Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 12: Complex analysis: applications-I

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1.1 The Riemann Zeta function

- Let s be a complex variable with $\mathrm{Re}(s) > 1$. The **comparison test** shows that the infinite series,

$$\zeta(s) = \Sigma_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

converges absolutely. Since each of its terms, $e^{-s \ln n}$ is an entire function of s, the series defines a holomorphic function in the $s-$plane to the right of $s = 1$.

- The Riemann Zeta function has many remarkable properties and still poses major puzzles to mathematicians! It is intimately connected with **prime numbers** as the following infinite product representation due to **Euler** shows:

$$\begin{aligned} \frac{1}{\zeta(s)} &= (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})..(1 - \frac{1}{P_n^s}).. \\ &= \Pi_{n=1}^{\infty}(1 - \frac{1}{P_n^s}) \end{aligned} \tag{2}$$

where P_n is the nth **prime number**: $P_1 = 2, P_2 = 3, P_3 = 5, ..$

1.2 Zeta function: properties

- **Proof of Euler's product:** If $\mathrm{Re}(s) > 1$, the infinite **series**, $\Sigma_{n=1}^{\infty} \frac{1}{P_n^s}$ is absolutely convergent, being a "sub-series" of $\zeta(s)$ itself! Hence the infinite product is absolutely convergent. Every factor is non-vanishing. Note that,

$$\zeta(s)(1 - \frac{1}{2^s}) \quad = \quad 1 + \frac{1}{3^s} + \frac{1}{5^s} + ..$$

 where the RHS has no terms involving multiples of $1/2^s$. Similarly, $\zeta(s)(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})$ has no multiples of $1/3^s$, etc. Since the product is convergent, $\zeta(s)\Pi_{n=1}^{\infty}(1 - \frac{1}{P_n^s}) = 1$. Here we use the fundamental theorem of **Euclid: every integer is expressible uniquely, apart from order, as a product of powers of prime numbers.**

- **Einstein's radiation integral:** Consider the integral ($\mathrm{Re}(s) > 1$):

$$\begin{aligned} E(s) &= \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \qquad (= \int_0^\infty \frac{x^s e^{-x}}{1 - e^{-x}} \frac{dx}{x}) \\ &= \int_0^\infty x^{s-1} (\Sigma_{n=1}^{\infty} e^{-nx}) dx = \Sigma_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} \\ &= \zeta(s)\Gamma(s) \end{aligned} \tag{3}$$

 where we set $nx = u$ and invert the order of summation and integration and use the definitions of Gamma and Zeta functions.

1.3 Hankel's integrals

- Consider the contour integral,

$$H(z;R) \quad = \quad \int_{C_R} e^{-t}(-t)^{z-1}dt \tag{4}$$

  Here $-t$ is positive on the negative real axis and $-\pi \leq \mathrm{Arg}(-t) \leq \pi$. The contour C_R runs from $R > 0$ on the positive real axis and approaches the origin from the right in the upper half-plane, winds once around the origin **anti-clockwise** and then returns to R in the lower half-plane.

- The integrand is analytic except at the origin and along the cut taken from 0 to ∞ along the positive real axis. We may write it as, $e^{-t+(z-1)\ln(-t)}$, choosing the principal branch, as indicated. We may deform the contour to go along the "upper lip" of the real line to $|t| = \rho < R$ and then describe this circle anti-clockwise once and then return to R along the lower real axis.

- Along upper lip, integrand is: $e^{-t+(z-1)\ln|t|-i\pi(z-1)}$
  Along lower lip, it is: $e^{-t+(z-1)\ln|t|+i\pi(z-1)}$
  On the circle, $-t = \rho e^{i\phi}$

4 Hankel's formula for the Gamma functio

$$\begin{aligned} H(z;R) &= \int_R^\rho e^{-t+(z-1)\ln|t|-i\pi(z-1)}dt + \int_{-\pi}^{\pi}(\rho e^{i\phi})^{z-1}e^{\rho e^{i\phi}}\rho e^{i\phi}id\phi \\ &\quad + \int_\rho^R e^{-t+(z-1)\ln|t|+i\pi(z-1)}dt \\ &= -2i\sin\pi z\int_\rho^R e^{-t}t^{z-1}dt + i\rho^z\int_{-\pi}^{\pi}e^{iz\phi+\rho(e^{i\phi})}d\phi \end{aligned}$$

