Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 11: Complex analysis: the Gamma function

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1.1 The Gamma function of Euler

- We have already seen how “elementary transcendental” functions can be studied in the complex plane. In physics and engineering, several other new functions turn up. We are going to study some of them.
- **Generalizing the factorial - the Gamma Function. Euler** noticed that the familiar factorial, $n!$ can be given an "analytical expression". Consider the simple real integral, where $a > 0$:

$$\int_0^\infty e^{-at} dt = 1/a \tag{1}$$

You can check easily that the integral converges absolutely and may therefore be repeatedly differentiated with respect to a. We obtain the formula:

$$\begin{aligned} \frac{d^n I}{da^n} &= \frac{n!}{a^n} \\ &= \int_0^\infty t^n e^{-at} dt \qquad \text{hence,} \\ n! &= \int_0^\infty t^n e^{-t} dt \end{aligned} \tag{2}$$

1.2 The "factorial integral"

- This Eulerian formula is clearly derived for **integer** values of n. **Euler** noted the remarkable fact that the key property of the factorial function, $(n+1)! = (n+1)n!$ can be extended to all **real values of** n. For this purpose, consider the integral,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, (s > 0)$$

- Obviously we have $\Gamma(n+1) = n!$ for integral n. Then, **Gamma** function satisfies the following **functional equation:**

$$\begin{aligned} \Gamma(s+1) &= s\Gamma(s) \\ \int_0^\infty t^s e^{-t} dt &= \left[-t^s e^{-t}\right]_0^\infty dt + s\int_0^\infty t^{s-1} e^{-t} dt \\ &= s\Gamma(s) \end{aligned} \tag{3}$$

upon integration by parts. The integral can be written in the form, $\Gamma(s) = \int_0^\infty e^{(s-1)\ln t} e^{-t} dt$. We can differentiate this with respect to s, obtaining,

$$\frac{d\Gamma}{ds} = \int_0^\infty (\ln t) e^{(s-1)\ln t} e^{-t} dt$$

.3 Gamma function: analytic continuation

- Evidently this integral is absolutely convergent also and hence we see that $\Gamma(s)$ is not only differentiable, but has **continuous derivatives of arbitrary order for all real** $s > 0$.
- The remarkable thing is, we can now **analytically continue** this function into the complex plane! Thus, let z be a complex variable with $\mathrm{Re}(z) > 0$. Consider the integral $\Gamma(z)$:

$$\Gamma(z) = \int_0^\infty e^{(z-1)\ln t} e^{-t} dt \tag{4}$$

  Note that this integral is **taken with respect to the real variable,** t.
- From what has already been established, the following statements are readily deduced: for real z, the function coincides with the Eulerian Gamma function; so long as z lies to the right of the imaginary axis, the integral is defined and absolutely convergent.
- Since the function so represented by the integral, Eq.(4) is differentiable with respect to z, it defines the analytic continuation of Euler's integral to the half plane. We also know that from the properties of analytic continuation, the functional equation for the function is also valid in this half-plane. We shall shortly locate the poles of this function and analytically continue it **maximally**.

1.4 “Reflection formula”

- Consider the expression (for real $0 < s < 1$)

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty t^{s-1}e^{-t}dt \int_0^\infty u^{-s}e^{-u}du$$

Substituting $u = vt; du = tdv$ in the inner integral, we get,

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty e^{-t}dt \int_0^\infty v^{-s}e^{-vt}dv$$

Interchanging the **order of integrations**(justified by absolute/uniform convergence of integrals!) we see that,

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^\infty v^{-s}dv \int_0^\infty e^{-t(1+v)}dt \\ &= \int_0^\infty \frac{v^{-s}}{1+v}dv \\ &= \frac{\pi}{\sin \pi s}\end{aligned} \tag{5}$$

where we make use of the previously established integral (Lecture 10, 3.4). Although we have proved the formula for real s, by the principles of **analytic continuation** the formula is valid for complex z in the right half plane. This formula provides a remarkable connection between the Gamma function and trigonometric functions.

