Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 14: Complex analysis: applications-3

A Thyagaraja

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1 Second-order linear differential equation

- Consider the following equation (named after **Airy**) with **variable coeffcients**:

$$\frac{d^2y}{dz^2} - zy(z) \quad = \quad 0 \tag{1}$$

We will seek a solution $y(z)$ in the form of a power series. Equations of this type are **linear and homogeneous.** Thus we set,

$$y(z) \quad = \quad a_0 + a_1 z + a_2 z^2 + ..$$

We simply substitute the series in the equation and equate like powers:

$$\begin{aligned} 2.1a_2 &= 0 \\ 3.2a_3 - a_0 &= 0 \\ 4.3a_4 - a_1 &= 0 \\ 5.4a_5 - a_2 &= 0 \\ .... &\quad .. \\ m(m-1)a_m - a_{m-3} &= 0 \end{aligned}$$

We see that $a_2 = 0; a_5 = 0; a_8 = 0...$ Suppose we **choose** $a_0 = 1; a_1 = 0$. It follows that $a_4 = a_7 = a_{10} = .. = 0$ too.

1.2 Series solutions

- However, the recurrence relations give,

$$
\begin{aligned}
a_3 &= \frac{1}{3.2} \\
a_6 &= \frac{a_3}{6.5} = \frac{1}{6.5.3.2} \\
a_9 &= \frac{a_6}{9.8} = \frac{1}{9.8.6.5.3.2} \\
a_{3n} &= \frac{a_{3n-3}}{3n(3n-1)}
\end{aligned}
$$

Thus the solution (denoted by $y_1(z)$) is obtained as a **power series**:

$$
\begin{aligned}
y_1(z) &= 1 + \frac{z^3}{2.3} + \frac{z^6}{2.3.5.6} + \frac{z^9}{2.3.5.6.8.9}.. \\
&= 1 + \frac{(z^3)}{2.3} + \frac{(z^3)^2}{2.3.5.6} + \frac{(z^3)^3}{2.3.5.6.8.9} +
\end{aligned}
$$

It is easily shown by **the Ratio Test** that this series is absolutely convergent for all z and thus represents an **entire function.** It is called the **Airy function.** It satisfies the equation and the conditions, $y(0) = 1; y'(0) = 0$.

1.3 Airy functions

- But this is not all! We could have chosen $a_0 = 0; a_1 = 1$. Then the recursion relations imply that, $a_3 = a_6 = .. = a_{3n} = 0$. However, we would have had recurrence relations like,

$$a_4 = \frac{1}{4.3}$$
$$a_7 = \frac{a_4}{7.6} = \frac{1}{7.6.4.3}$$

Leading to the power series:

$$y_2(z) = z + \frac{z^4}{3.4} + \frac{z^7}{3.4.6.7}..$$

$y_2(z)$, we see that $y_2(0) = 0; y_2'(0) = 1$. It is again proved by the ratio test that $y_2(z)$ is also an entire function and satisfies **Airy's** equation. It is plain that if a, b are **arbitrary constants**, $ay_2(z) + by_2(z)$ is also a solution of the Airy equation.

- It is clear that $y_1(z)$ and $y_2(z)$ defined by the above power series cannot be **constant, non-zero multiples** of each other. Suppose, for example, $y_1(z) = ky_2(z)$. This equation would fail to apply at $z = 0$ where the RHS vanishes but the LHS would not. They are a **fundamental set of solutions,** and all solutions can be expressed as a linear combination of these two.

.1 General theory of Frobenius and Fuchs

- **Definition 14.1:** A linear, homogeneous, second-order differential equation in the complex domain takes the following **standard form:**

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f \quad = \quad 0 \qquad (2)$$

where $p(z), q(z)$ are holomorphic (ie, single-valued, analytic) functions except for certain singular points. Any point z where they are both **analytic** is said to be an **ordinary point** of the differential equation. Any point which is a **singular point** of these coefficient functions is called a **singularity** of the equation. At such points, $p(z), q(z)$ can have **poles** or **isolated essential singularities.** I now present the principal **results** of the **Frobenius-Fuchs** theory without rigorous proofs.

- **Theorem 14.1:** If $z = 0$ is an ordinary point of the differential equation Eq.(2), a convergent power series solution, single-valued and analytic in z can be found taking on any given **initial values,** $y(0); y'(0)$. The series converges in a disk which must have at least one singularity of the equation on it. The solution represented by the power series is **unique.** If $p(z), q(z)$ are entire functions, so is the solution.

