Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 15 Complex analysis: applications-4

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1.1 Bessel coefficients: generating function

- We next consider a rather different approach to Bessel functions: consider the function of **two** complex variables z, t defined by:

$$G(z,t) \quad = \quad e^{\frac{z}{2}(t-\frac{1}{t})} \tag{1}$$

The function $G(z,t)$ is clearly a single-valued analytic function of t, for $0 < |t| < \infty$, for any z. It has essential singularities at $t = 0$ and at infinity. It can therefore be expanded in a **Laurent** series:

$$G(z,t) \quad = \quad \Sigma_{n=-\infty}^{\infty} J_n(z) t^n$$

The coefficients $J_n(z)$ appearing in this expansion are called **Bessel coefficients.** We shall shortly see that they are in fact Bessel functions of integral order in z. From **Laurent's theorem** we have,

$$J_n(z) \quad = \quad \frac{1}{2\pi i} \oint_C u^{-(n+1)} e^{\frac{z}{2}(u-\frac{1}{u})} du$$

where C is any closed curve encircling the origin once counter-clockwise.

1.2 Bessel coefficients: properties

- The power series for the coefficients can be obtained as follows. Setting $u = \frac{2v}{z}$ The above integral becomes,

$$J_n(z) = \frac{1}{2\pi i}(\frac{z}{2})^n \oint_C v^{-(n+1)} e^{(v - \frac{z^2}{4v})} dv$$

The contour can be taken as the unit circle. We may expand the uniformly convergent series in powers of z and obtain,

$$J_n(z) = \frac{1}{2\pi i}\Sigma_{j=0}^{\infty} \frac{(-1)^j}{j!}(\frac{z}{2})^{n+2j} \oint_{|v|=1} v^{-(n+1+j)} e^v dv$$

Evidently, if $n + j \geq 0$, the residue at $|v| = 0$ is $\frac{1}{(n+j)!}$. When $n + j$ is a negative integer, the residue is zero. Hence, we get the series expansion for $n \geq 0$ (and of course an integer!):

$$J_n(z) = \Sigma_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}(\frac{z}{2})^{n+2j}$$
$$= \frac{z^n}{2^n n!}[1 - \frac{1}{1!(n+1)}(\frac{z}{2})^2 + \frac{1}{2!(n+1)(n+2)}(\frac{z}{2})^4 ..] \quad (2)$$

Comparison with Eq.(12) of Lecture 14 shows that this **Bessel coefficient** is indeed identical with the “Bessel function” of integer order n we considered there.

1.3 Bessel functions: contd.

- When $n = -m$, a negative integer, we have similarly,

$$J_n(z) = \Sigma_{j=m}^{\infty} \frac{(-1)^j}{j!(j-m)!} (\frac{z}{2})^{2j-m}$$
$$= \Sigma_{k=0}^{\infty} \frac{(-1)^{k+m}}{k!(k+m)!} (\frac{z}{2})^{2k+m} \quad (3)$$

It follows directly from this that $J_{-n}(z) = (-1)^n J_n(z)$. We can also derive some useful **recurrence relations** from the generating function:

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z) \quad (4)$$
$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{dJ_n}{dz} \quad (5)$$

To prove these, first differentiate the generating formula,

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \Sigma_{n=-\infty}^{n=+\infty} J_n(z) t^n \quad (6)$$

with respect to t and equate like powers of t. Secondly, differentiate with respect to z and equate coefficients. We can also deduce the relations: $\frac{d}{dz}[z^n J_n(z)] = z^n J_{n-1}(z)$ $\frac{d}{dz}[z^{-n} J_n(z)] = -z^{-n} J_{n+1}(z)$.

2.1 Integral representations

- We can derive an interesting integral representation for $J_n(z)$ from the generating function. Putting $t = e^{i\theta}$ in Eq.(6), we obtain,

$$
\begin{aligned}
e^{iz\sin\theta} &= \Sigma_{n=-\infty}^{n=+\infty} J_n(z) e^{in\theta} \\
J_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta - in\theta}\, d\theta \qquad (7)
\end{aligned}
$$

- We consider (briefly!) a method of solving Bessel's equation by contour integrals which resembles the **Laplace transform** closely. We wish to write the solution in the form,

$$y(z) = z^{\nu}\int_a^b e^{izt}\hat{Y}(t)dt \qquad (8)$$

We have to determine the function $\hat{Y}(t)$ and the limits a, b so that $y(z)$ satisfies:

