Chennai Mathematical Institute
B.Sc Physics

Mathematical methods
Lecture 10: Complex analysis: more applications

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1.1 Cauchy principal values of integrals

- Hitherto we have considered contour integrals which have poles **within** the region they enclose; what happens if they have a **simple pole on the contour?** Consider the following integral:

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 - 1} \quad (1)$$

Clearly, there is a problem at $x = \pm 1$, although the integrand is well-behaved at infinity. If we examine the behaviour of the integrand near $x = 1$ (say), we note that it is **odd** about this point, changing sign from $-\infty$ to $+\infty$ as x increases from $1 - \epsilon$ to $1 + \epsilon$. **Cauchy** noticed that by **excluding** the interval, $(1 - \epsilon, 1 + \epsilon)$ one could integrate the function **and then take the limit** as $\epsilon \rightarrow 0$.

- If the limit **exists** we could then **define** the integral by this **second** limiting process, beyond that implicit in the concept of the integral as the limit of a sum. Obviously, if the definition is used at a **regular point** of the integrand, we'll simply get back the standard integral!

- Cauchy's limiting value for such **singular integrals** involves a **specific, double limiting process.** It is important to remember that the limit **may not always exist** but does, in many cases of physical interest. It is well-adapted to evaluation by residue calculus, as we shall discover.

1.2 Cauchy principal values: contd.

- **Definition 10.1:** Let $f(x)$ be continuous in some interval $[a,b]$, and there is a point $x_0 \in (a,b)$ such that the integrals, $I((a,x_0-\epsilon) = \int_a^{x_0-\epsilon} f(x)dx, I(x_0+\epsilon,b) = \int_{x_0+\epsilon}^b f(x)dx$ exist for arbitrary $\epsilon > 0$. If the **limit**,

$$\begin{aligned} \mathrm{Lim}_{\epsilon\to 0+}\left[\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx\right] &= \mathrm{Lim}_{\epsilon\to 0+}\left[I(a,x_0-\epsilon)+I(x_0+\epsilon,b)\right] \\ &= P\int_a^b f(x)dx \qquad (2) \end{aligned}$$

exists, then the RHS is called the **Cauchy principal value** of the integral, $\int_a^b f(x)dx$.

- **Example 1:** Evaluate the integral:

$$J = \int_{-\infty}^{\infty} \frac{dx}{x(x-ia)} \qquad (a>0)$$

The integrand has simple poles at the origin and also at $z = ia$. The integral is to be evaluated as a Cauchy Principal Value. Consider the contour $C(R;\epsilon)$ made up of: $-R, -\epsilon$ along the negative real axis, the "little semi-circle" $|z| = \epsilon$ in the upper half-plane, (ϵ, R) and the "large semi-circle" $|z| = R$ in the upper half-plane.

1.3 Cauchy principal values: contd.

- From the residue theorem we obtain ($f(z) = \frac{1}{z(z-ia)}$):

$$\begin{aligned}
\oint_{C(R;\epsilon)} \frac{dz}{z(z-ia)} &= 2\pi i \mathrm{Res}(z=ai) \\
&= \int_{-R}^{-\epsilon} f(x)dx + \int_{\pi}^{0} \frac{i\epsilon e^{i\phi} d\phi}{\epsilon e^{i\phi}(\epsilon e^{i\phi} - ia)} \\
&\quad + \int_{\epsilon}^{R} f(x)dx + \int_{0}^{\pi} \frac{iRe^{i\phi} d\phi}{Re^{i\phi}(Re^{i\phi} - ia)}
\end{aligned}$$

The residue is easily seen to be, $\frac{1}{ia}$. The integral over the "little semi-circle" equals $\frac{\pi}{a}$. The integral over the "large semi-circle" satisfies, $|\int_0^{\pi} \frac{iRe^{i\phi} d\phi}{Re^{i\phi}(Re^{i\phi}-ia)}| < \frac{\pi}{R-a}; R > a$. Hence, taking the limit as $R \to \infty; \epsilon \to 0+$, we get,

$$\begin{aligned}
P\int_{-\infty}^{\infty} \frac{dx}{x(x-ia)} &= \frac{\pi}{a} \\
&= 2\pi i[\mathrm{Res}(z=ai) + \frac{1}{2}\mathrm{Res}(z=0)]
\end{aligned}$$

What would have happened had we chosen to close the large semi-circle in the **lower half plane**?

