Chennai Mathematical Institute
B.Sc Physics

Mathematical methods
Lecture 16 Complex analysis: applications-5

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1.1 Some properties of Legendre functions

- **Special values:** Using the generating function in the form:

$$(1 - 2hz + h^2)^{-1/2} \quad = \quad 1 + hP_1(z) + h^2 P_2(z) + ..$$

We can show that $P_n(1) = 1$, by expanding $(1-h)^{-1}$. Similarly, $P_n(-1) = (-1)^n$ follows by expanding $(1+h)^{-1}$. We also have $P_n(0) = 0$ if n is **odd** and $P_n(0) = (-1)^{n/2} \frac{1.3.5...(n-1)}{2.4..n}$ when n is **even**. This follows by expanding $(1+h^2)^{-1/2}$.

- **Zeros:** The function, $(z^2-1)^n$ has n (multiple) zeros at $z = 1$ and an equal number at $z = -1$. It follows that $P_n(z) \propto \frac{d^n}{dz^n}[(z^2-1)^n] = 0$ must have n **real roots**, lying between ± 1. These roots must be **unequal**. If this is not true, let z_* be a double root. Then, $P_n(z_*) = 0 = P'_n(z_*)$. But the **Legendre equation** is a second-order differential equation with a **unique non-zero solution** through $z = z_*$. But this solution would be identically zero if **both** function and its derivative vanished there! Hence we have reached a contradiction.

- **Recurrence relations:** If we set $K(z,h) = (1 - 2zh + h^2)^{-1/2}$. We have:

$$\frac{\partial K}{\partial h} \quad = \quad -\frac{(h-z)}{(1-2zh+h^2)^{3/2}}$$

$$(1-2zh+h^2)\frac{\partial K}{\partial h} + (h-z)K \quad = \quad 0$$

1.2 Legendre functions: more properties

- If we substitute the expansion, $K = \Sigma h^n P_n(z)$ and equate like powers of h, we obtain the three-term recurrence:

$$nP_n - (2n-1)zP_{n-1} + (n-1)P_{n-2} = 0 \qquad (1)$$

- Differentiating both sides of the generating function with respect to z, we obtain,

$$\begin{aligned}
(1-2zh+h^2)^{-3/2} &= \Sigma_{n=0}^{\infty} h^{n-1} \frac{dP_n}{dz} \\
(z-h)\Sigma_{n=0}^{\infty} h^{n-1} \frac{dP_n}{dz} &= \Sigma_{n=0}^{\infty} n h^{n-1} P_n(z) \\
\text{Now,} \quad \frac{1-hz-h(z-h)}{(1-2zh+h^2)^{3/2}} &= \frac{1}{(1-2zh+h^2)^{1/2}} \\
(1-hz)\Sigma_{n=0}^{\infty} h^{n-1} P_n' - h\Sigma_{n=0}^{\infty} n h^{n-1} P_n &= K(z,h) = \Sigma_{n=0}^{\infty} h^n P_n
\end{aligned}$$

Equating like powers of h on both sides, we get,

$$nP_n - zP_n' = -P_{n-1}' \qquad (2)$$

$$(n+1)P_n = P_{n+1}' - P_{n-1}' \qquad (3)$$

1.3 Legendre functions: contd.

- Since Legendre polynomials are functions of $\cos\theta = z$, we may express them in terms of **trigonometric polynomials**:

$$
\begin{aligned}
P_0(\cos\theta) &= 1 \\
P_1(\cos\theta) &= \cos\theta \\
P_2(\cos\theta) &= \frac{1}{4}(3\cos 2\theta + 1) \\
P_3(\cos\theta) &= \frac{1}{8}[5\cos 3\theta + 3\cos\theta]... \\
P_n(\cos\theta) &= 2\frac{1.3.5...(2n-a)}{2.4.6...2n}[\cos n\theta + \frac{1.n}{1.(2n-a)}\cos(n-2)\theta + \\
&\quad \frac{1.3.n.(n-1)}{1.2.(2n-a)(2n-3)}\cos(n-4)\theta + ..]
\end{aligned}
$$

This is done by writing, $K(\cos\theta, h) = (1 - he^{i\theta})^{-1/2}(1 - he^{-i\theta})^{-1/2}$ and expanding the factors using the **Binomial series** and equating the coef£cients of h^n in both sides.

