. The result follows immediately from **Cauchy's Residue Theorem.**

- If we set, $f(z) = Q'(z)$, it is clear that the RHS equals n, the number of zeros of $Q(z)$ contained within the unit circle, exactly as would be predicted by the **Argument Principle**!

4.2 Evaluation of integrals-2

- **Example 2:** Evaluate, given $a > b > 0$, the integral,

$$I(a,b) \quad = \quad \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

If $z = \exp i\theta, \cos\theta = \frac{1}{2}(z+\frac{1}{z})$. Consider the integral J taken over the unit circle C,

$$\oint_C f(z)dz \quad = \quad \oint_C \frac{dz}{z\left[a+\frac{b}{2}(z+\frac{1}{z})\right]^2}$$

The poles occur at the zeros of the denominator (rationalized), when, $z^2+2(a/b)z+1=0; z=-a\pm(a^2-b^2)^{1/2})/b$. Only the real pole at $z=\frac{(a^2-b^2)^{1/2}-a}{b}=r_1$ lies within the unit circle and can possibly contribute. Note that $r_1+r_2=-2a/b; r_1-r_2=-2(a^2-b^2)^{1/2}/b$. The function is, $\frac{4z}{b^2(z-r_1)^2(z-r_2)^2}$. We must now calculate the residue at the pole enclosed by the unit circle.

4.3. Evaluation of integrals-2

- To calculate the residue at the pole, we set: $z - r_1 = u$. Then

$$\frac{4z}{b^2(z-r_1)^2(z-r_2)^2} = \frac{4}{b^2}\frac{u+r_1}{u^2(u-(r_2-r_1))^2}$$
$$= \frac{4}{b^2(r_2-r_1)^2}\frac{(u+r_1)(1+2\frac{u}{r_2-r_1}+..)}{u^2}$$

The residue then is given by,

$$\frac{4}{b^2(r_2-r_1)^3}(r_1+r_2) = \frac{a}{(a^2-b^2)^{3/2}}$$

- We may now apply Cauchy's Residue theorem and obtain the final result,

$$I(a,b) = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Obviously this method works for any function $Q(\cos\theta, \sin\theta)$ rational in its arguments and integrated over $[0, 2\pi]$.

- Although every integral obtained using the Residue Theorem can also be solved using other techniques, it is one of the most efficient, when applicable.

5.1 Singular and infinite integrals

- I recall some basic facts about "infinite integrals" of continuous functions of a real variable. These will turn out to be important and useful in our study of complex analysis.
- **Definition 9.1** If $f(x)$ is a **real, continuous function** of $a \leq x < \infty$, and if the limit,

$$\mathrm{Lim}_{b\to\infty} \int_a^b f(x)dx \quad = \quad I(a;f)$$

exists, we say that the infinite integral is **convergent** and that $I(a;f)$ is the value of the **infinite integral**, and write,

$$I(a;f) \quad = \quad \int_a^\infty f(x)dx$$

If the limit does not exist we say that the integral is **divergent.** Divergence can happen because the limiting value is either $\pm\infty$, or if there is no limit but the integral, $\int_a^b f(x)dx$ **oscillates** finitely or infinitely. We can also have integrals infinite at **both** upper and lower limits. Thus, if $\int_a^b f(x)dx = I(a,b;f)$ has the property:

$$\mathrm{Lim}_{b\to\infty}\mathrm{Lim}_{a\to-\infty} \int_a^b f(x)dx \quad = \quad I(f)$$

exists, irrespective of the order of limits, the infinite integral, $\int_{-\infty}^{\infty} f(x)dx$ is **convergent.**

5.2 Infinite integrals: examples

- **Example 1:**

$$\int_0^a e^{-t} dt = [-e^{-t}]_0^a = 1 - e^{-a}$$

Clearly, $\mathrm{Lim}_{a\to\infty}(1-e^{-a}) = 1$, hence the integral is convergent: $\int_0^\infty e^{-t} dt = 1$.