- The power series solutions and the proof of the above theorem can be obtained by the following method of **integral equations/iterative approximations** due to **Picard.**

2.2 Picard's method

- We recast the second-order Eq.(2) as an **equivalent** firt-order system: thus we put, $w(z) = y'(z)$ and consider the system:

$$\frac{dy}{dz} = w(z) \tag{3}$$

$$\frac{dw}{dz} = -p(z)w(z) - q(z)y(z) \tag{4}$$

The initial data are: $y(0) = y_0; w(0) = y'_0$. If $y(z), w(z)$ are any pair of analytic functions in the disk $D : |z| < R$, where the nearest singularity of $p(z), q(z)$ is at a radius R, the RHS would be analytic functions. We may therefore **formally** integrate the equations (along any path lying entirely in D, thanks to **Cauchy's theorem!**) and apply the boundary conditions to obtain the **linear, inhomogeneous integral equations:**

$$y(z) = y_0 + \int_0^z w(u)du \tag{5}$$

$$w(z) = y'_0 - \int_0^z [p(u)w(u) + q(u)y(u)]du \tag{6}$$

Picard's method of successive iterations starts with the crudest approximation to $y(z), w(z)$, namely $y^{(0)}(z) = y_0; w^{(0)}(z) = w_0 = y'_0$. Substituting on the RHS of Eqs.(5,6), we get the **next** approximants:

.3 Picard's method: higher approximation

$$y^{(1)}(z) = y_0 + \int_0^z w^{(0)}(u)du$$
$$w^{(1)}(z) = w_0 - \int_0^z [p(u)w^{(0)}(u) + q(u)y^{(0)}(u)]du$$

Furthermore, we see that in general we will have the equations:

$$y^{(n+1)}(z) = y_0 + \int_0^z w^{(n)}(u)du$$
$$w^{(n+1)}(z) = w_0 - \int_0^z [p(u)w^{(n)}(u) + q(u)y^{(n)}(u)]du$$

- Observe that that the zeroth approximations (ie the initial data) are analytic and this implies by induction that every pair $y^{(n)}(z); w^{(n)}(z)$ are also analytic and take on the correct initial conditions. If we can take the limit $n \to \infty$ on both sides of the above equations and the limits exist, we will have proved the required existence theorem!
- We obviously have, $[y^{(n+1)}(z) - y^{(n)}(z)] = \int_0^z [w^{(n)}(u) - w^{(n-1)}(u)]du$ and $[w^{(n+1)}(z) - w^{(n)}(z)] = -\int_0^z [p(u)(w^{(n)}(u) - w^{(n-1)}(u)) + q(u)(y^{(n)}(u) - y^{(n-1)}(u))]du$.

2.4 Picard estimates

- Let $z = |z|e^{i\theta}; u = te^{i\theta}$:

$$|y^{(n+1)}(|z|e^{i\theta}) - y^{(n)}(|z|e^{i\theta})| \leq \int_0^{|z|} |w^{(n)}(te^{i\theta}) - w^{(n-1)}(te^{i\theta})|dt$$

$$|w^{(n+1)}(|z|e^{i\theta}) - w^{(n)}(|z|e^{i\theta})| \leq \int_0^{|z|} [|p(te^{i\theta})(w^{(n)} - w^{(n-1)}) + q(te^{i\theta})(y^{(n)} - y^{(n-1)}$$

We set
$\phi_{n+1}(|z|) = |y^{(n+1)}(|z|e^{i\theta}) - y^{(n)}(|z|e^{i\theta})|; \psi_{n+1}(|z|) = |w^{(n+1)}(|z|e^{i\theta}) - w^{(n)}(|z|e^{i\theta})|$.
Then,

$$\phi_{n+1}(|z|) \leq \int_0^{|z|} \psi_n(t)dt < \int_0^{|z|} (\phi_n(t) + \psi_n(t))dt$$

$$\psi_{n+1}(|z|) \leq M \int_0^{|z|} [\psi_n(t) + \phi_n(t)]dt$$

$$(\phi_{n+1}(|z|) + \psi_{n+1}(|z|)) \leq (M+1) \int_0^{|z|} (\phi_n(t) + \psi_n(t))dt$$

where $|p(z)| < M; |q(z)| < M$.

