Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 13: Complex analysis: applications-2

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1.1 Ordinary linear differential equations

- Consider the problem of the falling particle again. Suppose it is held by a spring (satisfying **Hooke's Law**) attached to the **starting point**, $z = h$. We will assume that the force due to the spring is $\mathbf{F}_{\rm spring} = \Lambda(h - z)\mathbf{e}_z$, where $\Lambda > 0$ is the "spring constant". The equation of motion written this time for $u(t) = h - z(t)$ is:

$$\frac{d^2u}{dt^2} = -\nu\frac{du}{dt} - \lambda u + g \tag{1}$$

where $\lambda = \frac{\Lambda}{m}$. We must solve this equation, subject to $u(0) = 0; z(0) = h; u'(0) = 0$. We take Laplace transforms on both sides and obtain:

$$\begin{aligned} s^2\hat{u}(s) &= -\nu s\hat{u}(s) - \lambda\hat{u}(s) + \frac{g}{s} \\ \hat{u} &= \frac{g}{s(s^2 + \nu s + \lambda)} \end{aligned} \tag{2}$$

The **Laplace inversion formula** then gives for $t > 0$:

$$u(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}g}{s(s^2 + \nu s + \lambda)}ds \tag{3}$$

1.2 Falling particle

- We can evaluate the integral in Eq.(3) using the **Residue theorem** for $t > 0$, we close on the left-half plane on a large semi-circle. The integrand has simple poles at $s_0 = 0; s_\pm = -\frac{\nu}{2} \pm (\frac{\nu^2}{4} - \lambda)^{1/2}$ with residues,

$$
\begin{aligned}
r_0 &= \frac{g}{\lambda} \\
r_+ &= \frac{g e^{s_+ t}}{s_+(s_+ - s_-)} \\
r_- &= \frac{g e^{s_- t}}{s_-(s_- - s_+)}
\end{aligned}
$$

the complete solution is then given by:

$$
u(t) = \frac{g}{\lambda} + \frac{g}{2(\frac{\nu^2}{4} - \lambda)^{1/2}}[\frac{e^{s_+ t}}{s_+} - \frac{e^{s_- t}}{s_-}] \quad (4)
$$

- It is easily checked that the two initial conditions are indeed satisfied. In a problem, I ask you to consider various cases of this interesting solution based on the relative sizes of the damping rate ν and the “reduced spring constant”, λ.

1.3 Coupled oscillators

- Mechanical systems are often coupled: consider two oscillators connected to each other- an example is provided by two weights hanging from the same taut string. We will assume, for simplicity, the motion is in one dimension and consider the pair of equations:

$$m_1 \frac{d^2 x_1}{dt^2} + m_1 \beta_1 \frac{dx_1}{dt} + m_1 k_1^2 x_1 = b \frac{dx_2}{dt} \quad (5)$$

$$m_2 \frac{d^2 x_2}{dt^2} + m_2 \beta_2 \frac{dx_2}{dt} + m_2 k_2^2 x_2 = -b \frac{dx_1}{dt} \quad (6)$$

  Here $m_{1,2}$ are the masses, $\beta_{1,2}$, the "damping coefficients", $k_{1,2}^2$ are the spring constants) and b is an "interaction" constant, which dynamically links the two oscillators. Note that this interaction conserves the total kinetic energy of the two particles.

- We must solve this set of coupled equations, given initial data. For simplicity, I assume that $x_{1,2}(0) = 0; \frac{dx_1}{dt} = v_1, \frac{dx_2}{dt} = 0$ at the initial instant, as an illustration. Taking Laplace transforms, we get:

$$s^2 \hat{x}_1 + \beta_1 s \hat{x}_1 + k_1^2 \hat{x}_1 = \frac{b}{m_1} s \hat{x}_2 + v_1$$

$$s^2 \hat{x}_2 + \beta_2 s \hat{x}_2 + k_2^2 \hat{x}_2 = -\frac{b}{m_2} s \hat{x}_1$$

1.4 Coupled oscillators: formal solution

- The pair of simultaneous linear algebraic equations for $\hat{x}_{1,2}$ are easily solved. The inversion formula can be evaluated by using the residue theorem, noting the poles which occur:

$$\hat{x}_2 = -\frac{b}{m_2}\frac{s\hat{x}_1}{(s^2+\beta_2 s+k_2^2)}$$

$$\hat{x}_1 = \frac{v_1}{D(s)}$$

$$x_1(t) = \frac{v_1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{D(s)}ds$$

  where $D(s) = (s^2+\beta_1 s+k_1^2) + (\frac{b^2}{m_1 m_2})(\frac{s^2}{s^2+\beta_2 s+k_2^2})$.
  The zeros of the function $D(s)$ contribute, via residues there to the final solution. In the problems some special cases will be considered.