1.4 Legendre functions: orthogonality

- Legendre polynomials have the following important property:

$$\int_{-1}^{1} x^k P_n(x)dx = 0, k = 0, 1, 2, ..., n-1 \tag{4}$$

Proof: Using **Rodrigues' formula** we have,

$$\int_{-1}^{1} x^k P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1} x^k \frac{d^n}{dx^n}[(x^2-1)^n]dx$$

Repeated integration by parts using the fact that the derivatives of $(x^2-1)^n$ lower order than n vanish at both limits, gives the required result. This shows that,

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, m \neq n$$

This can be proved **directly** from the equations satis£ed by the Legendre functions:

$$\frac{d}{dx}[(1-x^2)\frac{dP_n}{dx}] + n(n+1)P_n = 0$$

$$\frac{d}{dx}[(1-x^2)\frac{dP_m}{dx}] + m(m+1)P_m = 0$$

.5 Legendre-Fourier orthogonal expansion

- Multiplying the £rst equation by $P_m(x)$ and the second by $P_n(x)$, subtracting and integrating over $[-1,1]$, we see that,

$$\int_{-1}^{1}(P_m[(1-x^2)P_n']' - P_n[(1-x^2)P_m']')dx = -\int_{-1}^{1}(1-x^2)[P_m'P_n' - P_n'P_m']dx = 0$$

$$= -(n-m)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)$$

One of the problems shows that:

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}, n = 0, 1, .. \quad (5)$$

It can be shown that an "arbitrary" differentiable function $f(x); x \in [-1,1]$ can be expanded in a **Legendre-Fourier orthogonal expansion**:

$$f(x) = \Sigma_{n=0}^{\infty}\hat{f}_n P_n(x) \quad (6)$$

$$\hat{f}_n = (n+\frac{1}{2})\int_{-1}^{1} f(x)P_n(x)dx \quad (7)$$

2.1 Introduction to Conformal Mapping

- Analytic functions have a remarkable geometrical interpretation: let $w = f(z)$ be analytic in some domain of the z-plane. For each value of z, we have a **complex image** in the number w. Thus we can regard this as a **mapping** of the domain into another domain in the w-plane. Here are some simple examples:

- **Ex. 1: "Translations"**
  Let $w = z + c$, where c is an arbitrary complex constant. Here, the origin in the z-plane is moved to $w = c$. The orientations of the axes in both planes are preserved, as well as lengths and angles between curves. This is a **Euclidean rigid translation.** It is one-one and **invertible**; $z = w - c$ being the **inverse mapping**.

- **Ex. 2: "Pure rotations"**
  Consider $w = ze^{i\theta}$, where, $\theta \in [0, 2\pi]$. This is also invertible and leaves the origin invariant. This is a **Euclidean rigid rotation**. The axes are rotated counter-clockwise through an angle θ.

- **Ex. 3: "Pure scalings"**
  Taking $w = \lambda z$, where, $\lambda > 0$. All lengths are "stretched" by the **scale-factor** λ, amounting to a magni£cation for $\lambda > 1$ and a shrinking if $\lambda < 1$. Again, the transformation is invertible and leaves the origin invariant. This is NOT a **Euclidean transformation.** However, it takes straight lines into straight lines and leaves angles invariant. Only the **scales** of £gures vary, not their **shapes.**