Taking $\mathrm{Re}(z) > 0$, and the limits, $R \to \infty, \rho \to 0$, we see that, the ρ -integral vanishes and the other two combine to give the Gamma integral:

$$\Gamma(z) = -\frac{1}{2i\sin\pi z}\int_C e^{-t}(-t)^{z-1}dt \tag{5}$$

- Here, C is a contour starting from $+\infty$ and running along in the upper half-plane, circling the origin counter-clockwise once and running back to infinity in the lower half plane. It follows that (from Lec 11, Eq.(5) and $z \to 1-z$):

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C e^{-t}(-t)^{-z}dt \tag{6}$$

5 Analytic continuation of the Zeta functio

- **Riemann** followed the **Hankel** procedure for the Gamma function to derive a formula for the Zeta function which will be valid everywhere in the complex plane, apart from a simple pole at $s = 1$. Thus, consider:

$$R = \int_C \frac{(-t)^{s-1}}{e^t - 1} dt \tag{7}$$

Where C is Hankel's contour, starting at $+\infty$, running in the upper half t-plane, circling the origin once anti clockwise and returning to infinity parallel to the real axis in the lower half plane, avoiding the poles of the denominator. As before, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
e^{(s-1)\ln(-t)} &= e^{(s-1)\ln|t| - i\pi(s-1)}; \qquad \mathrm{Im}(t) = \epsilon > 0; \mathrm{Arg}(-t) = -i\pi \\
e^{(s-1)\ln(-t)} &= e^{(s-1)\ln|t| + i\pi(s-1)}; \qquad \mathrm{Im}(t) = \epsilon < 0; \\
R &= -e^{-i\pi s}\int_{+\infty}^{\rho} \frac{t^{s-1}}{e^t - 1} dt - e^{i\pi s}\int_{\rho}^{+\infty} \frac{t^{s-1}}{e^t - 1} dt + i\rho^s \int_{-\pi}^{\pi} \frac{e^{is\phi} d\phi}{e^{\rho e^{i\phi}} - 1} \\
&= -2i \sin \pi s \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \qquad \text{thus, taking } \rho \to 0
\end{aligned}$$

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-t)^{s-1}}{e^t - 1} dt \qquad = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-t)^s}{e^t - 1} \frac{dt}{t} \tag{8}$$

1.6 Some deductions

- **Gamma function:** Although Eqs.(5,6) were **derived** for $\mathrm{Re}(z) > 0$, the contour integrals in them actually converge for **all** z and in Eq.(6), the RHS is an **entire** function. Thus we conclude that $\frac{1}{\Gamma(z)}$ is an entire function. This also shows that $\Gamma(z)$ cannot have any zeros. From the infinite product formulae (Lec. 11, Eqs.(11, 12))) we know that $\Gamma(z)$ must have simple poles at $z = 0, -1, -2, ...$

- **Zeta function:** The contour integral in Eq.(8) is valid for all s. Hence, the formula represents the Zeta function everywhere. It is clear that at $s = 1, 2, .., \Gamma(1-s)$ has poles. It follows from Cauchy's theorem that for $s = 2, 3, ..$ the contour integral vanishes. Hence, the Zeta function has no singularities there , as also shown by the **Einstein integral** Eq.(3). Putting $s = 1$ in the integral in Eq.(8), from the **Residue theorem**,

$$\begin{aligned} \frac{1}{2\pi i}\int_C \frac{(-t)}{e^t - 1}\frac{dt}{t} &= -1 \\ \mathrm{Lim}_{s\to 1}\frac{\zeta(s)}{\Gamma(1-s)} &= -1 \end{aligned}$$

  Since $\Gamma(1-s)$ has a **simple** pole at $s = 1$ with residue -1, $\zeta(s)$ must have also have simple pole there with residue 1. This shows that $(s-1)\zeta(s)$ is an **entire function.**

- The **Euler product**, Eq.(2) shows that for $\mathrm{Re}(s) > 1$, $\zeta(s)$ does not have any zeros.