1.5 Application of the reflection formula

- Setting $s = 1/2$, we obtain the result that $\Gamma(1/2) = \sqrt{\pi}$. However, this implies that,

$$\begin{aligned} \int_0^\infty \frac{e^{-t}}{t^{1/2}} dt &= 2\int_0^\infty e^{-u^2} du \\ &= \sqrt{\pi} \\ \int_{-\infty}^\infty e^{-u^2} du &= \sqrt{\pi} \end{aligned} \tag{6}$$

  We have thus evaluated the "Gaussian integral" by a method involving contour integration! We already know that $\Gamma(1) = 1; \Gamma(2) = 1$. If the formula applies for **arbitrary** s we would conclude that the function has simple poles at $s = 0, -1, -2, ...$ We shall find that this is indeed true.

- Let us consider the formula for $1/\tan z$: obtained from its "pole expansion" like Eq.(16) of Lecture 10:

$$\frac{1}{\tan z} = \frac{1}{z} + \Sigma_{n=1}^\infty \frac{2z}{z^2 - n^2\pi^2}$$

  Integrating, we get,

$$\ln \sin z = C + \ln z + \Sigma_{n=1}^\infty \ln(1 - \frac{z^2}{n^2\pi^2}) \tag{7}$$

2.1 Infinite products

- The constant C in Eq.(7) is evaluated using, $\text{Lim}_{z\to 0}\frac{\sin z}{z} = 1.$ Exponentiating we derive the famous **infinite product representation** for $\sin z$:

$$\frac{\sin \pi z}{\pi z} \quad = \quad \Pi_{n=1}^{\infty}(1 - \frac{z^2}{n^2}) \tag{8}$$

Putting $z = 1/2$, we see that $\pi/2$ is expressed as a product involving $4n^2$! This product expresses the **entire analytic function**, $\sin z$ in terms of its zeros located at $n\pi$ on the real axis.

- Some basic results are stated about the convergence of infinite products. The proofs are indicated in the problems for this Lecture.
Definition 11.1: Suppose that $a_n \geq 0$ for all n. Consider the expression, $p_N = \Pi_{n=1}^{N}(1 + a_n)$. We say that the **infinite product** $P = \Pi_{n=1}^{\infty}(1 + a_n)$ **converges** if the limit, $\text{Lim}_{N\to\infty} p_N = P$ exists.

- **Theorem 11.1:** The product $\Pi(1 + a_n)$ and the sum, Σa_n converge or diverge (to infinity) together.
Proof: We have the simple inequalities:

$$a_1 + .. + a_n \leq (1 + a_1)(1 + a_2)..(1 + a_n) \leq \exp\left[a_1 + a_2 + .. + a_n\right]$$

The stated result is immediate.

2.2 More about infinite products

- When a_n is not necessarily positive, we assume that $a_n \neq -1$. It can happen that $p_N \to 0$, in which case we say the infinite product **diverges to zero**. Thus, an infinite product is regarded as **convergent** only if its value tends to a non-zero limit. The following results are stated without proof, and may be used freely.
- **Theorem 11.2:** Suppose $a_n \leq 0$ for all n. Let $b_n = -a_n$. If $b_n \neq 1$ for all n, $\Pi(1-b_n)$ converges if Σb_n does and if $0 \leq b_n < 1$, it diverges to zero if Σb_n diverges (to infinity, as it is a series of positive terms).
- **Definition 11.2:** Let a_n be any sequence of numbers, real or complex. We assume $a_n \neq -1$. We say that the product $\Pi(1+a_n)$ is **absolutely convergent** if $\Pi(1+|a_n|)$ is convergent. Obviously, a necessary and sufficient condition for the absolute convergence of a product is that is that the series, Σa_n should be absolutely convergent.
- **Theorem 11.3:** An absolutely convergent infinite product is convergent.
- Referring to Eq.(8), since we know that $\Sigma_{n=1}^{\infty}|\frac{z^2}{n^2}|$ is always convergent, the infinite product converges absolutely for all z (not counting the zeros).