- If $b = 0$, the oscillators do not interact and the second oscillator is not "excited": $D(s)$ becomes a quadratic and we get a **damped simple harmonic motion** in the first oscillator. If $\beta_1 = \beta_2 = 0, D(s)$ becomes a quadratic in s^2 and the solution is easily obtained. The motion is interesting as the energy "sloshes" back and forth, undamped between the two oscillators!

1.5 Electrical circuit applications

- In electrical circuit theory, there are two **Kirchoff laws** which are ultimately derived from **Maxwell's equations** (see **Feynman Lectures on Physics** for a full discussion of elementary circuit theory). If $V(t)$ is an externally applied voltage (say, a battery) and $i(t)$ is the current flowing in the circuit, the laws lead to the following "circuit equation":

$$
\begin{aligned}
L\frac{di}{dt} + Ri + \frac{q}{C} &= V(t) \\
i &= \frac{dq}{dt}
\end{aligned}
\tag{7}
$$

  Where L is the **self-inductance** of the circuit, R is the **resistance** and C, the **capacitance.** $q(t)$ is the charge on the **capacitor** of the circuit. Here, the three elements (and the battery) are in **series.** This is the simplest electrical circuit imaginable.

- As an example, we will consider the following problem: suppose $V(t) = 0; t < 0$ and is turned "on" suddenly to be V_* for $t > 0$. Suppose further that $C = \infty$ (this means there is no capacitor in the circuit), and $i(0) = 0$. Describe the rise of the current in the circuit. We must solve for $t > 0$ the equation:

$$
L\frac{di}{dt} + Ri = V_* \tag{8}
$$

1.6 LR circuit

- Taking Laplace transforms, we obtain,

$$
\begin{aligned}
Ls\hat{i} + R\hat{i} &= \frac{V_*}{s} \\
\hat{i} &= \frac{V_*}{L}\frac{1}{s(\frac{R}{L} + s)}
\end{aligned}
$$

We have already solved this equation since Eq.(27) of Lecture 12 is exactly the same (with the identifications $-g \to \frac{V_*}{L}; \nu \to \frac{R}{L}; v_z(t) \to i(t)$) as Eq.(8)! Using the famous **Feynman Principle: "same equations, same solutions!"** we may write down the answer:

$$
i(t) = \frac{V_*}{R}[1 - e^{-\frac{R}{L}t}] \tag{9}
$$

Suppose that at some time $t = t_* > 0$, an "impulsive voltage" (say "surge") is applied. What is the solution? We simply add a **Dirac delta function** $V_{\text{surge}}\delta(t - t_*)$ to the right of Eq.(8)! The Laplace transformed equation is now:

$$
\begin{aligned}
Ls\hat{i} + R\hat{i} &= \frac{V_*}{s} + \int_0^\infty e^{-st} V_{\text{surge}}\delta(t - t_*)dt \\
&= \frac{V_*}{s} + V_{\text{surge}}e^{-st_*}
\end{aligned}
$$

1.7 LR circuit: impulsive response

- The complete solution is simply the sum of the solution Eq.(9) added to the "impulsive/surge response" obtained by the inversion of the second term. This is the **Principle of superposition** which applies to **all linear problems.** Consider then,

$$i_{\text{surge}}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{V_{\text{surge}}}{L} \frac{e^{s(t-t_*)}}{s+\frac{R}{L}} ds$$

We see something interesting: if $t < t_*$, ie **before the surge**, the integrand is analytic in the right-half plane and the contour may be closed by a large semi-circle there. **Cauchy's theorem** then gives $i_{\text{surge}}(t) = 0; t < t_*$, as it should be! For $t > t_*$, we can choose the **Bromwich contour** to the left and evaluate the integral using residues:

$$i_{\text{surge}}(t) = (\frac{V_{\text{surge}}}{L})e^{-\frac{R}{L}(t-t_*)}$$

The complete solution can now be wriiten down:

$$i(t) = \frac{V_*}{R}[1-e^{-\frac{R}{L}t}] + (\frac{V_{\text{surge}}}{L})H(t-t_*)e^{-\frac{R}{L}(t-t_*)} \tag{10}$$

where $H(t-t_*)$ is the **Heaviside function** (Lec. 12.2). Note that it is multiplied by the exponential term, which is the solution of the **homogeneous** circuit Eq.(8). It describes how a given current **resistively decays** in the circuit if no external voltage is applied.