1.7 Riemann's functional equation

- We apply the **Residue theorem** to the **single-valued function** $\frac{(-t)^{s-1}}{e^t-1}; \mathrm{Re}(s) < 0$ in the **slit region** R_N, where N is a large integer. R_N is bounded by the circle, $K_N : |t| = (2N+1)\pi$ and the "Hankel" contour C_N from $t = (2N+1)\pi$, circling the origin as usual. We suppose that R_N contains all the poles, $t = \pm 2n\pi i; n = 1, 2.., N$, where the residues are: r_n of the integrand in its **interior**. Then,

$$\frac{1}{2\pi i}\oint_{K_N}\frac{(-t)^{s-1}}{e^t-1}dt + \frac{1}{2\pi i}\oint_{C_N}\frac{(-t)^{s-1}}{e^t-1}dt \quad = \quad \Sigma_{n=1}^{N}(r_n + r_{-n})$$

Note: $(-t)^{s-1} = e^{(s-1)\ln|t| + i(s-1)\mathrm{Arg}(-t)}$ and, $\mathrm{Arg}(-t) = -\pi/2$ at $t = 2n\pi i; n > 0$. Hence,

$$r_n + r_{-n} \quad = \quad (2n\pi)^{s-1} 2\sin(\pi s/2)$$

Taking the limit $N \to \infty$, we see that the integral over K_N vanishes (show this!) and we obtain **Riemann's functional equation** for $\zeta(s)$,

$$\frac{\zeta(s)}{\Gamma(1-s)} \quad = \quad \frac{2^s \sin(\frac{s\pi}{2})}{\pi^{1-s}}\zeta(1-s) \tag{9}$$

1.8 Legendre's "duplication formula"

- We next obtain an important functional equation for the Gamma function:

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \quad = \quad \pi^{1/2}\Gamma(2z) \tag{10}$$

Proof: We use the **Euler product formula** for the **Gamma function**:

$$\mathrm{Lim}_{n\to\infty}\frac{n!n^z}{z(z+1)..(z+n)} \quad = \quad \Gamma(z) \qquad \text{then,}$$

$$\begin{aligned}
\Gamma(z)\Gamma(z+\frac{1}{2}) \quad &= \quad \mathrm{Lim}_{n\to\infty}\frac{(n!)^2 n^{2z+1/2}}{[z(z+1)..(z+n)][(z+1/2)(z+3/2)..(z+n+1/2)]} \\
&= \quad \mathrm{Lim}_{n\to\infty}\frac{n^{2z}n^{1/2}(n!)^2 2^{2n+2}}{2z(2z+1)(2z+2)..(2z+2n+1)} \\
&= \quad 2^{-2z}\Gamma(2z)\mathrm{Lim}_{n\to\infty}\frac{n^{1/2}(n!)^2 2^{2n+2}}{(2n+1)!}
\end{aligned}$$

The last term is **independent** of z and hence constant. Putting $z = 1/2$ and using $\Gamma(1/2) = \pi^{1/2}; \Gamma(1) = 1; \Gamma(2) = 1,$ we obtain Legendre's formula.

1.9 Riemann's "reflection formula"

- If we make use of the **Legendre duplication** formula for the **Gamma function** and Riemann's functional equation for the **Zeta function**, we obtain a **symmetrical form** of the latter called the "reflection formula":

$$\frac{\zeta(s)}{\Gamma(1-s)} = \frac{2^s \sin(\frac{s\pi}{2})}{\pi^{1-s}}\zeta(1-s)$$

$$\zeta(s)\Gamma(s)\cos(\frac{s\pi}{2}) = 2^{s-1}\pi^s\zeta(1-s) \quad (11)$$

But, $\pi^{1/2}\Gamma(s) = 2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}); \Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2}) = \frac{\pi}{\cos(\frac{\pi s}{2})}$.

- Substitution gives **Riemann's beautiful Reflection formula:**

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-s) \quad (12)$$

Although Eq.(9) was derived for $\mathrm{Re}(s) < 0$, by analytic continuation principles it is valid everywhere except at $s = 1$. For $\mathrm{Re}(s) > 1, \zeta(s)\Gamma(s)$ has no zeros. Hence the only zeros of $\zeta(1-s)$ for $\mathrm{Re}(1-s) < 0$ are at $s = 3, 5, ...$

- **Riemann** conjectured that all the other zeros of the Zeta function are on the "half-line", $\mathrm{Re}(s) = \frac{1}{2}$. Although it is known that there are an **infinity of zeros** on it, the **Riemann Hypothesis** is still unproved!.