2.3 Infinite product for entire functions

- **Theorem 11.4:** Let $f(z)$ be an entire function. We know that it may have zeros (or it may have no zeros at all like e^z) which must be isolated points in any finite part of the plane. Assume that the zeros are **simple** and denote them: $a_1, a_2, ...$ If $\frac{f'(z)}{f(z)}$ satisfies the conditions of **Theorem 10.1** of Lecture 10, we have the following **infinite product** representation:

$$f(z) \quad = \quad f(0)e^{z\frac{f'(0)}{f(0)}}\Pi_{n=1}^{\infty}(1-\frac{z}{a_n})e^{\frac{z}{a_n}} \tag{9}$$

- **Proof:** In the neighbourhood of a_n, the function has an expansion, $f(z) = (z-a_n)g(z)$, where $g(z)$ is analytic and does not vanish. Thus, $\frac{f'(z)}{f(z)} = \frac{1}{z-a_n} + \frac{g'(z)}{g(z)}$. Now if $\frac{f'(z)}{f(z)}$ satisfies the conditions of Theorem 10.1, on meromorphic functions, we have the "pole" expansion formula:

$$\frac{f'(z)}{f(z)} \quad = \quad \frac{f'(0)}{f(0)} + \Sigma_{n=1}^{\infty}\left[\frac{1}{z-a_n} + \frac{1}{a_n}\right] \tag{10}$$

Integrating and exponentiating (missing out some analytical subtleties!) we obtain the stated product representation. It is also possible to **construct** an entire function with prescribed zeros by generalising this basic idea.

3.1 Euler product formula

- We now develop the theory of the Gamma function from a different and historically interesting angle. Consider the finite product,

$$G_n(z) = \frac{1.2....(n-1)n^z}{z(z+1)..(z+n-1)}$$

where z is not equal to a negative integer. Then,
$\frac{G_{n+1}(z)}{G_n(z)} = \frac{n}{n+z}\frac{(n+1)^z}{n^z} = (1 - \frac{z}{n} + \frac{z^2}{n^2}..)(1 + \frac{z}{n} + \frac{z(z-1)}{2n^2} + ...) = 1 + \frac{A_n(z)}{n^2}$.
Evidently, $\mathrm{Lim}_{n\to\infty} A_n(z) = \frac{z(z-1)}{2}$. This implies,

$$\begin{aligned}\mathrm{Lim}_{n\to\infty} G_{n+1}(z) &= \mathrm{Lim}_{n\to\infty} G_2(z).\frac{G_3(z)}{G_2(z)}...\frac{G_{n+1}(z)}{G_n(z)} \\ &= \frac{1}{z}\Pi_{n=1}^{\infty}\left[(1+\frac{1}{n})^z(1+\frac{z}{n})^{-1}\right] \qquad (11)\end{aligned}$$

converges and defines an analytic function for all z except for $z = 0, -1, -2, ...$
Following **Euler** we set this limit equal to $\Gamma(z)$. We can rather easily demonstrate from this product definition of Euler's that $\Gamma(z+1) = z\Gamma(z)$. Thus:
$\frac{\Gamma(z+1)}{z\Gamma(z)} = \frac{1}{z+1}\left[\mathrm{Lim}_{m\to\infty}\Pi_{n=1}^{m}\frac{(1+\frac{1}{n})^{z+1}}{1+\frac{z+1}{n}}\right] / \left[\mathrm{Lim}_{m\to\infty}\Pi_{n=1}^{m}\frac{(1+\frac{1}{n})^{z}}{1+\frac{z}{n}}\right] =$
$\frac{1}{z+1}\mathrm{Lim}_{m\to\infty}\Pi_{n=1}^{m}\left[\frac{(1+\frac{1}{n})(z+n)}{z+n+1}\right] = \mathrm{Lim}_{m\to\infty}\frac{m+1}{z+m+1} = 1$.