1.4 Cauchy principal values: contd.

- This time, we retain the little semi-circle and close the large semi-circle in the **lower half-plane** enclosing the pole at the origin. From the residue theorem we obtain, for the same function:

$$\begin{aligned}\oint_{C_-(R;\epsilon)} \frac{dz}{z(z-ia)} &= 2\pi i \mathrm{Res}(z=0) \\ &= \int_R^{+\epsilon} f(x)dx + \int_0^{\pi} \frac{i\epsilon e^{i\phi} d\phi}{\epsilon e^{i\phi}(\epsilon e^{i\phi} - ia)} \\ &\quad + \int_{-\epsilon}^{-R} f(x)dx + \int_{\pi}^{2\pi} \frac{iRe^{i\phi} d\phi}{Re^{i\phi}(Re^{i\phi} - ia)}\end{aligned}$$

The residue at the origin is, $-\frac{1}{ia}$. The integral over the "little semi-circle" equals $-\frac{\pi}{a}$. The integral over the "large semi-circle" satisfies, $|\int_{\pi}^{2\pi} \frac{iRe^{i\phi} d\phi}{Re^{i\phi}(Re^{i\phi}-ia)}| < \frac{\pi}{R-a}; R > a$. Hence, taking the limit as $R \to \infty; \epsilon \to 0+$, and suitably rearranging we get,

$$P \int_{-\infty}^{\infty} \frac{dx}{x(x-ia)} = \frac{\pi}{a}$$

Exactly as before! This technique of avoiding the "pole on the contour" is called **indenting the contour".**

2.1 Hilbert transform, dispersion relations

- Let $f(z)$ be analytic in a region including the upper half-plane and the real axis. Let $|f(z)| \to 0; z = Re^{i\theta}; R \to \infty$, uniformly in θ. If a is in the upper half-plane and C is any contour enclosing a, Cauchy's integral gives:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} = f(a) \tag{3}$$

Letting $\mathrm{Im}(a) = 0$ and choosing $C_+(R,\epsilon)$ - real axis indented below by a little semi-circle,

$$f(a) = \frac{1}{2\pi i}\int_{-R}^{a-\epsilon}\frac{f(x)dx}{x-a} + \frac{1}{2\pi i}\int_{a+\epsilon}^{R}\frac{f(x)dx}{x-a} + \frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{f(a+\epsilon e^{i\phi})i\epsilon e^{i\phi}d\phi}{\epsilon e^{i\phi}} + I(R)$$

where the last term is the integral over the "large semi-circle".

- Taking the limits, $R \to \infty, \epsilon \to 0$, we get, upon noting that $I(R) \to 0$

$$P\int_{-\infty}^{+\infty}\frac{f(x)dx}{x-a} = \pi i f(a) \tag{4}$$

2.2 Hilbert transforms: crossing relation

- Separating real and imaginary parts in Eq.(4), we get the **Hilbert transform pair/dispersion relations**:

$$u(a) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{v(x)dx}{x-a} \tag{5}$$

$$v(a) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{u(x)dx}{x-a} \tag{6}$$

  Such relations play an important role in many areas of physics and mathematics, particularly in optics, particle physics/scattering theory. Note that to calculate $u(x)$ at a **single point,** we need $v(x)$ on the entire real axis!

- In physical problems one has a **crossing relation**: $f(-x) = \bar{f}(x) \rightarrow u(-x) + iv(-x) = u(x) - iv(x)$. This implies that $u(x)$ is even and $v(x)$ is odd (as a functions of x). Then, the dispersion relations take the forms:

$$\begin{aligned} u(a) &= \frac{1}{\pi} P \int_0^{\infty} v(x)[\frac{1}{x+a} + \frac{1}{x-a}]dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{xv(x)}{x^2-a^2} dx \end{aligned} \tag{7}$$

$$v(a) = -\frac{2}{\pi} \int_0^{\infty} \frac{au(x)}{x^2-a^2} dx \tag{8}$$

3.1 Residue calculus: further examples

- **Example 2:** Evaluate the integral, for $k > 0$,

$$I(k) \quad = \quad \int_{-\infty}^{\infty} \frac{\sin kx}{x} dx$$

We consider the function $\frac{e^{ikz}}{z}$, analytic in the upper half-plane except for the pole at the origin and"small" on the "large semi-circle" $|z| = R; 0 < \theta < \pi$. Consider the contour, $C(R, \epsilon)$ with two semi-circles and symmetric intervals on the real axis. The function is analytic in the region enclosed by $C(R, \epsilon)$.