2.4 Picard's method: convergence

- We "solve" these inequalities "recursively": $|\psi_1| \leq M(|y_0| + |y_0'|)|z|; |\phi_1| \leq M|y_0'||z| \rightarrow$

$$\begin{aligned}
\phi_1(t) + \psi_1(t) &< M(2|y_0'| + |y_0|)t \\
\phi_2(t) + \psi_2(t) &< \frac{(Mt)^2}{2!}(2|y_0'| + |y_0|) \\
&.. \\
\phi_n(t) + \psi_n(t) &< \frac{(Mt)^n}{n!}(2|y_0'| + |y_0|)
\end{aligned}$$

This **proves** that $\mathrm{Lim}_{n\rightarrow\infty}(\psi_n(|z|) + \phi_n(|z|)) = 0$. We see that this implies the uniform convergence and hence the analyticity of the infinite sums,

$$\begin{aligned}
\mathrm{Lim}_{n\rightarrow\infty} y^{(n+1)}(z) &= y_0 + \Sigma_{k=0}^{\infty}[y^{(k+1)}(z) - y^{(k)}(z)] \\
\mathrm{Lim}_{n\rightarrow\infty} w^{(n+1)}(z) &= w_0 + \Sigma_{k=0}^{\infty}[w^{(k+1)}(z) - w^{(k)}(z)]
\end{aligned}$$

Clearly, not only do the solutions exist, but the same technique shows that uniqueness is assured. It is clearly seen that the solutions constructed this way depend **linearly** on the initial data. Thus choosing $y_0 = 1; y_0' = 0$ we can generate a solution pair, $(y_I(z), w_I(z))$. Choosing $y_0 = 0; y_0' = 1$ gives a second pair, $y_{II}(z), w_{II}(z))$. The functions, $y_I(z), y_{II}(z)$ form a fundamental set of solutions to Eq.(2), in effect proving Theorem 14.1.

2.5 Regular singular points

- **Definition 14.2:** A point $z = a$ is a **regular singular point/regular singularity** if the functions, $(z-a)p(z), (z-a)^2 q(z)$ are **analytic** at the point but at least one of $p(z), q(z)$ has a pole there. If, at $z = a$ either p or q has a singularity which is not regular, the equation is said to have an **irregular** singular point there.
- The following equation arises in **potential theory** in cylindrical polars:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 0$$

This equation has no singularity in the finite r -plane except at $r = 0$. Evidently, the singularity there is a **regular** one. Note it has two solutions, $f_I(r) = 1; f_{II}(r) = \ln r$. Note also that the second of these is **not** a holomorphic function while the first is actually an entire function. This shows that at a regular singularity of the equation not all of the **solutions** need have singularities. On the other hand, a solution may not be single-valued in the region surrounding the singular point.

Bessel's equation arises in mathematical physics in many contexts and has a regular singularity at $z = 0$:

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + (1 - \frac{\nu^2}{z^2})y = 0 \qquad (7)$$

where ν is a parameter called **order**. It often is, but need not necessarily be, an integer.

2.6 Linear independence of solutions

- We know from previous examples that second-order, homogeneous linear ode's have an infinity of solutions which can be expressed as a linear combination (with constant coefficients) of **two** "fundamental solutions". The following discussion clarifies the issues and general concepts involved. They will be needed in our analysis of solutions near **regular singularities** of such equations.

- **Definition 14.3:** If $y_1(z), y_2(z)$ are any two solutions of a second-order linear differential equation in a common domain, D, they are said to be **linearly dependent** if two non-zero constants a, b exist such that the equation,

$$ay_1(z) + by_2(z) \quad = \quad 0$$

is satisfied **at every point** in D. If no such constants can be found, the solutions are said to be **linearly independent** in D.

Examples: 1. The equation, $\frac{d^2y}{dz^2} = y$ has solutions, $y_1 = e^z; y_2 = e^{-z}$, valid for all finite z. It is easy to show that they are **linearly independent.**
2. The **Airy equation** (Eq.(1)) has two entire function solutions $y_0(z); y_1(z)$. It follows from the different intial conditions they satisfy at the origin that they are linearly independent.
3. It must be very carefully noted that **linear independence** does **not** mean that the solutions are **functionally independent**! In Example 1 above, we see that $e^{-z} = \frac{1}{e^z}$. Thus, each solution can be expressed as a **function** (in this case reciprocal) of the other.