3.2 Euler product formula: contd.

- We now have to show that this is exactly the function we previously defined via Eq.(4). Note that in that equation we assumed that the real part of z has to be positive. To do this, we consider the following **finite** integral, for $\mathrm{Re}(z) > 0$ and integrate it by parts repeatedly to obtain,

$$\begin{aligned} g_n(z) &= \int_0^n (1-\frac{t}{n})^n t^{z-1} dt \\ &= n^z \int_0^1 (1-u)^n u^{z-1} du \\ \int_0^1 (1-u)^n u^{z-1} du &= \left[\frac{1}{z} u^z (1-u)n\right] + \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z du \\ &= ..... \\ &= \frac{n(n-1)...1}{z(z+1)..(z+n-1)} \int_0^1 u^{z+n-1} du \end{aligned}$$

This proves that $g_n(z) = \frac{n}{n+z} G_n(z)$. It can be shown directly that,

$$\mathrm{Lim}_{n\to\infty} g_z(z) = \int_0^n (1-\frac{t}{n})^n t^{z-1} dt \quad = \quad \int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$$

3 Euler's constant and Weierstrass' produ

- **Euler's constant:**

$$\mathrm{Lim}_{n\to\infty}[\frac{1}{1}+\frac{1}{2}+..+\frac{1}{n}]-\ln n \quad = \quad \gamma = 0.5772157..$$

Proof: Consider,

$$\begin{aligned} u_n &= \int_0^1 \frac{t}{n(n+t)}dt \\ &= \frac{1}{n}-\ln\frac{n+1}{n} < \int_0^1 \frac{dt}{n^2} = \frac{1}{n^2} \end{aligned}$$

Now, $0 < u_n < \frac{1}{n^2}$. Hence, by the **comparison text,** $\Sigma_{n=1}^{\infty} u_n$ converges. This proves the result.

- **Weierstrass** defined the Gamma function by the infinite product:

$$\frac{1}{\Gamma(z)} \quad = \quad z e^{\gamma z}\Pi_{n=1}^{\infty}(1+\frac{z}{n})e^{-\frac{z}{n}} \tag{12}$$

This is readily shown to be equivalent to Euler's product, Eq.(11). It shows that $\frac{1}{\Gamma(z)}$ is an **entire function** with simple zeros at $z = 0, -1, \ldots$ It also shows that $\Gamma(z)$ can have no zeros.

3.4 Asymptotic expansions

- If $S_n(z) = \Sigma_{m=1}^{n} u_m(z)$ is an infinite series of functions, we know that it **converges** at some $z = z_0$, if the limit, $\mathrm{Lim}_{n\to\infty} S_n(z_0)$ exists.
- **Definition 11.3:** Suppose a sequence of functions $(u_n(z))$ has the properties, $\mathrm{Lim}_{z\to z_0} u_n(z) = 0; \mathrm{Lim}_{z\to z_0} \frac{u_{n+1}(z)}{u_n(z)} = 0$, for **each** n. Then the sequence is said to be an **asymptotic gauge sequence.** If $f(z)$ is a function and $F_n(z) = \Sigma_{m=1}^{n} f_m u_n(z)$, we say that the **partial sums** $F_n(z)$ are **asymptotic** to $f(z)$ at $z = z_0$ if the following relation is true for any **fixed n**:

$$\mathrm{Lim}_{z\to z_0} \frac{f(z) - F_n(z)}{u_n(z)} = 0$$

Note that the sequence $F_n(z)$ could actually be **divergent at any fixed** z as $n \to \infty$!

Example:

$$\begin{aligned} f(x) &= e^x \int_x^\infty e^{-t} \frac{dt}{t} \qquad (x > 0) \\ &= \frac{1}{x} - \frac{1}{x^2} + ..(-1)^{n-1} \frac{(n-1)!}{x^n} + (-1)^n e^x \int_x^\infty e^{-t} \frac{dt}{t^{n+1}} \end{aligned}$$

repeatedly integrating by parts. The series is divergent for **fixed** x, but is **asymptotic** for fixed n and large x.

3.5 Asymptotic expansions: contd.

- **Some observations:** In most cases, we are interested in the behaviour of a function for large $|z|$. The idea is to **approximate** the function by some simple, elementary functions whose behaviour is well-understood. An **asymptotic series** is usually a **divergent** series which gives an extremely good approximation to the function as $|z| \to \infty$, for any **fixed number** of terms. Thus, given $f(z)$, we say, $\Sigma_{n=0}^{\infty} \frac{a_n}{z^n}$ is **asymptotic** to f if, for any given n we have:

$$\mathrm{Lim}_{z \to \infty} z^n (f(z) - \Sigma_{m=0}^{n} \frac{a_m}{z^m}) \quad = \quad 0$$

provided $\mathrm{Arg}(z)$ is suitably restricted to some sector. We then write, $f(z) \simeq \Sigma_{n=0}^{\infty} \frac{a_n}{z^n}$, although the series may actually be divergent.