$$\begin{aligned} \oint_{C(R,\epsilon)} \frac{e^{ikz}}{z} dz \quad &= \quad \int_{-R}^{-\epsilon} \frac{e^{ikx}}{x} dx + \int_{\epsilon}^{R} \frac{e^{ikx}}{x} dx \\ &\quad + \int_{\pi}^{0} \frac{e^{ik\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta + \int_{0}^{\pi} \frac{e^{ikRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= \quad 0 \end{aligned}$$

We can combine the first two integrals on the RHS to obtain, $2i \int_{\epsilon}^{R} \frac{\sin kx}{x} dx$ and take the limits, $R \to \infty; \epsilon \to 0$, and obtain, after using the evenness of the integrand and Jordan's Lemma:

$$I(k) = \int_{-\infty}^{\infty} \frac{\sin kx}{x} dx \quad = \quad \pi$$

.2 Residue calculus: multi-valued function

- What about infinite integrals over the range $(0, \infty)$? The next example shows the use of **multi-valued** functions in contour integrals.

  Example 3: Evaluate the integral,

$$J = \int_0^\infty \frac{x}{1+x^3} dx$$

We begin by considering the analytic function, $f(z) = \frac{z \ln z}{1+z^3}$. We note that $\ln x$ is real for $x > 0$ and consider $\ln z$ in the plane cut along the positive real axis. The function is holomorphic in this cut plane and takes the value, $\ln x + 2\pi i$ on the "lower edge" of the cut. Let $C(R,\epsilon)$ be a contour comprising a "little circle" around the origin of radius $\epsilon > 0$ and a "large circle" of radius $R > 1$. The function $f(z)$ is holomorphic in this region except for simple poles at $z = e^{i\pi/3}, -1, e^{i5\pi/3}$. The residue theorem then gives:

$$\begin{aligned}
\oint_{C(R,\epsilon)} f(z) dz &= \int_\epsilon^R \frac{x \ln x}{1+x^3} dx + \int_0^{2\pi} \frac{Re^{i\theta} \ln(Re^{i\theta}) iRe^{i\theta}}{1+R^3 e^{i3\theta}} d\theta \\
&\quad + \int_R^\epsilon \frac{x(\ln x + 2\pi i)}{1+x^3} dx - \int_0^{2\pi} \frac{\epsilon e^{i\theta} \ln(\epsilon e^{i\theta}) i\epsilon e^{i\theta}}{1+\epsilon^3 e^{i3\theta}} d\theta \\
&= 2\pi i (r_1 + r_2 + r_3)
\end{aligned}$$

3.3 Multi-valued functions: contd.

- We see that the integrals over the large and little circles go to zero when we take limits and observe that the integrals along the cuts “combine” and give a contribution only from the **discontinuity** of the logarithm at the cut. The residues are:
  $r_1 = \frac{i\pi}{3}\frac{e^{i\pi/3}}{3e^{2i\pi/3}} = i\frac{\pi}{9}e^{-i\pi/3} = i\frac{\pi}{9}[\frac{1}{2} - i\frac{\sqrt{3}}{2}]$
  $r_2 = -\frac{i\pi}{3}$
  $r_3 = \frac{i5\pi}{3}\frac{e^{i5\pi/3}}{3e^{i10\pi/3}} = \frac{i5\pi}{9}[\frac{1}{2} + i\frac{\sqrt{3}}{2}]$ hence,

$$\begin{aligned} J &= \int_0^\infty \frac{x}{1+x^3}dx \\ &= \frac{2\pi}{3\sqrt{3}} \end{aligned}$$