2.7 The Wronskian

- The following theorem gives necessary and sufficient conditions for any two solutions of Eq.(2) to be linearly independent.

- **Theorem 14.2:** If $y_1(z); y_2(z)$ are any two solutions of Eq.(2) defined in a common domain D of analyticity, they are **linearly independent** if and only if their **Wronskian** defined by, $W(z) = y_1(z)y_2'(z) - y_1'(z)y_2(z)$ does not vanish in D.

- **Proof:** Suppose non-zero a, b exist such that $ay_1(z) + by_2(z) = 0$. Since this equation holds throughout D, we may differentiate and obtain, $ay_1'(z) + by_2'(z) = 0$ as a consequence, at each point $z \in D$. But this implies that the **determinant** of the homogeneous linear algebraic equations for the constants a, b must vanish! This determinant is none other than the **Wronskian**, $W(z)$. Hence, if $y_1(z), y_2(z)$ are linearly dependent in D, then, $W(z) = 0$.
  Next, suppose $W(z) \neq 0$ in D. Then, the homogeneous linear equations,

$$\begin{aligned} ay_1(z) + by_2(z) &= 0 \\ ay_1'(z) + by_2'(z) &= 0 \end{aligned}$$

  have only the **trivial solution**, $a = b = 0$ at any point in D. Hence, the solutions must be linearly independent.

2.8 Equation for the Wronskian

- I will derive a remarkable equation satisfied by any two solutions of Eq.(2). Let $y_1(z), y_2(z)$ be any two solutions in D. We know that,

$$\begin{aligned} y_1'' + p(z)y_1' + q(z)y_1 &= 0 \\ y_2'' + p(z)y_2' + q(z)y_2 &= 0 \end{aligned}$$

  Multiply the second equation by $y_1(z)$ and the first by $y_2(z)$ and subtract the latter from the former to obtain:

$$\begin{aligned} [y_1 y_2'' - y_2 y_1''] + p[y_1 y_2' - y_2 y_1'] &= 0 \ \text{hence,} \\ \frac{dW}{dz} + p(z)W(z) &= 0 \end{aligned} \qquad (8)$$

  This **linear, first order, homogeneous** equation can be immediately integrated! Thus let us suppose that at some point z_0, we know $y_1(z), y_2(z)$. We may immediately calculate $W(z_0)$ there. Equation (8) can be integrated to give,

$$W(z) = W(z_0)e^{-\int_{z_0}^{z} p(u)du} \qquad (9)$$

  The integral may be taken along any path in D where $p(z)$ is assumed to be analytic. This implies that if $W(z_0)$ vanishes, it vanishes **everywhere** in D. If it doesn't vanish at any one point, it can vanish nowhere.

3.1 The Bessel equation

- I want to consider the Bessel equation as a case study for the **Frobenius-Fuchs series expansion of the solution** at a **regular singularity.** We assume a power series with a new twist and write,

$$y(z) \quad = \quad a_0 z^c + a_1 z^{c+1} + a_2 z^{c+2} + .. \tag{10}$$

where c is a complex constant to be determined. We substitute in **Bessel's** Eq.(7) and follow the usual procedure of equating the coefficients of the various powers of z to zero. We then obtain the following set of recurrence relations:

$$\begin{aligned}
a_0[c^2 - \nu^2] &= 0 \qquad (z^{c-2}) \\
a_1[(c+1)^2 - \nu^2] &= 0 \qquad (z^{c-1}) \\
a_2[(c+2)^2 - \nu^2] + a_0 &= 0 \qquad (z^{c}) \\
a_n[(c+n)^2 - \nu^2] + a_{n-2} &= 0 \qquad (z^{c+n-2})
\end{aligned}$$

We can satisfy the first equation choosing $a_0 = 1$ without loss of generality if we set,

$$c^2 \quad = \quad \nu^2 \tag{11}$$

This is called the **indicial equation** at the regular singularity. We can solve it easily: $c = \pm\nu$.