- **Given** a function, and the gauge sequence, the asymptotic expansion of the function is **unique**: For example,

$$a_n \quad = \quad \mathrm{Lim}_{|z| \to \infty} z^n (f(z) - \Sigma_{n=1}^{n-1} \frac{a_{n-1}}{z^{n-1}})$$

However, the **same** asymptotic expansion can represent two **different functions!**

- Asymptotic series can be added and multiplied and integrated. They can be differentiated if the function can be differentiated and the derivative has an asymptotic expansion.

3.6 Stirling's formula

- **Laplace's method:** Let $s > 0$ be large and real. We wish to obtain an asymprtotic formula for $\Gamma(s+1) = \int_0^\infty e^{s \ln t - t} dt$. We make a preliminary transformation, $t = su; \Gamma(s+1) = s^{s+1} \int_0^\infty e^{s(\ln u - u)} du$.
- The function, $\ln u - u$ has a **maximum** at $u = 1$. Thus, if s is large and positive, most of the contribution to the integral comes from this point. Expanding the exponent in a **Taylor series** about the maximum, we have,

$$s(\ln u - u) \quad \simeq \quad -s - s\frac{(u-1)^2}{2} + ..$$

It then follows that,

$$\Gamma(s+1) \simeq s^{s+1} e^{-s} \int_0^\infty e^{-s\frac{(u-1)^2}{2}} du$$

When s is large, the integrand is nearly zero at $u = 0$. We may take the lower limit to be $-\infty$ to this degree of approximation. Then the integral is readily evaluated (using Lecture 10!) and we obtain the famous **Stirling formula** for the factorial:

$$\Gamma(s+1) \simeq \sqrt{(2\pi)} s^{s+1/2} e^{-s}$$

3.7 Watson's lemma

- The following lemma due to **Watson** can often be used to justify asymptotic expansions. The proof is not needed in this Course.
- **Watson's Lemma:** Let $0 < \lambda_1 < \lambda_2 < ..; \lambda_n \to \infty$; we further assume that $\phi(t) = \Sigma_{n=1}^{\infty} a_n t^{\lambda_n - 1}$; $|t| < a$ and $|\phi(t)| < Ke^{bt}$; $K, B > 0$ for $t > a$. Then,

$$\begin{aligned} f(z) &\equiv \int_0^\infty e^{-zt}\phi(t)dt \\ &\simeq \Sigma_{n=1}^{\infty} a_n \Gamma(\lambda_n) z^{-\lambda_n} \end{aligned} \tag{13}$$

when $|z|$ is large and $|\mathrm{Arg}(z)| \leq \frac{\pi}{2} - \Delta$, where $\Delta > 0$ is an arbitrary positive number.

- **Example:**

$$\begin{aligned} f(z) &= \int_0^\infty \frac{e^{-zt}}{1+t}dt \\ &= \int_0^\infty e^{-zt}(\Sigma_{n=0}^{\infty}(-1)^n t^n)dt \\ &= \frac{1}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} + .. \qquad (|z| \to \infty; |\mathrm{Arg}(z)| < \pi/2) \end{aligned} \tag{14}$$