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + (1 - \frac{\nu^2}{z^2})y = 0 \qquad (9)$$

2.2 Solution by contour integrals

- We then find that,

$$
\begin{aligned}
z\frac{dy}{dz} &= \nu y + z^{\nu+1}\int_a^b e^{izt}\hat{Y}(t)itdt \\
z\frac{d}{dz}(z\frac{dy}{dz}) + (z^2-\nu^2)y &= (2\nu+1)z^{\nu+1}\int_a^b e^{izt}\hat{Y}(t)itdt + z^{\nu+2}\int_a^b e^{izt}\hat{Y}(t)(1-t^2)dt \\
&= -iz^{\nu+1}[e^{izt}\hat{Y}(t)(1-t^2)]_a^b \\
&\quad +iz^{\nu+1}\int_a^b e^{izt}[(2\nu+1)\hat{Y}t + \frac{d}{dt}(\hat{Y}(1-t^2))]dt
\end{aligned}
$$

This shows that to satisfy Bessel's equation, we must solve,

$$
\begin{aligned}
\frac{d}{dt}[\hat{Y}(1-t^2)] + (2\nu+1)t\hat{Y} &= 0 \quad \text{namely,} \\
(t^2-1)\frac{d\hat{Y}}{dt} &= (2\nu-1)t\hat{Y}
\end{aligned}
$$

The solution is easy: $\hat{Y} = (t^2-1)^{\nu-\frac{1}{2}}$. We must also choose the limits so that the integrated term vanishes. There are many ways of doing this, leading to different integral representations.

2.3 Hankel's formula

- As an example, let us consider the case when, $\mathrm{Re}(z) > 0$ and $\nu + 1/2$ is not a positive integer. We take a contour which runs from $t = +i\infty$ to $t = (1+r)i, r > 0,$ goes round the origin counter-clockwise on the circle $|t| = 1 + r$ and returns to $i\infty$. It is clear that the integrated term vanishes at $t = a = b = i\infty$. We see that this contour contains within it $t = \pm 1$ and plainly, the function, $(t^2-1)^{\nu-\frac{1}{2}}$ can be expanded in the **binomial series in** $1/t^2$:

$$(t^2-1)^{\nu-\frac{1}{2}} = \Sigma_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}-\nu+m)}{\Gamma(m+1)\Gamma(\frac{1}{2}-\nu)} t^{2\nu-1-2m}$$

We may multiply this by e^{izt} and integrate term-by-term and obtain:

$$z^{\nu} \int_{i\infty}^{-1,+1} e^{izt}(t^2-1)^{\nu-\frac{1}{2}} dt = \Sigma_{m=0}^{\infty} \frac{z^{\nu}\Gamma(\frac{1}{2}-\nu+m)}{\Gamma(m+1)\Gamma(\frac{1}{2}-\nu)} \int_{i\infty}^{-1,+1} t^{2\nu-1-2m} e^{izt} dt$$

Using the properties of the Gamma function, it is easily shown that,

$$\int_{i\infty}^{-1,+1} t^{2\nu-1-2m} e^{izt} dt = -2\pi i \frac{(-1)^{m+1} e^{-\nu\pi i} z^{2m-2\nu}}{\Gamma(2m-2v+1)}$$

$$J_{-\nu}(z) = \frac{\Gamma(\frac{1}{2}-\nu) e^{\nu\pi i} (\frac{z}{2})^{\nu}}{2\pi i \Gamma(\frac{1}{2})} \int_{i\infty}^{-1,+1} e^{izt}(t^2-1)^{\nu-\frac{1}{2}} dt \quad (10)$$

3.1 Wave equation: Sommerfeld integrals

- We consider the solutions of the 2-d **D'Alembert Wave Equation** in cylindrical polar coordinates, (r, θ):

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (11)$$

where c is the **constant wave speed** and $u(r, \theta, t)$ is the amplitude of this scalar wave. We look for solutions of the form, $u \simeq U e^{-i\omega t}$. We then see that u_* satisfies **Helmholtz's equation**, where $k = \frac{\omega}{c}$ is called the wave number:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial \theta^2} + k^2 U = 0 \quad (12)$$