1.8 Green's function

- The linearity of the problem and the nature of the impulsive response suggests the following enormously fruitful idea: suppose we are given a **continuous forcing voltage**, $V(t)$ (non-zero only for $t > 0$) applied to the circuit. From the properties of the **Dirac delta**, it is clear that we may write this as follows:

$$V(t) = \int_0^\infty V(\tau)\delta(t-\tau)d\tau$$

  This says that the voltage function can be regarded as a **continuous series of impulses!** Since we know that the system response to each "elementary impulse" $V(\tau)\delta(t-\tau)d\tau$ is none other than, $V(\tau)d\tau\frac{H(t-\tau)}{L}e^{-\frac{R}{L}(t-\tau)}$, the **complete solution** must be a simple superposition of the individual impulsive responses:

$$\begin{aligned} i(t) &= \int_0^\infty V(\tau)\frac{H(t-\tau)}{L}e^{-\frac{R}{L}(t-\tau)}d\tau \\ &= \int_0^t V(\tau)\frac{e^{-\frac{R}{L}(t-\tau)}}{L}d\tau \end{aligned} \qquad (11)$$

- You should show, by **direct differentiation with respect to** t, that $i(t)$ satisfies the equation, $L\frac{di}{dt} + Ri = V(t)$ for $t > 0$ and $i(0) = 0$. The solution for $V(t) = V_*$ is also immediate!

1.9 Convolution of functions

- **Definition 13.1:** If $f(t), g(t)$ are any two Laplace-transformable functions defined over $[0, \infty)$, their **convolution or "faltung"** is a function of t denoted by $f * g[t]$, defined by the integral:

$$f * g[t] = \int_0^t f(u)g(t-u)du \qquad (12)$$

Plainly, the solution we found in the last section is a **convolution** of the applied voltage, $V(t)$ and what is called the **Green's function/system response function**, $g(t) = \frac{e^{-\frac{R}{L}t}}{L}$

Theorem 13.1: For any pair of **admissible functions** $f(t), g(t)$:

$$f * g = g * f = \int_0^t g(u)f(t-u)du$$

Proof: We may write without loss of generality, using the Heaviside functions properties and by setting $t - u = v$ and adjusting the limits accordingly, we obtain:

$$\begin{aligned} f * g[t] &= \int_0^\infty f(u)H(t-u)g(t-u)du = -\int_t^{-\infty} f(t-v)H(v)g(v)dv \\ &= \int_0^t f(t-v)g(v)dv = g * f[t] \qquad (\textbf{commutativity}) \end{aligned}$$

1.10 The convolution theorem

- **Theorem 13.2 (The Laplace convolution theorem):** For any pair of **admissible functions** $f(t), g(t)$,

$$L[f * g](s) \quad = \quad \hat{f}(s)\hat{g}(s) \tag{13}$$

Proof: Consider,

$$\begin{aligned} L[f * g](s) &= \int_0^\infty e^{-st}[\int_0^t f(u)g(t-u)du]dt \\ &= \int_0^\infty e^{-st}[\int_0^\infty f(u)H(t-u)g(t-u)du]dt \\ &= \int_0^\infty \int_0^\infty e^{-s(t-u)}e^{-su}f(u)H(t-u)g(t-u)dudt (= I(f,g)(s), \quad \text{say} \end{aligned}$$

Setting $v = t - u$ and inverting the order of integration, we get upon using the properties of $H(x)$, the stated result:

$$\begin{aligned} I(f,g)(s) &= \int_0^\infty f(u)e^{-su}(\int_{-u}^\infty e^{-sv}g(v)H(v)dv)du \\ &= \int_0^\infty f(u)e^{-su}(\int_0^\infty e^{-sv}g(v)dv)du \\ &= \hat{f}(s)\hat{g}(s) \end{aligned}$$

2.1 Applications of Laplace transforms

- We can apply the convolution theorem to solve the following **inverse problem**: if we **know** the voltage $V(t)$ and the current $i(t)$ in an LR circuit, can we calculate the circuit response function, $G(t)$**?** We are required, in this case, to solve a **Volterra convolution integral equation:**

$$i(t) = \int_0^t V(t-\tau)G(\tau)d\tau \tag{14}$$

Note that we have re-written Eq.(11) using the commutativity of the convolution. The solution is immediate upon taking Laplace transforms and using the **Convolution theorem**, we have:

$$\begin{aligned} \hat{G}(s) &= \frac{\hat{i}(s)}{\hat{V}(s)} \\ G(t) &= \frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty} e^{st}\hat{G}(s)ds \end{aligned} \tag{15}$$

This is a typical **inverse problem:** we supply an **input** to some “black box” **linear system** and we record the output. We can, subject to some assumptions, **determine** how the system will respond to **any** input! This “input-output” analysis can be, and has been very widely applied in science, enegineering and even economics!