2.1 Transform methods

- Complex analysis provides a very powerful tool for solving many problems in physics and engineering using “transform methods”. We take a brief look at these techniques to give you a feel for what is involved.

- **Definition 12.1: Laplace transform: Suppose $f(t)$ is a complex function of the real variable $t \in [0, \infty)$, such that for a complex s, the integral,**

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{13}$$

is defined. Then $\hat{f}(s)$ is called the Laplace transform of $f(t)$. We term the function, $f(t)$ the inverse transform of $\hat{f}(s)$. (Nb. Some texts use the variable p instead of s!)

Ex. 1: Find the Laplace transform of $e^{\lambda t}$, where λ is any complex number.

$$\begin{aligned} \hat{f}(s) &= \int_0^\infty e^{-st} e^{\lambda t}\,dt \\ &= \frac{1}{s-\lambda} \end{aligned}$$

The integral converges for all $\mathrm{Re}(s) > \mathrm{Re}(\lambda)$. Sometimes we may write the result as, $L(e^{\lambda t}) = \frac{1}{s-\lambda}$.Taking $\lambda = 0$, we see that $L(1) = \frac{1}{s}$. Clearly, $L^{-1}(\frac{1}{s-\lambda}) = e^{\lambda t}$.

2.2 Laplace transform theory

- **Theorem 12.1:** Let $f(t), g(t)$ be arbitrary functions defined on $[0,\infty)$ having Laplace transforms, $L(f), L(g)$. Then, if a, b, c are any constants, and $f'(t)$ is the derivative of f then:

$$\begin{aligned}
L(af) + L(bg) &= L(af + bg) && (14)\\
L(t^c) &= \int_0^\infty e^{-st} t^c dt \\
&= \frac{\Gamma(c+1)}{s^{c+1}} && (15)\\
L(\frac{df}{dt}) &= \int_0^\infty e^{-st} f'(t) dt \\
&= -f(0) + s\hat{f}(s) && (16)\\
L(\frac{d^2f}{dt^2}) &= \int_0^\infty e^{-st} \frac{d^2f}{dt^2} dt \\
&= -f'(0) - sf(0) + s^2\hat{f}(s) && (17)
\end{aligned}$$

The first equation says that the Laplace transfor is a **linear transformation.** The second is based on the Gamma function and demonstrates that **large** t behaviour of $f(t)$ is related to **small** s behaviour of $\hat{f}(s)$. The third and fourth equations say that differentiation w.r.t t is related to multiplication by s and involves “initial values”.

2.3 More about Laplace transforms

- The differentiation operator on $f(t)$ (in the "t"-domain) transforms into a multiplication of $\hat{f}(s)$ (in the "s"-domain) by s. It is this property which makes Laplace transforms useful in many problems. The following theorem concerns the **analyticity** of $\hat{f}(s)$.
- **Theorem 12.2:** Suppose $f(t)$ is a function such that $\int_0^\infty e^{-\sigma t} f(t)dt$ exists. Then, $\int_0^\infty e^{-st} f(t)dt$ exists as an analytic function of s for all $\mathrm{Re}(s) > \mathrm{Re}(\sigma)$.
- **Proof:** Consider $I(T) = \int_0^T e^{-st} f(t)dt = \int_0^T e^{-(s-\sigma)t} e^{-\sigma t} f(t)dt$. Then,

$$I(T) \quad = \quad (s-\sigma)\int_0^T e^{-(s-\sigma)t}\phi(t)dt + e^{-(s-\sigma)T}\phi(T)$$

where, $\int_0^T e^{-\sigma t} f(t)dt = \phi(T)$. and upon integrating by parts. Now, since the integral for ϕ converges as $T \to \infty$, $|\phi(t)|$ is **bounded** and the integral on the RHS absolutely is convergent for $\mathrm{Re}(s) > \mathrm{Re}(\sigma)$. For the same reason, the second term goes to zero. The absolute and uniform convergence in this **right half-plane** guarantees analyticity in s. Thus, Laplace transforms have a typical **abscissa of convergence**. Although the integral is defined for certain s, the transform function may be analytically continued for other values.