Using the **Cartesian form**, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, we can verify that the "plane wave" $U = Ae^{ixk_x + iyk_y}; k^2 = k_x^2 + k_y^2$, satisfies the equation, where A, k_x, k_y are any constants. This becomes in polars, $U = Ae^{ikr\cos(\theta - \alpha)}$, where $k_x = k\cos\alpha; k_y = k\sin\alpha$. This can be checked by direct substitution in Eq.(12). Setting $\rho = kr$, we look for solutions of the form, $U = Z_n(\rho)e^{in\theta}$. We can get solutions of this type by **superposing** several plane waves: $U = A\int_a^b e^{i\rho\cos(\theta-\alpha)}e^{in\alpha}d\alpha$. Put, $\alpha = v + \theta; a = v_0 + \theta; v_1 = b + \theta \rightarrow U = Ae^{in\theta}\int_{v_0}^{v_1} e^{i\rho\cos v + inv}dv$. The idea is to choose v_0, v_1 and a suitable contour so that the integral becomes **only a function of** ρ.

3.2 Sommerfeld-Debye integrals

- The **Sommerfeld contours** are chosen as follows: we first apply a simple shift and express the integral in terms of $\lambda = v - \pi/2$. The integral becomes, apart from a constant, $w(\rho) = \int_C e^{in\lambda - i\rho\sin\lambda} d\lambda$ for a suitable contour C. We then take $\mathrm{Re}(\rho) > 0$ and consider the contour $C(-\pi + i\infty, \pi + i\infty)$: this consists of the vertical line in the upper half-plane, $\mathrm{Re}(\lambda) = -\pi; \mathrm{Im}(\lambda) \geq 0$, the segment of the real axis, $-\pi \leq \lambda \leq \pi$ and the parallel verical line, $\mathrm{Re}(\lambda) = \pi; \mathrm{Im}(\lambda) \geq 0$. Integrating along these lines,

$$\begin{aligned} w(\rho) &= \int_{C:-\pi+i\infty}^{-\pi} + \int_{C:-\pi}^{\pi} + \int_{C:\pi}^{\pi+i\infty} \\ &= -ie^{-in\pi}\int_0^\infty e^{-(\rho\sinh t + nt)} dt + \int_{-\pi}^{\pi} e^{-i\rho\sin\lambda + in\lambda} d\lambda \\ &\quad + ie^{in\pi}\int_0^\infty e^{-(\rho\sinh t + nt)} dt \\ J_n(\rho) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\lambda - \rho\sin\lambda)} d\lambda - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-(\rho\sinh t + nt)} dt \qquad (13) \end{aligned}$$

  where $J_n(\rho) = \frac{w(\rho)}{2\pi}$ is the normalisation needed to conform to standard expressions. Note that when n is an integer, this reduces to Eq.(7), but now represents, by analytic continuation, a solution of Bessel's equation for any n and ρ!

.3 Hankel functions: Sommerfeld integrals

- Suppose we take C_1 to run from $-\pi/2 + i\infty \to -\pi/2 \to \pi/2 \to \pi/2 + i\infty$ and integrate $e^{iz\cos t + i\nu(t-\pi/2)}; \mathrm{Re}(z) > 0$. This defines a new linear combination of Bessel functions called a **Hankel function**:

$$H_\nu^{(1)}(z) = \frac{1}{\pi}\int_{C_1} e^{i[z\cos t + \nu(t-\pi/2)]}dt = J_\nu(z) + iY_\nu(z) \quad (14)$$

Similarly, when $C_2 : \pi/2 - i\infty \to \pi/2 \to 3\pi/2 \to 3\pi/2 + i\infty$ we get,

$$H_\nu^{(2)}(z) = \frac{1}{\pi}\int_{C_2} e^{i[z\cos t + \nu(t-\pi/2)]}dt = J_\nu(z) - iY_\nu(z) \quad (15)$$

where $Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$.

- Two special cases for $0 < \mathrm{Arg}(z) < \pi$ where C_1 is the imaginary axis and $C_2 : -i\infty \to 0 \to 2\pi \to 2\pi + i\infty$ are:

$$H_\nu^{(1)}(z) = -\frac{i}{\pi}e^{-i\nu\pi/2}\int_{-\infty}^{\infty} e^{iz\cosh t - \nu t}dt \quad (16)$$

$$H_\nu^{(2)}(z) = \frac{2}{\pi}e^{i\nu\pi/2}[\int_0^{\pi} e^{-izcost}\cos(\nu t) + i\int_0^{\infty} e^{iz\cosh t}\cosh(\nu t - i\nu\pi)dt] \quad (17)$$