2.2 Analytic mappings: conformality

- Let D be a domain in the z-plane and let $w = f(z)$ be analytic there. Consider a point $z_0 \in D$. Its image in the w-plane is obviously $w_0 = f(z_0)$. Let $z_0 + \delta z$ be a neighbouring point in D to z_0. Its image is of course, $w(z_0 + \delta z) \simeq w_0 + f'(z_0)\delta z$.
- This shows that **so long as we restrict ourselves to the immediate neighbourhood of a point** the mapping is a combination of a "stretch" with $\lambda = |f'(z_0)|$ and rotation through an angle, $\theta = \mathrm{Arg}[f'(z_0)]$, both of which only depend upon $f'(z_0)$. Provided the latter is non-zero we note that the mapping is **locally invertible** in the neighbourhood.
- Now consider **two** in£nitesimal segments, $\delta z, \Delta z$ drawn from z_0. The respective image segments are, $\delta w = f'(z_0)\delta z; \Delta w = f'(z_0)\Delta z$.
- We now have the obvious equations:

$$|\delta w| = |\delta z||f'(z_0)| \quad (8)$$

$$|\Delta w| = |\Delta z||f'(z_0)| \quad (9)$$

$$\mathrm{Arg}[\delta w] = \mathrm{Arg}[\delta z] + \mathrm{Arg}[f'(z_0)] \quad (10)$$

$$\mathrm{Arg}[\Delta w] = \mathrm{Arg}[\Delta z] + \mathrm{Arg}[f'(z_0)] \quad (11)$$

With the remarkable consequence that the **angle** between the segments $\delta w, \Delta w$ is **exactly the same as** that between their respective **pre-images**, $\delta z, \Delta z$.

2.3 Linear mappings, inversion

- The simplest analytic functions are **linear**. Thus, the transformation,

$$w(z) \quad = \quad az + b \tag{12}$$

combines the rigid translation, rotation and magni£cation. Plainly it transforms straight lines into straight lines and circles into circles, but does generally change location, areas, lengths and orientation relative to the original £gure.

- Consider next, the function,

$$w(z) \quad = \quad \frac{1}{z} \tag{13}$$

Clearly this interchanges the origin of the z-plane with the in£nity of the w-plane and vice versa. It is also clear that the unit circle is invariant under the mapping (ie every point $e^{i\theta}$ on it goes into $e^{-i\theta}$).

- Note that the **interior** of the unit circle in the z-plane (ie the set $|z| < 1$) goes into the **exterior of the unit circle** in the w-plane. This transformation is clearly **conformal** except possibly at $z = 0, \infty$ and is called **inversion** with respect to the unit circle. It is its own inverse transformation.

4 Inversion and linear fractional mapping

- **Inversion takes straight lines and circles into straight lines and circles:** If t is a real parameter, $z = at + b$, (where $a = \alpha + i\beta; b = \gamma + i\delta$ are complex constants) describes a straight line in the z-plane. Inversion in the unit circle gives,

$$
\begin{aligned}
w &= \frac{1}{at+b} \\
t &= \frac{1}{a}(\frac{1}{w} - b) = \frac{1}{\bar{a}}(\frac{1}{\bar{w}} - \bar{b})
\end{aligned}
$$

If b/a is real, this is a straight line through $w = 0$. Otherwise, it is a circle, also passing through this point. It is not hard to show that the circle, $|z - c| = R$ is transformed by inversion into a circle, or exceptionally into a straight line. We next consider an important class of transformations called **linear fractional/bilinear/homographic/Möbius mappings**:

$$
w = \frac{az+b}{cz+d} \tag{14}
$$

where a, b, c, d are arbitrary complex constants subject **only** to the restriction $ad - bc \neq 0$. If the condition is not satis£ed and $ad = bc$, it is clear that the mapping becomes $w = const$, and is thus not a mapping at all, so henceforth this condition is always implied.