3.1 Bessel functions

- If the square roots of ν^2 have unequal real parts, we shall denote by $+\nu$ the root with the larger real part, the other being $-\nu$. We will assume that the **difference of the two roots**, 2ν is **not an integer.** This implies that $(c+n)^2 - \nu^2$ cannot vanish for any $n > 0$. The second equation now requires, $a_1 = 0$ and subsequently $a_{2n+1} = 0; n = 1, 2, ...$ We can now solve for the coefficients, $a_2, a_4, ..$ successively as none of the denominators can vanish.

$$
\begin{aligned}
a_2 &= -\frac{1}{4\nu + 4} = \frac{(-1)}{(\nu+1).2^2} \\
a_4 &= \frac{1}{(8\nu + 16)(4\nu + 4)} = \frac{(-1)^2}{2!(\nu+1)(\nu+2).2^4} \\
a_6 &= -\frac{1}{(12\nu + 36)(8\nu + 4)(4\nu + 4)} = \frac{(-1)^3}{3!(\nu+1)(\nu+2)(\nu+3).2^6}
\end{aligned}
$$

The power series becomes:

$$
y(z;\nu) = z^\nu [1 - \frac{1}{1!(\nu+1)}(\frac{z}{2})^2 + \frac{1}{2!(\nu+1)(\nu+2)}(\frac{z}{2})^4 - ..]
$$

2 Bessel functions: standard series solution

- In the preceding work, we could have taken a_0 to be an arbitrary non-zero constant. It is conventional to take $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$. We then obtain the series solution for the **Bessel function of order** ν:

$$J_\nu(z) \quad = \quad (\frac{z}{2})^\nu \Sigma_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} (\frac{z}{2})^{2n} \tag{12}$$

- **We make some important observations:**
  a) The solution, $y(z;\nu)$ makes sense for any $+\nu$ since the denominators in the recurrence relations following the indicial equation cannot vanish.
  b) If ν happens to be an **integer** $J_\nu(z)$ is single-valued, otherwise z^ν has a **branch point** at the regular singularity, $z=0$ of Bessel's equation, Eq.(7). In the latter case, it can be written as $z^\nu \Phi_\nu(z^2/4)$, where $\Phi_\nu(z^2/4)$ is **an even, entire function**, since the series for $\Phi_\nu(z^2/4)$ is absolutely and uniformly convergent for any finite z, as can be shown by the **Weierstrass M-test** and the series for $e^{-z^2/4}$.
  c) If 2ν is not an integer, the series makes sense with the choice $-\nu$. Hence $J_{-\nu}(z)$ is also a valid solution. It is **linearly independent** of $J_\nu(z)$, since the latter tends to zero at the origin whereas $J_{-\nu}(z)$ "blows up" like $z^{-\nu}$.
  d) Thus, when 2ν is not an integer, $J_{\pm\nu}(z)$ form a **fundamental set** and any solution can be expressed as a linear combination of these two.

3.3 Bessel functions of integer order

- We next consider what happens when $+\nu \geq 0$ is an integer. Consider $\nu = 0$ first. We proceed as before and obtain,

$$J_0(z) = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (\frac{z}{2})^{2n} \tag{13}$$

To find the second, linearly independent solution, we use the following trick: write, $y(z) = J_0(z)w(z)$, where $w(z)$ must be determined. Substituting in the Bessel equation of zeroth order,

$$\begin{aligned} (J_0 w)'' + \frac{1}{z}(J_0 w)' + J_0 w &= [J_0'' + \frac{1}{z} J_0' + J_0]w + w'' J_0 + (2J_0' + \frac{J_0}{z})w' \\ &= 0 \end{aligned}$$

The first term on the RHS vanishes. The second and third give the following **linear, first-order, homogeneous** o.d.e for $w'(z)$:

$$\begin{aligned} \frac{dw'}{dz} &= -(\frac{2J_0'}{J_0} + \frac{1}{z})w' \\ w' &= e^{-(2 \ln J_0 + \ln z)} \qquad \text{hence} \\ w(z) &= \int_{z_0}^{z} \frac{du}{u(J_0(u))^2} \end{aligned} \tag{14}$$

3.4 Bessel functions: contd.

- Let us calculate the Wronskian of $J_\nu(z), J_{-\nu}(z)$ when ν is s not an integer. Note that as $z \to 0$,

$$
\begin{aligned}
J_\nu(z) &\simeq \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \\
J_{-\nu}(z) &\simeq \frac{z^{-\nu}}{2^{-\nu} \Gamma(-\nu+1)} \\
W(J_\nu, J_{-\nu}) &= -\frac{2\nu}{z} \frac{1}{\Gamma(\nu+1)\Gamma(-\nu+1)} = -\frac{2\sin\pi\nu}{\pi z}
\end{aligned}
$$