3.4 Asymptotic expansions

- The behaviour of Bessel functions for **fixed** ν and large $|z|$ can be guessed from the defining Eq.(9). Make the substitution (this is called a **Liouville transformation**) $y(z) = (z)^{-1/2}Y(z)$. Then, $Y(z)$ satisfies the equation,

$$Y'' + Y[1 + \frac{1/4 - \nu^2}{z^2}] \quad = \quad 0 \tag{18}$$

For $|z|$ large, the $\frac{1}{z^2}$ term in the equation is negligible and we see that $Y \simeq A_+ e^{iz} + A_- e^{-iz} \rightarrow y(z) \simeq \frac{A_+ e^{iz} + A_- e^{-iz}}{z^{1/2}}$. The problem is to precisely determine the constants and find higher order corrections. The **method of steepest descents** can be applied and one find the important formulae:

$$J_\nu(z) \quad \simeq \quad (\frac{2}{\pi z})^{1/2}[\cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) - \frac{\nu^2 - \frac{1}{4}}{2z}\sin(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + ..] \tag{19}$$

$$H_\nu^{(1)}(z) \quad \simeq \quad (\frac{2}{\pi z})^{1/2} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}[1 + \frac{i(\nu^2 - \frac{1}{4})}{2z} + ..] \tag{20}$$

$$H_\nu^{(2)}(z) \quad \simeq \quad (\frac{2}{\pi z})^{1/2} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}[1 - \frac{i(\nu^2 - \frac{1}{4})}{2z} + ..] \tag{21}$$

These are valid for $0 < \mathrm{Arg}(z) < \pi$.

4.1 Laplace's equation: spherical polars

- We have seen that **Bessel functions** arise naturally when we consider the wave equation in a cylinder. If we wish to solve **Laplace's equation** in spherical coordinates, we write $x = r\sin\theta\cos\phi; y = r\sin\theta\sin\phi; z = r\cos\theta$, and obtain the form:

$$\nabla^2\Phi = \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}] = 0$$

Upon separating variables, we encounter **Legendre functions** which are also related to functions called **spherical harmonics** which prove useful in mathematical physics. Thus setting $\Phi = F(r)G(\theta)H(\phi)$ and substituting in Laplace's equation we get,

$$\frac{d}{dr}(r^2\frac{dF}{dr})GH + \frac{FH}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dG}{d\theta}) + \frac{FG}{\sin^2\theta}\frac{d^2H}{d\phi^2} = 0$$

If we divide this by FGH, the equation can only be satisfied if the first term is a constant, which we take to be the complex number $n(n+1)$:

$$\frac{d}{dr}(r^2\frac{dF}{dr}) = n(n+1)F \tag{22}$$

$$\frac{1}{H}\frac{d^2H}{d\phi^2} + \frac{\sin\theta}{G}\frac{d}{d\theta}(\sin\theta\frac{dG}{d\theta}) + n(n+1)\sin^2\theta = 0 \tag{23}$$

4.2 Legendre's associated equation

- In Eq.(23) the first term can be separated by equating it to a new separation constant, $-m^2$. Then we find that, G, H satisfy the equations:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dG}{d\theta}) + [n(n+1) - \frac{m^2}{\sin^2\theta}]G = 0 \qquad (24)$$

$$\frac{d^2H}{d\phi^2} = -m^2H \qquad (25)$$

The equations for F, H are easily solved: thus,

$$\begin{aligned} F(r) &= Ar^n + Br^{-(n+1)} \\ H(\phi) &= Ce^{im\phi} + De^{-im\phi} \end{aligned}$$

where, A, B, C, D are arbitrary complex constants. To solve Eq.(24) for G, we put, $\mu = \cos\theta; \frac{d}{d\mu} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$ and obtain **Legendre's Associated Equation**:

$$(1-\mu^2)\frac{d^2G}{d\mu^2} - 2\mu\frac{dG}{d\mu} + [n(n+1) - \frac{m^2}{1-\mu^2}]G = 0 \qquad (26)$$

4.3 Legendre's equation

- Consider first the case, $m^2 = 0$. Setting $\mu \equiv z; G \equiv w(z)$, Eq.(26) reduces to **Legendre's equation**:

$$(1 - z^2)\frac{d^2 w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)w = 0 \qquad (27)$$