2.5 Properties of linear fractional maps

- We can solve for z in terms of w and obtain,

$$\begin{aligned} (cw-a)z &= b-dw \\ z &= \frac{-dw+b}{cw-a} \end{aligned} \tag{15}$$

Thus, the inverse of a linear fractional transformation is also a linear fractional transformation. We note that if $c=0$, the transformation becomes **linear.** Otherwise,

$$w = \frac{a}{c} + \frac{(bc-ad)}{c(cz+d)}$$

If we apply a linear fractional transformation T_1 and follow it up with another, T_2, we can see that this "product" must also be a linear fractional transformation:

$$\begin{aligned} w_1(z) &= \frac{a_1 z + b_1}{c_1 z + d_1} \\ w_2(w_1(z)) &= \frac{a_2 w_1 + b_2}{c_2 w_1 + d_2} \\ w_1 * w_2[z] &= \frac{a_2(a_1 z + b_1) + b_2(c_1 z + d_1)}{c_2(a_1 z + b_1) + d_2(c_1 z + d_1)} \\ &= \frac{Az+B}{Cz+D} \end{aligned}$$

2.6 Linear fractional maps

- In “composing” two linear fractional maps, we £nd that:

$$\begin{aligned} A &= a_2a_1 + b_2c_1 \\ B &= a_2b_1 + b_2d_1 \\ C &= c_2a_1 + d_2c_1 \\ D &= c_2b_1 + d_2d_1 \end{aligned}$$

  This is **exactly** the **matrix multiplication rule** for the 2×2 matrices of coef£cients of the linear fractional transformations. Furthermore, the condition imposed amounts to saying that the matrices must be non-singular since $ad - bc$ is the determinant of the matrix. It follows that the set of all linear fractional transformations form a **group.** The **identity** transformation, $w = z$ is represented by the identity matrix: $a = 1; b = 0; c = 0; d = 1$.

- You should note that a non-singular matrix and its multiple by a non-zero complex number represent the **same** linear fractional transformation. Thus only **three** parameters are needed to specify a bilinear map. Note also that a linear fractional transformation has the property,

$$\frac{dw}{dz} = \frac{ad - bc}{(cz + d)^2}$$

  It is therefore **conformal** except possibly at $z = -d/c; \infty$.

2.7 More about bilinear maps

- Every bilinear map can be obtained through a sequence of **translation, rotation, stretching and inversion.** Thus,

$$w = \frac{az+b}{cz+d}$$

is equivalent to,

$$\zeta = z + \frac{d}{c}; \xi = \frac{1}{\zeta}$$

$$w = \frac{a}{c} + (\frac{bc-ad}{c^2})\xi$$

- The equation to a circle of radius R and centre ζ is clearly,

$$(z-\zeta)(\bar{z}-\bar{\zeta}) = R^2$$

$$z\bar{z} - z\bar{\zeta} - \bar{z}\zeta - R^2 + \zeta\bar{\zeta} = 0$$

If we "scale" $z = \alpha R z_*; \zeta = R\zeta_*,$ the equation becomes,

$$\alpha\bar{\alpha} z_* \bar{z_*} - z_*\bar{\zeta_*} - \bar{z_*}\zeta_* + \zeta_*\bar{\zeta_*} - 1 = 0$$

2.8 Bilinear maps of circles

- Thus any equation of the form,

$$A z\bar{z} + B\bar{z} + \bar{B}z + C = 0 \qquad (16)$$

with $A > 0$, C real and B complex represents a circle. If $A = 0$, it degenerates to a straight line. Since the "primitive" transformations making up a bilinear transformation takes such an equation into one of the same form, so do bilinear maps.