Example 2: The integral, $\int_0^\infty \cos x dx$ is divergent (show why this is so!)

Example 3: If $s > 0$, $\int_0^a x^s dx = \frac{a^{s+1}}{s+1}$, for $a > 0$. Obviously, $\int_0^\infty x^s dx$ diverges to infinity (it is unbounded as the upper limit tends to infinity).

Example 4: It is easy to show that for $-1 \leq s \leq 0$, the infinite integral, $\int_1^\infty x^s dx$ is divergent.

Example 5: If $s < -1$, $\int_1^a x^s dx = \frac{1}{s+1}[a^{s+1} - 1]$. In this case, we have convergence and, $\int_1^\infty x^s dx = \frac{1}{|s+1|}$.

- The above infinite integrals have continuous **integrands but the limits tend to infinity.** We could also have **singular integrals** in which the integrand becomes unbounded (and hence discontinuous) at either a limit or at an internal point. The following definition and examples illustrate this:

5.3 Singular integrals

- **Definition 9.2:** Let $f(x)$ be continuous in $x \in (a, b]$. Consider the integral, $I(u, b; f) = \int_u^b f(x)dx; a < u \le b$. Suppose the limit,

$$\mathrm{Lim}_{u \to a+} \int_u^b f(x)dx \tag{15}$$

exists as u tends to a from above. Then we say that the **singular integral** $\int_a^b f(x)dx$ is **convergent** and is equal to this limit. It is implicit here that $\mathrm{Lim}_{x \to a} f(x)$ does **not** exist, as x approaches a from above.
Example 6: Consider the integral: $\int_u^1 x^{-1/2}dx = 2[1 - u^{1/2}]; u > 0$. Obviously we may take the limit as $u \to 0+$. However, the integrand "blows up" at $x = 0$. Thus, we say that the singular integral, $\int_0^1 \frac{dx}{x^{1/2}}$ **converges** to the value, 2.
Example 7: The integral, $\int_u^1 \frac{dx}{x} = \ln(\frac{1}{u})$ evidently tends to infinity as $u \to 0+$. Hence, we say that the singular integral, $\int_0^1 \frac{dx}{x}$ is **divergent.**

- I next formulate some simple facts about infinite and singular integrals we shall need in our work. The proofs will be omitted but are quite simple.

4. Comparison test for absolute convergen

- **Proposition 9.1:** If $f(x)$, assumed continuous in $[a,\infty)$ is such that its absolute value leads to the convergent infinite integral, $\int_a^\infty |f(x)|dx$, the infinite integral, $\int_a^\infty f(x)dx$ must also exist. The integral is said to be **absolutely convergent.** If the latter integral exists, but not the former, we say that the infinite integral, $\int_a^\infty f(x)dx$ is **conditionally convergent.**
- Note that there are conditionally convergent integrals which are **not absolutely convergent.**
- **Proposition 9.2:** If $f(x) \geq 0$ and continuous in $[a,\infty)$ and $\int_a^R f(x) = g(R) < K$, where $R > a$ and K is a fixed constant for any R, the infinite integral, $\int_a^\infty f(x)dx$ is convergent and we also must have, $\int_a^\infty f(x)dx < K$.
- **Proposition 9.3:** If $|f(x)| < g(x)$, where f, g are defined and continuous in $[a,\infty)$, and $\int_a^\infty g(x)dx$ is **convergent.** Then, $\int_a^\infty f(x)dx$ is **absolutely convergent.** This is the integral analogue of the **comparison test** for the convergence of infinite series.
  Example: Consider the important integral, $I = \int_1^\infty e^{-x^2}dx$. How can we prove this infinite integral is convergent without evaluating it? Note that for $x \geq 1, x^2 \geq x$. Hence, $e^{-x^2} \leq e^{-x}$. Since, the integral, $\int_1^\infty e^{-x}dx = e^{-1}$, the comparison test shows that I is convergent and $I < e^{-1}$. We will later evaluate the related, convergent integral, $\int_0^\infty e^{-x^2}dx$. Note that our comparison function cannot be directly used in proving the last integral is convergent! We can easily extend the results to doubly infinite and singular integrals.