2.2 Solving Volterra integral equations

- We can formally solve the **Volterra convolution integral equations** using Laplace transforms. The equation is of the form:

$$f(t) = g(t) + \int_0^t K(t-\tau) f(\tau) d\tau, \quad (t \geq 0)$$

where $g(t)$ is a **known** function which vanishes for $t < 0$ and $K(u)$ is called the **kernel function** of the equation, also defined for $t \geq 0$. We are required to solve this equation for the **solution**, $f(t)$. Taking Laplace transform of the equation and using the **Convolution theorem**, we have:

$$\begin{aligned}
\hat{f}(s) &= \hat{g}(s) + \hat{K}(s) \hat{f}(s) \\
\hat{f}(s) &= \frac{\hat{g}(s)}{1 - \hat{K}(s)} \\
f(t) &= \int_0^t g(t-\tau) K^*(\tau) d\tau \\
K^*(\tau) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{s\tau}}{1 - \hat{K}(s)} ds
\end{aligned}$$

For obvious reasons we call $K^*(\tau)$ the **resolvent kernel** of the equation.

2.3 Examples

- **Abel's equation:** The following integral equation occurs widely in physics, particularly in plasma diagnostics:

$$g(t) = \int_0^t \frac{f(u)}{(t-u)^\alpha}du \quad (0<\alpha<1)$$

where $g(t)$ is a given function for $t>0$ and we are required to find $f(t)$. Evidently, this is particular case of the Volterra equation. Taking Laplace transforms, and setting $K(u)=u^{-\alpha}$

$$\hat{f}(s) = \frac{\hat{g}(s)}{\hat{K}(s)}$$

$$\hat{K}(s) = \int_0^\infty e^{-st}\frac{dt}{t^\alpha} = s^{\alpha-1}\Gamma(1-\alpha)$$

$$\frac{\hat{f}(s)}{s} = \frac{\hat{g}(s)s^{-\alpha}}{\Gamma(1-\alpha)}$$

Observe that $\beta-1=-\alpha \to, \beta=1-\alpha; \frac{s^{\beta-1}}{\Gamma(1-\beta)}=\hat{K}_*(s)$, $K_*(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

$$\int_0^t f(u)du = \frac{\sin\pi\alpha}{\pi}\int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}}d\tau \tag{16}$$

2.4 Examples

- Integrating the RHS by parts, we see that,

$$\int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad = \quad \frac{t^\alpha g(0)}{\alpha} + \frac{1}{\alpha}\int_0^t g'(\tau)(t-\tau)^\alpha d\tau$$

  Differentiating Eq.(16) w.r.t t, we obtain the solution:

$$f(t) \quad = \quad \frac{\sin \pi\alpha}{\pi}[g(0)t^{\alpha-1} + \int_0^t g'(\tau)(t-\tau)^{\alpha-1} d\tau] \tag{17}$$

- **Kinematic waves:** Hitherto we have considered **ordinary differential equations** with **constant coefficients.** Let us consider some examples of **partial differential equations.** The following equation describes **traffic flow** in a model and also occurs in collisionless kinetc theories. We are required to solve for $f(x,t)$ where the function satisfies the so-called **kinematic wave** equation:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} \quad = \quad 0 \tag{18}$$

  where u is a constant "wave speed". We require that at $t = 0$, $f(x,0) = g(x)$ be a given function of $x \in (-\infty, \infty)$.

2.5 Wave propagation: kinematic waves

- Let us take Laplace transforms with respect to t, after setting, $\hat{f}(s,x) = \int_0^\infty f(x,t)e^{-st}dt$. Then, we derive:

$$s\hat{f} + u\frac{\partial \hat{f}}{\partial x} = g(x)$$

To solve this, we must multiply both sides by the "integrating factor", $\frac{1}{u}e^{\frac{sx}{u}}$ and obtain,

$$\begin{aligned}
\frac{\partial}{\partial x}[e^{\frac{sx}{u}}\hat{f}] &= \frac{g(x)}{u}e^{\frac{sx}{u}} \\
\hat{f} &= e^{\frac{-sx}{u}}\int_0^x \frac{g(v)}{u}e^{\frac{sv}{u}}dv \\
f(x,t) &= \frac{1}{2\pi i u}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^x g(v)e^{-\frac{s}{u}(x-v)}dv]ds \\
&= \int_0^x g(v)\delta(x-ut-v)dv = g(x-ut)
\end{aligned}$$

where use is made of $\delta(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}ds$ and some simple manipulations with the integrals. This result shows that the solution after a time t is simply the initial disturbance, $g(x)$ **translated** to the right with the wave speed $u > 0$, **without any change in shape!** It can be directly checked by substitution in the kinematic wave equation.