- We **can** regard a bilinear map as one which takes a point z to the point $T(z)$ in the **same complex plane.**

$$w(z) = z = \frac{az+b}{cz+d}$$

$$cz^2 + (d-a)z - b = 0$$

$$z = \frac{(a-d) \pm [(a-d)^2 + 4bc]^{1/2}}{2c}$$

These are called the **invariant points** of the map. We can show also that every bilinear transformation is **univalent** in the entire complex plane. Thus let $z_1 \neq z_2$ be any two distinct points. Then,

$$w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \neq 0$$

2.9 Bilinear invariant: the cross-ratio

- Let $z_i : i = 1, 2, 3, 4$ be any four distinct points and let w_i be their images under a bilinear transformation. Then, we have:

$$
\begin{aligned}
(w_1 - w_2)(w_3 - w_4) &= [\frac{(ad-bc)^2}{\Pi_{i=1}^{4}(cz_i + d)}](z_1 - z_2)(z_3 - z_4) \\
(w_1 - w_4)(w_3 - w_2) &= [\frac{(ad-bc)^2}{\Pi_{i=1}^{4}(cz_i + d)}](z_1 - z_4)(z_3 - z_2)
\end{aligned}
$$

Dividing the two equations and cancelling the common fact, we obtain the remarkable relation called the **cross-ratio invariant** of a bilinear transformation:

$$
\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \qquad (17)
$$

If three of the points are taken as **given**, together with their images, the fourth point can be considered to be $z = z_4$ and correspondingly $w = w_4$. Thus we can write the original bilnear map in the equivalent form:

$$
\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} \qquad (18)
$$

3.1 Bilinear maps and invariant points

- In general, bilinear maps have two distinct invariant points. Take them to be α, β. The most general bilinear map with these invariant points is:

$$\frac{w-\alpha}{w-\beta} = \lambda \frac{z-\alpha}{z-\beta} \quad (19)$$

where λ is a complex constant.

- We can also £nd a bilinear transformation which maps the unit disk $|z| < 1$ into $|w| < 1$. If the point $z = \alpha, |\alpha| < 1$ goes to $w = 0$, we must have,

$$w = \lambda \frac{z-\alpha}{1-\bar{\alpha}z} \quad (20)$$

has the property that $|z| = 1$ goes to $|w| = 1$ if and only if $\lambda = e^{i\phi}; 0 \leq \phi \leq 2\pi$.

- The bilinear transformation which takes the upper half-plane, $\mathrm{Im}(z) > 0$ into the unit disk, $|w| < 1$ must obviously take the form,

$$w = e^{i\phi} \frac{z-\alpha}{z-\bar{\alpha}} \quad (21)$$

provided we have $\mathrm{Im}(\alpha) > 0$ since if $\mathrm{Im}(z) = 0, |w| = 1$, and α in the upper half-plane simultaneously is mapped into $w = 0$. Obviously $|w| > 1$ corresponds to the lower half-plane.

3.2 More general conformal mappings

- Consider the analytic function, $f(z) = z^m$, where $m > 1$ is an integer. Evidently this has **critical points** at $z = 0, \infty$. The mapping, $w = z^m$ leaves the positive real axis invariant. It takes the ray, $z = re^{\frac{i\pi}{m}}, r > 0$ into the negative real axis, $w = -r$. Thus the interior of the sector in the z-plane, $0 \leq \theta \leq \frac{\pi}{m}$ is mapped onto the upper half w-plane leaving the origin and in£nity as invariant points. The inverse mapping correspondingly maps the upper half plane to the sector.
- The following idea is sometimes useful: if a smooth curve C in the z-plane (may or may not be closed) is given parametrically by $z = F(t) + iG(t); -\infty < t \leq \infty$, the transformation, $z = F(w) + iG(w)$ maps the real axis in the w plane to C.
- **Exponential map:** Consider $w = e^{\frac{\pi z}{a}} : a > 0$. We see that the entire real axis, $\mathrm{Im}(z) = 0$, is mapped onto the **positive** real axis, with $z = 0 \rightarrow w = 1$ and $z = -\infty \rightarrow w = 0$. The parallel line $\mathrm{Im}(z) = ia$ transforms into the **negative** real axis in the w-plane, with $z = ia \rightarrow w = -1$ and $z = ia + \infty \rightarrow w = -\infty$. We see that this mapping takes the **strip** $0 \leq \mathrm{Im}(z) \leq a$ to the **upper-half** w-plane.
- **The sine map:** Consider the mapping, $w = \sin(\frac{\pi z}{a}); a > 0$. Obviously the lines $z = \frac{(\pm a)}{2} + it; t > 0$ go into $w = \sin(\pm\frac{\pi}{2} + i\frac{\pi t}{a}) = \pm\cosh\frac{\pi t}{a}$