5.5 Uniform convergence

- **Proposition 9.5:** Let $f(z,w)$ be a continuous function of complex variables z, w, where z belongs to a region R and w lies on some contour C. Let $f(z,w)$ be holomorphic in R as a function of z for every w on C. Then, $I(z;C) = \int_C f(z,w)dw$ is analytic in R and, $\frac{dI}{dz} = \int_C \frac{\partial f}{\partial z} dw$.
- This justifies "differentiating" under the integral sign" in complex analysis. The proof is omitted.
- **Definition 9.3:** An infinite integral, $\int_0^\infty f(z,x)dx$ is said to be **uniformly convergent** (here $f(z,x)$ is defined and continuous with respect to the real variable x and the complex variable w in suitable regions) if given $\epsilon > 0$ arbitrarily small, we can find an R depending **only** upon ϵ and not on z such that, $|\int_{x_1}^{x_2} f(z,x)dx| < \epsilon$ for arbitrary $R < x_1 < x_2$.
- **Proposition 9.6:** Let C be a contour going to infinity, and any bounded part of it is rectifiable. If the conditions of the preceding proposition hold on any bounded part of C and the integral, $F(z) = \int_C f(z,w)dw$ is **uniformly convergent**, the results of Proposition 9.5 apply. Thus, **uniformly convergent infinite integrals of analytic functions are analytic.**
- We will encounter several instances where these results will be useful and also the analogue of Weierstrass' M-test for absolute and uniform convergence of infinite/singular integrals.

6.1 Infinite integrals and residue calculus

- **Example 1:** Evaluate the definite integral:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

It is well-known that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. It follows that the integral converges to π. We shall integrate this using Cauchy's residue calculus. Consider the holomorphic function, $f(z) = \frac{1}{1+z^2}$. It is analytic in the upper half-plane, except for the simple-pole at $z = +i$. Consider a semi-circular contour $C(R)$ running from $-R$ to R along the real axis and closed by the semi-circle $Re^{i\phi}, 0 \leq \phi \leq \pi$. If $R > 1$, this encloses the pole at $z = i$ where the function has the residue, $\mathrm{Lim}_{z \to i}(z-i)\frac{1}{1+z^2} = \frac{1}{2i}$. From **Cauchy's residue theorem**, we have:

$$\begin{aligned} \oint_{C(R)} \frac{dz}{1+z^2} &= \int_{-R}^{R} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{Re^{i\phi} i d\phi}{1+R^2 e^{2i\phi}} \\ &= (\frac{2\pi i}{2i}) \end{aligned}$$

If we let $R \to \infty$, the first integral tends to I whilst the second goes to zero since, $|\int_0^{\pi} \frac{Re^{i\phi} i d\phi}{1+R^2 e^{2i\phi}}| < R \int_0^{\pi} \frac{d\phi}{R^2-1} < \frac{\pi R}{R^2-1}$ upon using standard inequalities. Thus, $I = \pi$.

6.2 Infinite integrals and residue calculus

- **Example 2:** Evaluate the definite integral:

$$F(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}dx}{1+x^2} \qquad (k > 0)$$

Consider the analytic function, $f(z) = \frac{e^{ikz}}{1+z^2}$. It has simple poles at $z = \pm i$. If $z = Re^{i\theta}; 0 < \theta < \pi$, we have on $R > 1, f(Re^{i\theta}) = \frac{e^{(ikR\cos\theta - kR\sin\theta)}}{1+R^2e^{2i\theta}}$. Thus, we have the following deductions from Cauchy's residue theorem, upon using the contour $C(R)$ of Ex. 1:

$$\begin{aligned} \oint_{C(R)} \frac{e^{ikz}dz}{1+z^2} &= \int_{-R}^{R} \frac{e^{ikx}dx}{1+x^2} + \int_0^{\pi} f(Re^{i\theta})iRe^{i\theta}d\theta \\ &= 2\pi i \mathrm{Res}(z = +i) \\ &= 2\pi i(e^{-k}/2i) \end{aligned}$$

Taking the limit as $R \to \infty$, we get the result:

$$F(k) = \pi e^{-k}$$

since, $|\int_0^{\pi} f(Re^{i\theta})iRe^{i\theta}d\theta| < R\int_0^{\pi} \frac{e^{-kR\sin\theta}d\theta}{R^2-1} < \frac{\pi R}{R^2-1} \to 0$.