- **Example 4:** Suppose that $0 < a < 1$. Evaluate the integral,

$$G(a) = \int_0^\infty \frac{x^{a-1}}{1+x}dx$$

3.4 More about multi-valued functions

- We consider the integral, $\int \frac{z^{a-1}}{1+z} dz$. The contour, $C(R,\epsilon)$ used in the previous problem turns out to be relevant. This integral is absolutely convergent and has the correct behaviour on both “little” and “large” circles. The integrand $z^{a-1} = \frac{e^{i(a-1)\ln z}}{1+z}$ is not **holomorphic** in the whole plane, but is, in the **cut** plane region enclosed by $C(R,\epsilon)$ apart from the simple pole at $z=-1$. Applying the residue theorem we get:

$$\begin{aligned}
\oint_{C(R,\epsilon)} \frac{z^{a-1}}{1+z} dz &= \int_0^{2\pi} \frac{(Re^{i\theta})^{a-1}}{1+(Re^{i\theta})} iRe^{i\theta} d\theta - \oint_0^{2\pi} \frac{(\epsilon e^{i\theta})^{a-1}}{1+(\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta + \\
& \int_\epsilon^R \frac{x^{a-1}}{1+x} dx + \int_R^\epsilon \frac{(xe^{2\pi i})^{a-1}}{1+x} dx \\
&= 2\pi i \mathrm{Res}(z=-1)
\end{aligned}$$

Taking the limits and re-arranging after calculating the residue, we see that:

$$\begin{aligned}
(1-e^{2\pi i(a-1)}) \int_0^\infty \frac{x^{a-1}}{1+x} dx &= 2\pi i e^{(a-1)i\pi} \qquad \text{consequently,} \\
\int_0^\infty \frac{x^{a-1}}{1+x} dx &= \frac{\pi}{\sin a\pi} \qquad (10)
\end{aligned}$$

We will need this result later, in the theory of the Gamma function.

4.1 More examples!

- **Example 5:** Show that,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (11)$$

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-2ibx} = \sqrt{\pi} e^{-b^2} \quad (12)$$

Proof: The first integral is not suitable for contour integration but can be done as follows: let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Then,

$$\begin{aligned} I^2 &= (\int_{-\infty}^{\infty} e^{-x^2} dx)(\int_{-\infty}^{\infty} e^{-x^2} dx) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\phi \\ &= \pi[-e^{-r^2}]_0^{\infty} \end{aligned}$$

The result follows upon taking square roots on both sides. To do the second integral, we consider a rectangular contour, $C(R, b) : \pm R, \pm R + ib$, and the entire function, e^{-z^2}.

4.2 More examples: contd.

- We are required to show that,

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-2ibx} = \sqrt{\pi} e^{-b^2}$$

Proof: Consider:

$$\begin{aligned}
\oint_{C(R,b)} e^{-z^2} dz &= \int_{-R}^{R} e^{-x^2} dx + \int_0^b e^{-(R+iy)^2} i dy \\
&\quad - \int_{-R}^{R} e^{-(x+ib)^2} dx - \int_0^b e^{-(R-iy)^2} i dy \\
&= 0
\end{aligned}$$

However, we observe that,

$$\begin{aligned}
\left| \int_0^b e^{-(R+iy)^2} i dy \right| &\leq \int_0^{|b|} e^{-R^2+y^2} dy \\
&< |b| e^{-R^2+b^2} \\
&= 0
\end{aligned}$$

Taking the limit as $R \to \infty$ we get the required result-a **very important** one in **Fourier transform theory.**

4.3 More examples: contd.

- **Example 6:** Show that for $-\pi/4 \le \phi \le \pi/4$,

$$\int_0^\infty e^{-r^2\cos(2\phi)-ir^2\sin(2\phi)}dr = \frac{\sqrt{\pi}}{2}e^{-i\phi} \tag{13}$$

Note that $\oint_{C(R,\phi} e^{-z^2}dz = 0$, where $C(R,\phi)$ is the closed contour bounding the sector, $|z| \le R; 0 \le \theta = \mathrm{Arg}(z) \le \phi$. Then,

$$\int_0^R e^{-x^2}dx + \int_0^\phi e^{-R^2\cos 2\theta - iR^2\sin 2\theta} iRe^{i\theta}d\theta = \int_0^R e^{-r^2\cos 2\phi - ir^2\sin 2\phi}e^{i\phi}dr$$