3.3 Riemann’s Mapping Theorem

- **The fundamental problem of conformal mapping:** Given a domain D in the z-plane, can it be mapped conformally into any given domain of the w-plane? In particular, can we map D into the unit disk $U_w : |w| < 1$? The answer is given by an amazing theorem due to **Riemann.**

- **Caveat:** D cannot be **completely arbitrary!** There is a restriction on the boundary: suppose D has a **single** boundary point. We can take this to be the point at in£nity. Then if we had a mapping function, $w = f(z)$, it would map the whole **£nite** complex plane into $|w| < 1$. It is therefore a **bounded entire function!** By **Liouville’s Theorem** it would have to be a constant, meaning there is no such mapping.

- **Riemann’s Mapping Theorem:** If D is a domain bounded by a simple closed contour C, there exists a **unique** analytic function, $w = f(z)$, which maps D conformally into $|w| < 1$ and also transforms a point $z = a$ within C to the origin ($w = 0$). and a given direction at $z = a$ into the real w-axis direction there.

- The proof is dif£cult and not required! **Riemann’s Theorem** is intimately connected with **Dirichlet’s problem** for the domain D: to £nd a **harmonic function** $U(x, y) : (x, y) \in D$ which takes on given boundary values on C and satis£es **Laplace’s equation** in D:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (22)$$

3.4 Dirichlet's Principle

- **Motivation for Riemann's Theorem:** Assume that $w = f(z)$ exists. Clearly $\frac{f(z)}{z-a} \neq 0$ and is regular in D. We therefore write, $e^{\phi(z)} = \frac{f(z)}{z-a}$, and recognize that since $\phi(z) = U(x,y) + iV(x,y)$ then it is analytic in D. Furthermore, since $w = f(z) = (z-a)e^{\phi(z)}$ must satisfy $|w| = 1$ on C, $\ln|z-a| + U(x,y) = 0$ on it.

- **All** we have to do to get $w(z)$, are the following steps, setting $a = \alpha + i\beta$:
  1. Solve the **special Dirichlet problem** in D for $U(x,y;\alpha,\beta)$ with $U = -\ln|z-a|$ on C.
  2. Construct $V(x,y)$, the **conjugate harmonic function** associated with U.
  3. Put $\phi(z) = U + iV, w(z) = (z-a)e^{\phi(z)}$.

- **An important observation:** The **real function** de£ned by,

$$G(x,y;\alpha,\beta) \quad = \quad \frac{1}{2}\ln[(x-\alpha)^2 + (y-\beta)^2] + U(x,y;\alpha,\beta) \tag{23}$$

  is harmonic in D except at (α,β) (a logarithmic singularity) and **zero** on C. It is called the **Green's function** for the domain and Laplace's equation, invented by **George Green** himself. Thus G implies $w(z)$ and vice versa. This has a **physical meaning!** In **2-d electrostatics** the electric £eld $\mathbf{E} = -\frac{\partial\Phi}{\partial x}\mathbf{i} - \frac{\partial\Phi}{\partial y}\mathbf{j}$. In the absence of space-charge, $\nabla.\mathbf{E} = 0 \rightarrow \Phi(x,y)$ is a harmonic function. Given D we can ask, "what is the electric potential due to a (line) charge Q placed at $z = a = \alpha + i\beta$ if C is an **equipotential**?"