- In a problem, you are asked to show that this equation has three **regular singularities** at $z = +1, -1, \infty$. From the **Frobenius-Fuchs Theorem** we know that it will have two linearly independent analytic solutions in the finite plane. The exponents at $z = \pm 1$ are zero. Hence they are **logarithmic branch points.** Since $z = 0$ is an **ordinary point** of the equation we can find power (Taylor) series solutions:

$$\begin{aligned} w_1(z) &= 1 - \frac{n(n+1)}{2!}z^2 + \frac{n(n-2)(n+1)(n+3)}{4!}z^4 \\ &\quad - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}z^6 .. \\ w_2(z) &= z[1 - \frac{(n-1)(n+2)}{3!}z^2 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}z^4 - ..] \end{aligned}$$

4.4 Legendre functions

- These series can be shown to diverge logarithmically at $z = \pm 1$ for **general** n. Note that the series for $w_1(z)$, an **even** function of z, **terminates** whenever n is an **even integer**; similarly the series for $w_2(z)$ terminates whenever n is an **odd** integer. Thus we have **polynomial solutions** to the Legendre equation for n taking **integer values**. Thus we have the **Legendre polynomials** which are **normalized solutions** such that they are equal to unity at $z = 1$:

$$
\begin{aligned}
P_0(z) &= 1 \\
P_1(z) &= z \\
P_2(z) &= \frac{1}{2}(3z^2 - 1) \\
P_3(z) &= \frac{1}{2}(5z^3 - 3z)...
\end{aligned}
$$

We can directly obtain these polynomials as follows: clearly, the Newtonian potential, $\Phi = \frac{1}{r}$ satisfies **Laplace's equation**. In Cartesian coordinates, the potential at $\mathbf{x} = (x, y, z)$ due to a point mass at $\mathbf{x}_0 = (x_0, y_0, z = z_0)$ is proportional to, $\Phi(x, y, z : 0, 0, z_0) = \frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{1/2}}$. This can be written as:

$$
\Phi = \frac{1}{[r^2 - 2rr_0 \cos\theta + r_0^2]^{1/2}}
$$

Legendre polynomials: generating functi

- Now, suppose that $r > r_0$, we then have, using the **Binomial expansion:**

$$(1-x)^{-p/q} = 1 + \frac{p}{1!}(\frac{x}{q}) + \frac{p(p+q)}{2!}(\frac{x}{q})^2 + \frac{p(p+q)(p+2q)}{3!}(\frac{x}{q})^3 + ..$$

$$\begin{aligned}
\Phi(r, \cos\theta; r_0) &= \frac{1}{r}[1 - \frac{r_0}{r}(2\cos\theta - \frac{r_0}{r})]^{-1/2} \\
&= \frac{1}{r}[1 + \frac{1}{2}(\frac{r_0}{r})(2\cos\theta - \frac{r_0}{r}) + \frac{1.3}{2!2^2}(\frac{r_0}{r})^2(2\cos\theta - \frac{r_0}{r})^2..] \\
&= \frac{1}{r}[1 + (\frac{r_0}{r})(\cos\theta - \frac{r_0}{2r}) + \frac{1.3}{2!}(\frac{r_0}{r})^2(\cos\theta - \frac{r_0}{2r})^2 \\
&\quad + \frac{1.3.5}{3!}(\frac{r_0}{r})^3(\cos\theta - \frac{r_0}{2r})^3 + ..] \\
&= \frac{1}{r}[1 + (\frac{r_0}{r})P_1(\cos\theta) + (\frac{r_0}{r})^2 P_2(\cos\theta) + (\frac{r_0}{r})^3 P_3(\cos\theta)..] \quad (28)
\end{aligned}$$

Similarly, for $r < r_0$, we may expand in powers of r/r_0 and obtain,

$$\Phi(r, \cos\theta; r_0) = \frac{1}{r_0}\Sigma_{n=0}^{\infty}(\frac{r}{r_0})^n P_n(\cos\theta) \quad (29)$$

4.5 The Schläfli Integral

- Following **Schläfli** we consider the function defined by the contour integral taken around a contour which includes within it $t = z$:

$$\begin{aligned}
g_n(z) &= \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}dt \qquad (30)\\
\frac{dg_n}{dz} &= \frac{n+1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+2}}dt\\
\frac{d^2g_n}{dz^2} &= \frac{(n+1)(n+2)}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+3}}dt\\
(1-z^2)g_n'' - 2zg_n' + n(n+1)g_n &= \frac{n+1}{2\pi i}\oint_C \frac{(t^2-1)^n dt}{2^n(t-z)^{n+3}}\\
&\times[(n+2)(1-z^2) - 2z(t-z) + n(t-z)^2]
\end{aligned}$$