Since,

$$\begin{aligned}|\int_0^\phi e^{-R^2\cos 2\theta - iR^2\sin 2\theta} iRe^{i\theta}d\theta| &\le R\int_0^{\pi/4} e^{-R^2\cos 2\theta}d\theta \\ &= R\int_0^{\pi/2} e^{-R^2\sin u}du < \frac{\pi}{4R}\end{aligned}$$

Using **Jordan's Lemma.** Taking the limit as $R \to \infty$, we get the stated result. Putting $\phi = \pi/4$, we get the famous **Fresnel integral** of diffraction theory:

$$\int_0^\infty e^{-ir^2}dr = \frac{\sqrt{\pi}}{2}e^{-i\pi/4} \tag{14}$$

5.1 Meromorphic functions

- We have seen how analytic functions may be represented by **power series** and by **contour integrals.** We next consider a new type of representation which brings the **poles** of the function into full view.

- **Definition 10.2:** A function is said to be **meromorphic** in a region if it is holomorphic in the region except for a **finite** number of poles within it.

- The simplest meromorphic functions are rational functions. We know that all rational functions can be expressed in terms of **partial fractions.** An interesting question is: can we do something similar to a partial fraction expansion for meromorphic functions?

- **Theorem 10.1:** Let $f(z)$ be a function whose only singularities, except at infinity, are poles. Let all the poles be **simple** and are to be located at $a_1, a_2, ..$ where, $0 < |a_1| \leq |a_2| \leq |a_3| \leq ...$ with residues, $r_1, r_2, ...$ Suppose there is a sequence of contours C_n such that C_n includes $a_1, a_2, .., a_n$ but no other poles. We also assume that the minimum distance R_n of C_n from the origin tends to infinity with n and the length L_n of C_n, where $\frac{L_n}{R_n} < K$, a **constant independent of** n. We further assume that $|f(z)|$ is bounded on the entire system of contours C_n. Under these conditions, the following **simple pole expansion formula** holds for all values of z except at the poles.

$$f(z) \quad = \quad f(0) + \Sigma_{m=1}^{\infty} r_m \left[\frac{1}{z - a_m} + \frac{1}{a_m} \right] \tag{15}$$

.2 Meromorphic functions: representation

- **Proof:** We start by considering the integral $I_n(z)$, where z is inside C_n, not coincident with any of the interior poles or the origin:

$$I_n = \frac{1}{2\pi i}\oint_{C_n} \frac{f(w)}{w(w-z)}dw$$

Note that the integrand has two sets of poles in the domain enclosed by C_n: the poles of $f(w), a_1, .., a_n$ and the poles at $w = 0, w = z$. At the first set of poles a_m, the integrand has residues, $\frac{r_m}{a_m(a_m-z)}$. At $w = 0$, the residue is, $-\frac{f(0)}{z}$. Similarly, at $w = z$, the residue is $\frac{f(z)}{z}$. **Cauchy's residue theorem** then gives:

$$I_n = \Sigma_{m=1}^{n} \frac{r_m}{a_m(a_m - z)} - \frac{f(0)}{z} + \frac{f(z)}{z}$$

We can directly estimate the integral:

$$|I_n| \leq \frac{L_n}{2\pi R_n(R_n - |z|)}\text{Max}_{C_n}|f(w)|$$

5.3 Meromorphic functions: contd.

- If we now let $n \to \infty$, we see that $I_n \to 0$, thanks to our assumptions about $f(w)$. This implies that,

$$\begin{aligned} \frac{f(z)}{z} &= \frac{f(0)}{z} - \mathrm{Lim}_{n\to\infty}\Sigma_{m=1}^{n} \frac{r_m}{a_m(a_m - z)} \\ f(z) &= f(0) + \Sigma_{m=1}^{\infty} r_m \left[\frac{1}{z - a_m} + \frac{1}{a_m}\right] \end{aligned}$$

  as asserted. Note that the series converges uniformly inside any contour which does not contain any of the poles.