6.3 Infinite integrals and residue calculus

- **Example 3:** Evaluate the definite integrals for $k > 0$:

$$G(k) = \int_{-\infty}^{\infty} \frac{x^3 \sin kx dx}{1+x^4}$$
$$H(k) = \int_{-\infty}^{\infty} \frac{x^3 \cos kx dx}{1+x^4}$$

Consider the analytic function, $g(z) = \frac{z^3 e^{ikz}}{1+z^4}$. It has simple poles at $z = e^{i\pi/4}, e^{i3\pi/4}$, in the **upper half-plane.** If $z = Re^{i\theta}; 0 < \theta < \pi$, we have on $R > 1, g(Re^{i\theta}) = \frac{R^3 e^{3i\theta} e^{(ikR\cos\theta - kR\sin\theta)}}{1+R^4 e^{4i\theta}}$. Thus, we infer using the residue theorem and the contour $C(R)$ of Ex. 1,2:

$$\oint_{C(R)} \frac{z^3 e^{ikz} dz}{1+z^4} = \int_{-R}^{R} \frac{x^3 e^{ikx} dx}{1+x^4} + \int_0^{\pi} g(Re^{i\theta}) iRe^{i\theta} d\theta$$
$$= 2\pi i[\mathrm{Res}(z = e^{i\pi/4}) + \mathrm{Res}(z = e^{i3\pi/4})]$$

Taking the limit as $R \to \infty$, and separating real and imaginary parts, we get the required result. First we must calculate the residues at the two simple poles and then we must show that $\mathrm{Lim}_{R\to\infty} \int_0^{\pi} g(Re^{i\theta}) iRe^{i\theta} d\theta = 0$. The first part is left as an exercise for you, but the second part is an important result called **Jordan's Lemma.**

7 Jordan's Lemma

- **Theorem 9.4:** Let $f(z)$ be analytic in the upper half plane with the property, $\text{Lim}_{R\to\infty} f(Re^{i\theta}) = 0$ **uniformly** in θ. Then, for any $k > 0$,

$$\text{Lim}_{R\to\infty} \int_0^{\pi} e^{ikRe^{i\theta}} f(Re^{i\theta}) iRe^{i\theta} d\theta \quad = \quad 0$$

- **Proof:** Consider R sufficiently large so that $|f(Re^{i\theta})| < \epsilon$ uniformly for $0 < \theta < \pi$. Then,

$$|\int_0^{\pi} e^{ikRe^{i\theta}} f(Re^{i\theta}) iRe^{i\theta} d\theta| \quad < \quad 2\epsilon R \int_0^{\pi/2} e^{-kR\sin\theta} d\theta \tag{16}$$

Now, either by drwing the graphs of $y = \sin\theta$ and $y = \frac{2\theta}{\pi}$, or by calculating the derivative, we can show that $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$ for $0 \leq \theta \leq \pi/2$. Hence, $\sin\theta \geq \frac{2\theta}{\pi}$. Using this in the integrand on the RHS of the inequality (16), we get,

$$\begin{aligned} |\int_0^{\pi} e^{ikRe^{i\theta}} f(Re^{i\theta}) iRe^{i\theta} d\theta| \quad &< \quad 2\epsilon R \int_0^{\pi/2} e^{-(2kR/\pi)\theta} d\theta \\ &< \quad \frac{\pi\epsilon}{k} \end{aligned}$$

This proves Jordan's Lemma since ϵ is arbitrary.