The terms within the braces may be re-arranged: $(n+2)(1-z^2) - 2z(t-z) + n(t-z)^2 = -(n+2)(t^2-1) + 2(n+1)t(t-z)$. This leads to the amazing result that $g_n(z)$ satisfies the **Legendre equation** (this follows from the fact that the integrand is single-valued for integer n):

$$(1-z^2)g_n'' - 2zg_n' + n(n+1)g_n = \frac{n+1}{2\pi i}\oint_C \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right]dt = 0$$

4.6 Proof of the Schläfli representation

- Here is a new way to look at our result: we consider, for real $|u| > 1$ the infinite series,

$$
\begin{aligned}
K(z,u) &= \Sigma_{n=0}^{\infty} \frac{g_n(z)}{u^n} \\
&= \Sigma_{n=0}^{\infty} \frac{1}{2\pi i} \oint_C \frac{(t^2-1)^n}{2^n u^n (t-z)^n} \frac{dt}{t-z} \\
&= \frac{1}{2\pi i} \oint_C [\frac{1}{1-\frac{t^2-1}{2u(t-z)}}] \frac{dt}{t-z} \\
&= \frac{1}{2\pi i} \oint_C [\frac{2u}{2u(t-z)-(t^2-1)}] dt \\
&= -\frac{2u}{2\pi i} \oint_C \frac{dt}{(t-u)^2-(1-2uz+u^2)}
\end{aligned}
\tag{31}
$$

We know how to do the contour integral! We note that the integrand has poles at $t_+ = u + (1-2uz+u^2)^{1/2}; t_- = u - (1-2uz+u^2)^{1/2}$.The **Residue theorem** then gives (for large u): the contour C enclosing $t = z$ contributes the residue at $t = t_-$:

$$K(z,u) = \frac{u}{(u^2-2uz+1)^{1/2}} \tag{32}$$

4.6 Rodrigues' formula

- Consider once again,

$$K(z,u) \quad = \quad \frac{1}{[1-(\frac{2z}{u})+(\frac{1}{u})^2]^{1/2}}$$

Put, $u = \frac{r}{r_0}; z = \cos\theta$. We then obtain, $r\Phi(r,\cos\theta;r_0) = K(z,u)$. It is then immediately clear that $g_n(z) = \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}dt \equiv P_n(\cos\theta)$.

- We can now obtain a remarkable formula for the Legendre polynomials. We see from **Cauchy's integral theorem** using a suitable contour C, the relation:

$$(z^2-1)^n \quad = \quad \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{t-z}dt$$

It follows by differentiating under the integral sign,

$$\frac{d^n}{dz^n}[(z^2-1)^n] \quad = \quad \frac{n!}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt$$

$$P_n(z) \quad = \quad \frac{1}{2^n n!}\frac{d^n}{dz^n}[(z^2-1)^n] \qquad (33)$$

This important formula due to **Rodrigues** can also be proved directly from the (terminating) power series expansion for $P_n(z)$.

4.7 Laplace's integral

- In **Schläfli's formula**, we take $C : t = z + (z^2-1)^{1/2}e^{i\phi}$, namely, a circle with centre $t = z$ and radius $|z^2-1|^{1/2}$. We then have,

$$
\begin{aligned}
P_n(z) &= \frac{1}{2^{n+1}\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt \\
&= \frac{1}{2^{n+1}\pi i}\int_{-\pi}^{\pi} \frac{[(z-1+(z^2-1)^{1/2}e^{i\phi})(z+1+(z^2-1)^{1/2}e^{i\phi})^n}{[(z^2-1)^{1/2}e^{i\phi}]^n} \\
&\quad \times i d\phi \\
&= \frac{1}{2^{n+1}\pi}\int_{-\pi}^{\pi} \frac{[z^2-1+2z(z^2-1)^{1/2}e^{i\phi}+(z^2-1)e^{2i\phi}]^n}{[(z^2-1)^{1/2}e^{i\phi}]^n}d\phi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}[z+(z^2-1)^{1/2}\cos\phi]^n d\phi \\
P_n(z) &= \frac{1}{\pi}\int_0^{\pi}[z+(z^2-1)^{1/2}\cos\phi]^n d\phi \qquad (34)
\end{aligned}
$$

This called **Laplace's First Integral** for Legendre polynomials.