2.4 Important special cases

- **Definition 12.2:** Let $\delta(t-a)$ be the **Dirac delta function**: thus, by definition, $\delta(t-a)=0$ for $t \neq a$ and $\delta(0)=\infty$ in such a way that $\int_c^d f(t)\delta(t-a)dt = f(a)$, if $a \in (c,d)$ and zero if a is outside the interval of integration. It is not really a "function" but a "distribution" or linear functional. The **Heaviside function** is defined by,

$$\begin{aligned} H(x) &= 0,(x<0) \\ &= 1,(x>1) \end{aligned} \tag{18}$$

Then, it follows that, $\int_{-\infty}^{x} \delta(u)du = H(x)$. Furthermore, $H(0)$ can be **defined** as $1/2$.

- **Theorem 12.3:** We have the following formal results ($a \geq 0$; proofs are exercises!):

$$L(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-as} \tag{19}$$

$$L(H(t-a)) = \int_a^\infty e^{-st}dt = \frac{e^{-as}}{s} \tag{20}$$

$$L(e^{ct}f(t)) = \hat{f}(s-c) \tag{21}$$

$$L(f(t-a)H(t-a)) = e^{-as}\int_a^\infty f(t-a)dt = e^{-as}\hat{f}(s) \tag{22}$$

$$L(f(\frac{t}{a})) = a\hat{f}(as) \tag{23}$$

2.5 Laplace inversion formula

- Consider the function, $\hat{f}(s) = \frac{a}{s-b}$, where a, b are arbitrary complex constants. Let us try to find its **inverse transform.** This means: we must find $f(t)$ such that $f(t) = 0; t < 0$ and $\int_0^\infty e^{-st} f(t) dt = \hat{f}(s)$.

- **Bromwich contour integral**: Note that $\hat{f}(s)$ has a simple pole at $s = b$. Let c be **any real number** such that $c > \mathrm{Re}(b)$, and $\lambda > 0$ is any positive number and t is real. Let us consider the integral,

$$I_{c,\lambda}(t) = \frac{1}{2\pi i} \int_{c-i\lambda}^{c+i\lambda} e^{ts} \frac{a}{s-b} ds$$

  Note that the integral is taken on a line parallel to the **imaginary axis** is the s-plane to the **right** of the singularity of the integrand at $s = b$.

- Now suppose $t < 0$. We close the contour with a semi-circle centre $s = c$, and radius λ, running from $c - i\lambda$ to $c + i\lambda$ to the **right of the line of integration.** Evidently the integrand is **analytic** in the region bounded by the vertical line $\mathrm{Re}(s) = c$ and the semi-circle. By **Cauchy's theorem** it vanishes. If now we let $\lambda \to \infty$, it is easy to see that $|e^{ts}| = e^{-|t|R\cos\theta} \to 0; R \simeq \lambda; \cos\theta > 0$ on the semi-circle. Hence, defining $f(t) = \mathrm{Lim}_{\lambda\to\infty} \frac{1}{2\pi i} \int_{c-i\lambda}^{c+i\lambda} e^{st} \hat{f}(s) ds$, we see that $f(t) = 0$ for $t < 0$,as required!

2.6 The inverse transform

- Next we consider what happens when $t > 0$. Since in this case, e^{st} would tend to zero for s on the semi-circle closing the contour in the **left half plane**, we consider the contour integral,

$$I^{-}_{c,\lambda}(t) \quad = \quad \frac{1}{2\pi i}\oint_{C_-} e^{ts}\frac{a}{s-b}ds$$

where C_- is the contour including the straight line from $c - i\lambda$ to $c + i\lambda$ as before, but closed this time by the semi-circle on the **left**. This contour **includes** the pole at $s = b$ in its **interior**. Hence, by the **Residue Theorem**, we have, making use of the fact that the integrand goes to zero on the semi-circle as $\lambda \to \infty$,

$$\begin{aligned} f(t) &= \mathrm{Lim}_{\lambda\to\infty} I^{-}_{c,\lambda}(t) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\frac{a}{s-b}ds \\ &= ae^{bt} \end{aligned}$$

Of course, we can check this **Bromwich prescription** for getting the inverse transform is correct by directly taking the Laplace transform of $f(t)$ and verifying that indeed we get back the given $\hat{f}(s)$.