3.5 Examples

- Plainly, $\Phi(x, y; \alpha, \beta) = -\frac{Q}{2\pi\epsilon_0} G$, as a simple calculation shows. It is also known from electrostatics that the **harmonic function** U which solves the Dirichlet problem stated has the following property: among all possible differentiable functions $u(x, y)$ in D which satisfy the **same boundary condition** on C as U, the electrostatic energy, proportional to,

$$\mathcal{E}[u] \quad = \quad \int_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2] dx dy$$

  assumes the **least value** for the harmonic function U. This is known as **Dirichlet's Principle**. On physical grounds it seems obvious but is hard to prove!

- **Example 1:** Consider the function, $f(z) = -\frac{Q}{2\pi\epsilon_0} \ln z = \frac{Q}{2\pi\epsilon_0} [\ln r - i\theta]$, Q being the charge per unit length (in Coulombs). The electric £eld is purely radial, $E_r = \frac{Q}{2\pi\epsilon_0 r}$ (V/m). The equipotentials are circles. Now consider, $f(z) = -\frac{Q}{2\pi\epsilon_0} \ln[\frac{(z-a)}{(1-z\bar{a})}]$.If $z = e^{i\theta}$, we see that $f(e^{i\theta}) = -\frac{Q}{2\pi\epsilon_0} \ln[\frac{(e^{i\theta}-a)}{(e^{-i\theta}-\bar{a})}] + i\frac{Q}{2\pi\epsilon_0}\theta$, ie $\mathrm{Re}[f(e^{i\theta})] = 0$.

- **Example 2:** The **complex potential** $\Omega(z) = \Phi(x, y) + i\Psi(x, y)$ in **hydrodynamics:** we have, $\frac{d\Omega}{dz} = \Phi_x + i\Psi_x = \Phi_x - i\Phi_y = V_x - iV_y$ gives the veocity components. The curves $\Psi = const$ are called **stream lines**. The potential $\Omega = V_\infty z; V_\infty > 0$, describes **uniform ¤ow** in the upper/lower half spaces and has $\Psi = 0$ on the real axis. If $z = \zeta + \frac{a^2}{\zeta}$, what happens to $\Psi = 0$ in $\Omega(\zeta) = V_\infty[\zeta + \frac{a^2}{\zeta}]; a > 0$?

4.1 Concluding remarks

- In this course I have tried to introduce you to the **elementary** principles and techniques of **Complex Analysis.** This is, as I hope you appreciate, a very beautiful and powerful branch of mathematics with countless applications in physics and engineering.
- In view of the limitations on time and the fact that this is a course of **mathematical methods**, I have had to present many topics at possibly insuf£cient depth.
- However, if you have a committed approach to problem-solving and tackle all the exercises, you will have acquired enough mastery to apply the methods described in this course with con£dence.
- As this is huge subject which is still actively being studied, there is an enormous literature. I have consulted the many excellent texts available in preparing this set of 16 lectures. I attach a list for your further reference and study.

References and suggested further reading

1. **Theory and problems of Complex Variables**, S. Lipschutz„ Schaum's Outline Series, McGraw-Hill, 1974. (Extremely useful compendium!)
2. **Modern Analysis**, E.T. Whittaker and G.N. Watson, Cambridge U.P, 4th Edn. 1992. (A classic text, but quite advanced later chapters).
3. **Theory of functions of a complex variable**, E.T. Copson, Oxford U.P, 1960. (Possibly the best all round pure mathematics text.)
4. **Mathematical methods of physics**, J. Mathews and R.L. Walker, Benjamin, 1965. (Somewhat advanced but full of valuable examples).
5. **Functions of a complex variable**, G.F. Carrier, M. Krook and C.E. Pearson, McGraw-Hill, 1966. (Dif£cult in places, but well worth detailed study).
6. **Applied functions of a complex variable**, A. Kyrala, Wiley-Interscience, 1972. (Very good, intermediate to advanced level text.)
7. **Functions of a complex variable**, G. Moretti, Prentice-Hall, 1964. (Excellent, with many illustrations).
8. **Partial differential equations in physics**, A. Sommerfeld, Volume **6**, Academic Press, 1964. (A wonderful text by a great physicist and teacher: Ch. 4 is suitable for this course.)