4.1 The saddle point method

- I am going to discuss a very powerful method of deriving asymptotic expansions of certain types of integrals. This is called the **saddle-point method**, or sometimes, the **method of steepest descent.**
- Consider the analytic function defined by the contour integral:

$$F(z) = \int_C e^{zf(t)} dt \tag{15}$$

  We suppose that $f(t)$ is analytic and the path of integration (not closed but usually a simple contour) is such that the **integrand** tends to zero at the end points. Consider first the case when z is a large positive number, s.
- The maximum contribution to the integral must come from where the real part of $f(t)$ is largest. Evidently $\mathrm{Re}[f(t)] \to -\infty$ near the endpoints. Now, since this is an analytic function, from the maximum modulus principle, we know it cannot have a true maximum or minimum, but only a "saddle point".
- Thus consider $f(t) = u(\xi, \eta) + iv(\xi, \eta); t = \xi + i\eta$. The function, $u(\xi, \eta)$ is **harmonic** and therefore has a saddle point at (ξ_0, η_0), say, where $u_\xi = u_\eta = 0$. From the **Cauchy-Riemann equations** we know that, $v_\xi = v_\eta = 0$ and hence, $f'(t_0) = 0$. We **assume** that $f''(t_0) \neq 0$.

4.2 Saddle geometry

- Using **Taylor's theorem** locally,

$$f(t) - f(t_0) = \frac{f''(t_0)}{2!}(t - t_0)^2 + ..$$

Let us set: $f''(t_0) = \alpha e^{i\beta}; \alpha = |f''(t_0)|, t - t_0 = \rho e^{i\theta}; \rho = |t - t_0|$. Then, in the neighbourhood of the saddle-point, neglecting higher order terms, we must have, $u(\xi, \eta) - u(\xi_0, \eta_0) = \frac{\alpha}{2}\rho^2 \cos(\beta + 2\theta); v(\xi, \eta) - v(\xi_0, \eta_0) = \frac{\alpha}{2}\rho^2 \sin(\beta + 2\theta)$. If we write, $\phi = \theta + \beta/2$,

$$\begin{aligned} u - u_0 &= \frac{\alpha}{2}\rho^2 \cos(2\phi) = \frac{\alpha}{2}(\lambda^2 - \mu^2) \\ v - v_0 &= \frac{\alpha}{2}\rho^2 \sin(2\phi) = \frac{\alpha}{2}(2\lambda\mu) \end{aligned}$$

where $\lambda = \rho \cos \phi; \mu = \rho \sin \phi,$. Then, the contours, $u = const$ are rectangular hyperbolas with asymptotes, $\lambda = \pm\mu$. The curves of the **conjugate harmonic function** $v = const$ is the **orthogonal system** of rectangular hyperbolas with asymptotes, $\lambda = 0, \mu = 0$. The surfaces, $u(\lambda, \mu) = const$ clearly exhibit the "saddle" nature. The regions, bounded by the two asymptotes($\lambda = \pm\mu$) of u including the line, $\mu = 0$, correspond to "mountains" and the complementary sectors, including the line, $\lambda = 0$, are "valleys" in this "local landscape" around the "saddle point", $\lambda = \mu = \rho = 0$.

4.3 Steepest descent evaluation

- Let us return to Eq.(15). We assume that $f(t)$ has a **single** saddle point at t_0 and set $z = s >> 1$, the end point conditions imply that C must run from one valley to another. Using **Cauchy's theorem** we deform C to run through t_0. If we choose the path C^*, corresponding to $\lambda = 0 \rightarrow v = v_0$, $sf(t) = sf(t_0) - \frac{s\alpha}{2}\mu^2$, the real part decreases as fast as possible and the imaginary part is constant. We then obtain:

$$
\begin{aligned}
F(s) &\simeq e^{sf(t_0)} \int_{C^*} e^{-s\frac{\alpha}{2}\mu^2} d\mu \\
&\simeq e^{sf(t_0)} \int_{-\infty}^{\infty} e^{-\frac{s\alpha}{2}\mu^2} d\mu
\end{aligned}
$$

We may take the end points to be infinity and integrate with respect to μ, a real variable along the curve of "steepest descent", $\lambda = 0; v = v_0$. Using the results of Lecture 10, Section 4.1, we obtain,

$$
F(s) \simeq [\frac{2\pi}{s|f''(t_0)|}]^{1/2} e^{sf(t_0)}
$$

Why do we choose the "contour of steepest descent"? On any **other contour**, the equation, $v(\xi, \eta) = v_0$ does not hold. The integrand is highly oscillatory and all parts of the contour contribute. On the path of steepest descent, we have a **Gaussian/monotonic** decrease of the integrand, facilitating the evaluation.