2.7 The Bromwich contour integral

- The general formula for the Laplace inverse transform is stated below without proof.
- **Laplace inversion formula:** If $f(t)$ vanishes for $t < 0$ and is suitably well-behaved, its **direct Laplace transform** $\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt$ exists as an analytic function, free of singularities for $\mathrm{Re}(s) > c$ (the abscissa of convergence). One can obtain $f(t)$ from $\hat{f}(s)$ by evaluating the **Bromwich contour integral**:

$$f(t) \quad = \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} \hat{f}(s) ds \tag{24}$$

  where the contour is any curve lying to the right of all singularities of $\hat{f}(s)$. In particular, it may be chosen to be a vertical line, $\mathrm{Re}(s) = c$.

- **Example 1:** Suppose $\hat{f}(s) = \Sigma_{n=1}^{N} \frac{a_n}{s-b_n}$. Then, $f(t) = \Sigma_{n=1}^{N} a_n e^{b_n t}; t > 0$ and zero for negative t.

  Example 2: Let a_n, b_n be such that $\mathrm{Re}(b_n) < c; n = 1, 2, ..$ and $\hat{f}(s) = \Sigma_{n=1}^{\infty} \frac{a_n}{s-b_n}$ is a **meromorphic function** which is bounded as $s \to \infty$, then, $f(t) = \Sigma_{n=1}^{\infty} a_n e^{b_n t}; t > 0$.

2.8 Derivation of inversion formula

- The following argument (**not rigorous!**) shows how the inversion formula may be derived. Suppose $\hat{f}(s)$ is a given analytic function, with no singularities for $\mathrm{Re}(s) > c$, for some c. We also assume that $\hat{f}(s)$ behaves "suitably" at infinity. Then, by **Cauchy's integral theorem** we have:

$$\hat{f}(s) = \frac{1}{2\pi i}\oint_C \frac{\hat{f}(u)}{s-u}du = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\hat{f}(u)}{s-u}du$$

where C is a contour composed of a vertical line to the right of c and an infinitely large semi-circle. The integral is described in the **clock-wise** direction! Observe that for $\mathrm{Re}(s) > c$, $\int_0^\infty e^{-t(s-u)}dt = \frac{1}{s-u}$. Substituting this in the above integral relation and interchanging the order of integration, we obtain:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\hat{f}(u)}{s-u}du &= \int_0^\infty e^{-st}dt[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tu}\hat{f}(u)du], \qquad \text{hence,} \\
&= \int_0^\infty e^{-st}f(t)dt, \qquad \text{where,} \\
f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tu}\hat{f}(u)du \qquad (25)
\end{aligned}$$

This is the required inversion formula. The vertical line can be replaced by any contour running to the right of c from $-i\infty$ to $i\infty$.

2.9 Simplest application

- Consider a particle of mass m dropped from a height h above ground assuming a constant gravitational acceleration $-g\mathbf{e}_z$, and that the air resistance during its **vertical motion** is $-m\nu v_z$, where the constant ν is called the **momentum relaxation rate** and is a measure of the drag on the particle. Calculate the motion, given **initial data** $z(0) = h; v_z(0) = 0$ We have to solve **Newton's equations of motion:**

$$\frac{dz}{dt} = v_z \tag{26}$$

$$m\frac{dv_z}{dt} = -m\nu v_z - mg \tag{27}$$

To solve these equations, we take **Laplace transforms** of both equations. Using the rules and initial conditions we get (check for yourselves!):

$$\begin{aligned}
-h + s\hat{z}(s) &= \hat{v_z}(s) \\
(\nu + s)\hat{v_z}(s) &= -\frac{g}{s}, \qquad \text{whence,} \\
v_z(t) &= -g\frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty} \frac{e^{st}}{s(s+\nu)} ds = \frac{g}{\nu}[e^{-\nu t} - 1] \\
z(t) &= \frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty} e^{st}[\frac{h}{s} - \frac{g}{s^2(\nu + s)}]ds = h - \frac{g}{\nu}(t + \frac{e^{-\nu t}}{\nu})
\end{aligned}$$