2.6 D'Alembert's Wave Equation: 1-d

- The one-dimensional propagation of waves (sound or electromagnetic disturbances) is governed by **D'Alembert's Wave Equation:**

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} \qquad (19)$$

  where c is the constant wave speed. Since the equation is **second order** in t, we must specify, at $t = 0$ **both** $\psi(x,0) = F(x)$ and $\frac{\partial\psi}{\partial t}(x,0) = G(x)$ in the domain, $-\infty < x < \infty$. This example illustrates the **combined** use of **Laplace and Fourier transforms.**

- We take simple conditions. At $t = 0$ we assume an "impulsive" disturbance at $x = x_0$ such that $F(x) = 0; G(x) = c\delta(x - x_0)$. Taking **Laplace transform** of the equation w.r.t t, we have,

$$\int_0^\infty e^{-st}\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}dt = \int_0^\infty e^{-st}\frac{\partial^2\psi}{\partial x^2}dt$$

$$-\frac{\delta(x-x_0)}{c} + \frac{s^2}{c^2}\hat{\psi}(x,s) = \frac{\partial^2\hat{\psi}}{\partial x^2}$$

  where we use the fact that $F(x) = 0$ and the Laplace transform w.r.t "commutes" with spatial differentiation.

2.7 Solving D'Alembert's equation

- Note that we now have to solve a **second-order, inhomogeneous o.d.e**! Let us set $\hat{\psi}(x,s) = W(x,s)$; treating s, as a parameter, we must solve,

$$\frac{d^2W}{dx^2} = (\frac{s}{c})^2 W - \frac{\delta(x-x_0)}{c}$$

We now take **Fourier transforms** w.r.t x on both sides by multiplying by $e^{-i\lambda x}$ and integrating from $-\infty$ to $+\infty$, assuming that the function goes to zero at infinity:

$$-\lambda^2 \hat{W} = \frac{s^2 \hat{W}}{c^2} - \frac{e^{-i\lambda x_0}}{c}$$

$$\hat{W} = \frac{e^{-i\lambda x_0}}{c} \frac{1}{\lambda^2 + \frac{s^2}{c^2}}$$

Using **Fourier's inversion formula** derived in the Problem set 12, we obtain,

$$\hat{\psi}(x,s) = \int_{-\infty}^{\infty} \frac{e^{i\lambda(x-x_0)}}{c} \frac{d\lambda}{2\pi[\lambda^2 + \frac{s^2}{c^2}]}$$

2.8 Completing the solution-1

- Applying next the **Laplace inversion formula**, we see that,

$$\psi(x,t;x_0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_{-\infty}^{\infty}\frac{e^{i\lambda(x-x_0)}}{c}\frac{d\lambda}{2\pi(\lambda^2+\frac{s^2}{c^2})}]ds$$

We invert the order of integration and do the s integral first:

$$\begin{aligned}\xi(\lambda,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{ds}{\lambda^2+\frac{s^2}{c^2}}\\ &= \frac{c}{2i\lambda}[e^{ic\lambda t}-e^{-ic\lambda t}]\end{aligned}$$

using the **Residue theorem** for $t > 0$. As usual, for $t < 0$, $\xi(\lambda,t)\equiv 0$ by **Cauchy's theorem** upon closing the contour in the right-half plane. We can now evaluate the **Fourier inverse transform:**

$$\begin{aligned}\psi(x,t;x_0) &= \int_{-\infty}^{\infty}\frac{e^{i\lambda(x-x_0)}}{c}\xi(\lambda,t)\frac{d\lambda}{2\pi}\\ &= \frac{1}{2i}\int_{-\infty}^{\infty}[e^{i\lambda(x-x_0+ct)}-e^{i\lambda(x-x_0-ct)}]\frac{d\lambda}{2\pi\lambda}\end{aligned}$$

2.9 Completing the solution-2

- Differentiating w.r.t t, we see upon using $\delta(x) = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{d\lambda}{2\pi}$ that,

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{c}{2} \int_{-\infty}^{\infty} [e^{i\lambda(x-x_0+ct)} + e^{i\lambda(x-x_0-ct)}] \frac{d\lambda}{2\pi} \\
&= \frac{c}{2} [\delta(x - x_0 - ct) + \delta(x - x_0 + ct)] \\
\psi(x, t; x_0) &= \frac{1}{2} [H(x - x_0 + ct) - H(x - x_0 - ct)]
\end{aligned}
$$