- What if the function $f(z)$ has a **multiple pole** of finite order $p_n > 1$ at a_n, for example? How do we deal with this case? To take a specific example, let us consider a_1 and assume that the function has a pole of finite order $p_1 > 1$ there. For example, let us take $p_1 = 2$. We shall take the other poles to be simple, as before. By a slight generalisation of the argument, it can be shown that:

$$f(z) = f(0) + \Sigma_{m=1}^{\infty} r_m \left[\frac{1}{z - a_m} + \frac{1}{a_m}\right] + b_{-2}\left[\frac{1}{(z - a_1)^2} - \frac{1}{a_1^2}\right]$$

5.4 Mittag-Leffler expansions

- We can turn this around and ask: "given a set of points, $a_1, a_2, ..$ in the plane, can we construct a meromorphic function with only poles at these points and having **prescribed principal parts** at the poles?" The answer is "yes" and the construction is given by the **Mittag-Leffler Theorem.** First we consider an easy special case:

- **Theorem 10.2:** Let $a_1, .., a_n$ be a **finite** set of points in the complex plane. We consider the polynomials, $p_1(z), .., p_n(z)$ with $p_j(0) = 0$. Then, a meromorphic function (in fact a rational function!) with poles at the a_j and principal parts, $p_j(\frac{1}{z-a_j})$ is given by,

$$f(z) \quad = \quad \Sigma_{j=1}^{n} p_j(\frac{1}{z-a_j})$$

  This construction is not **unique** since any **entire function** added to the RHS gives another function with the same "singularity structure". If there are an infinity of poles, we must not have any finite limit points, as then the function would have an essential singularity. As long as $a_n \to \infty$, we may use the above construction **if the series converges.**

- **Theorem 10.3:** If $|a_r| \leq |a_s|; r \leq s$, and the poles tend to infinity, there exists a set of polynomials $q_j(z)$ such that,

$$f(z) \quad = \quad p_0(\frac{1}{z}) + \Sigma_{j=1}^{\infty} [p_j(\frac{1}{z-a_j}) - q_j(z)]$$

5.5 Example 1

- **Example 1:** Consider the function,

$$f(z) = \frac{1}{\sin z} - \frac{1}{z}$$

The function is taken to be zero at $z = 0$. Since at $z = n\pi$, where n is a non-zero positive or negative integer, $\sin z$ has a zero, $f(z)$ has simple poles. The origin is a **removeable singularity**. The residue at $z = n\pi$:

$$\mathrm{Lim}_{z \to n\pi}(z - n\pi)f(z) = (-1)^n$$

It can be shown that the function satisfies the conditions of **Theorem 10.1**. Thus we have the expansion:

$$\frac{1}{\sin z} - \frac{1}{z} = \Sigma_{n=1}^{n=\infty}(-1)^n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) + \Sigma_{n=-\infty}^{n=-1}(-1)^n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$$

Each of the series on the RHS separately converges. We can combine them in pairs and obtain the remarkable formula:

$$\frac{1}{\sin z} = \frac{1}{z} + 2z\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2\pi^2 - z^2} \tag{16}$$

5.6 Example 2

- **Example 2:** Consider,

$$g(z) \quad = \quad \frac{1}{e^z - 1} - \frac{1}{z}$$

There is a removeable singularity at the origin. The poles occur at $z = 2n\pi i$ where n is a non-zero integer. The residue there is:

$$\mathrm{Lim}_{z\to 2n\pi i}(z - 2n\pi i)g(z) \quad = \quad 1$$

Verifying the other conditions of the expansion theorem, we find,

$$\begin{aligned} g(z) \quad &= \quad g(0) + \Sigma_{n=1}^{\infty}\left(\frac{1}{z - 2n\pi i} + \frac{1}{2n\pi i}\right) \\ &\quad\quad + \Sigma_{n=-\infty}^{-1}\left(\frac{1}{z - 2n\pi i} + \frac{1}{2n\pi i}\right) \end{aligned}$$

As before, we may combine the $\pm n$ terms and obtain the result:

$$\frac{1}{e^z - 1} \quad = \quad \frac{1}{z} - \frac{1}{2} + 2z\Sigma_{n=1}^{\infty}\frac{1}{z^2 + 4n^2\pi^2} \tag{17}$$

- Other important expansions of this type will be treated in the problem set.