Using the property of the **Gamma function.** This shows that $J_\nu, J_{-\nu}$ form a fundamental set of solutions when ν is not an integer. Note also that from Eq.(12), the series, considered as a **function of the order**, ν is an entire function for any fixed $z \neq 0$. Thus it can be differentiated with respect to ν and will satisfy the equation,

$$
\frac{d^2}{dz^2}\left(\frac{\partial J_\nu}{\partial \nu}\right) + \frac{1}{z}\frac{d}{dz}\left(\frac{\partial J_\nu}{\partial \nu}\right) + (1-\nu^2)\frac{\partial J_\nu}{\partial \nu} = \frac{2\nu}{z^2} J_\nu \qquad (15)
$$

Plainly, by changing $\nu \to -\nu$, we see that,

$$
\frac{d^2}{dz^2}\left(\frac{\partial J_{-\nu}}{\partial \nu}\right) + \frac{1}{z}\frac{d}{dz}\left(\frac{\partial J_{-\nu}}{\partial \nu}\right) + (1-\nu^2)\frac{\partial J_{-\nu}}{\partial \nu} = \frac{2\nu}{z^2} J_{-\nu} \qquad (16)
$$

3.5 The second solution for integer ν

- Although the "trick" used earlier for integer ν works and shows that the second solution has a logarithmic branch point at $z = 0$, there is a different method used by **Hankel**. We note that if $\nu = n$ is an integer, $J_{-n}(z) = (-1)^n J_n(z)$. This can be proved using Eq.(12) and the fact that $\frac{1}{\Gamma(p)} = 0$ for $p = 0, -1, -2, -3, \ldots$ Now, **define**, for **non-integer values of** ν

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \quad (17)$$

Using previous results, we see that, $W(J_\nu, Y_\nu) = \frac{2}{\pi z}$. Hence J_ν, Y_ν form a fundamental set. If $\nu \rightarrow n$, an integer, it is seen that the numerator and denominator vanish. We can apply L'Hospital's rule and write,

$$\begin{aligned} Y_n(z) &= \mathrm{Lim}_{\nu\rightarrow n} Y_\nu(z) \\ &= \frac{1}{\pi}[\frac{\partial J_\nu}{\partial\nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial\nu}]_{\nu=n} \end{aligned} \quad (18)$$

It is easily shown from Equations (15,16) that $Y_n(z)$ is indeed a solution of Eq.(7) for integer n.It also follows by continuity that $W(J_n, Y_n) = \frac{2}{\pi z}$.

3.6 Summary: Frobenius-Fuchs Theorem

- **Theorem 14.2:** The second-order linear o.d.e Eq.(2) is said to have a **regular singularity** at $z = z_0$, if $p(z)$ has at most a simple pole and/or $q(z)$ a double pole. Then, 1: The equation has two linearly independent solutions in a "punctured disk" around z_0 of the forms, $y_1(z) = (z - z_0)^{c_1}[1 + \Sigma_{n=1}^{\infty} a_n (z - z_0)^n]$,$y_2 = (z - z_0)^{c_2}[1 + \Sigma_{n=1}^{\infty} b_n (z - z_0)^n]$ where $c_1, c_2; \mathrm{Re}(c_1) \geq \mathrm{Re}(c_2)$ are the roots of the **indicial equation** at $z = z_0$, obtained by substituting the series into the equation, **provided** the **exponents** $c_{1,2}$ do not differ by a positive integer or zero. 2. If $s = c_1 - c_2$ is a positive integer or zero, the second solution must take the form,

$$y_2(z) = \lambda y_1(z) \ln(z - z_0) + (z - z_0)^{c_2}[1 + \Sigma_{n=1}^{\infty} b_n (z - z_0)^n] \tag{19}$$

where λ is determined by p, q. Both series converge uniformly to analytic (but not generally single-valued) functions in disk aroundz_0 extending to the nearest singularity of the equation to z_0.
To understand the behaviour at ∞ we put $z = 1/u; (-\frac{du}{u^2} = dz)$. Then we get,

$$\begin{aligned} u^2 \frac{d}{du}[u^2 \frac{dy}{du}] - u^3 \frac{dy}{du} + (1 - \nu^2 u^2) y &= 0 \\ y'' + \frac{1}{u} y' + [\frac{1}{u^4} - \frac{\nu^2}{u^2}] y &= 0 \end{aligned}$$

Infinity (ie, $u = 0$) is therefore an **irregular singularity** of the Bessel equation.