Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 8: Complex analysis: multi-valued functions

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1. Multi-valuedness and branch points

- Let us consider the function $w(z)$ **defined** by the equation, $w^2(z) = z$. For real positive $z = x$, we know that there are **two** solutions, $w_+(x) = \sqrt{x}; w_-(x) = -\sqrt{x}$. There seems to be no connection between them. Indeed, for $x > 0$, the two values, $w_+(x), w_-(x)$ differ by a finite amount. Now let us consider the behaviour in the complex plane.
- Let us start by using the "Modulus-amplitude" representation for $z = re^{i\theta}$, with $r > 0; 0 \leq \theta < 2\pi$. Then, $w_+(z) = r^{1/2}e^{i\theta/2}; w_-(z) = -r^{1/2}e^{i\theta/2}$.
- If we start at the point $r; \theta = 0$, where $w_+ = r^{1/2}$ and move **around the origin counter-clockwise** in the direction of θ increasing, and approach the starting point from **below** the real axis with $\theta = 2\pi - \epsilon$, we see that $w_+ \simeq -r^{1/2}$. In fact, we **do not return** to the value we started from but end up with $w_-(r) = -r^{1/2}$!
- **However** if we continuously move around along a closed contour which **does not** contain the origin in its **interior**, **both** $w_\pm(z)$ clearly return to their initial values.
- **Remarks:** A branch point of an analytic function is a singular point around which the function fails to be **single-valued/holomorphic**. The function may be differentiable there: e.g $w(z) = z^{5/2}$. This implies the function **can be bounded** at a branch point.

2. Branch points of analytic functions

- We note that **both** $w_+(z), w_-(z)$ are **differentiable** at any point $z \neq 0$. The origin is a point where they not only fail to have a derivative, but also if we go round it and return to the starting point (wherever we start from, as you can convince yourselves) **continuously moving along a simple closed contour**, the two functions **turn into each other.**
- It is easy to show that around the "point at infinity" the behaviour is similar. The upshot is that at each z, we seem to have **two branches** of the **same function, both analytic (except at the origin and infinity).** This **double-valuedness** invalidates many theorems on analytic functions. We now look at methods devised by **Riemann** and his successors to get around this difficulty.
- **Definition 8.1:** A point in the complex plane in the neighbourhood of which an analytic function fails to be single-valued when continuously following its values around a simple closed contour enclosing (but **not passing through**) the point is called a **branch point** of the function. Each complete circuit around the branch point will "transmute" the function value to a possibly different branch. If we return to the original branch after a **finite** number of circuits, the branch point is of **finite order.**

3. Branch cuts: examples

- To get over the difficulty of multi-valuedness, a simple device called a **branch cut** can be used. Thus, imagine we cut the complex plane from the origin to infinity **along the positive real axis.**
- This is a line which joins the two **branch points** of the function, $w(z) = \sqrt{z}$. If we start from any point and describe any curve **which does not pass through any branch point and does not cross the cut** and returns to the point, each branch, w_+, w_- is single-valued.
- If we start at a point $x + i\epsilon$, just above the real axis and come to its “neighbour” $x - i\epsilon$ which lies just below it “under” the cut, the function is **discontinuous** across the **cut**. However, so long as we agree to treat the cut as a **barrier** which we may not cross, both branches remain **holomorphic** in the **cut plane.** The branch cut is to be treated like a solid “wall”.
- With these conventions, we may treat either branch of the function as a legitimate holomorphic function in the cut-plane and apply all our theorems.
- However, we note that we **cannot** encircle the **origin** (or the point at infinity) by any closed curve in the cut plane! This implies that both those points are **not** isolated singularities of the function and the function does not have **Laurent-Taylor expansions** about them.

3. Branch cuts: (contd.)

- It should be very clearly understood that we **are allowed to put the "cut" in an infinity of ways!** For example, we can choose **any ray** from the origin to infinity to be the cut. In fact, we can draw **any simple contour** from the origin to infinity and make that the cut. Then all closed contours used by us must **never** cross the cut in order to maintain the "holomorphy" of our function branches.

- If you think about this carefully, it is clear we can define two **holomorphic branches** in the cut plane which are **discontinuous** across the cut and which "transmute" into each other there. In physical problems, the location and the nature of the cut will be dictated by the conditions of the problem, as we will discover. To summarize, only the **topology** of the cut plane matters: in the present example, the two branch points of the function are connected by a **cut** which makes the **domain of holomorphy multiply-connected**.

- **Example:** Consider $w^2(z) = (z-a)(z-b)$, where $a, b \neq 0$ are arbitrary complex constants. By considering circles in the neighbourhood of $z=0, z=\infty$, we see that the function is single-valued there: if any curve C encloses/excludes **both points** a, b, the two branches of the function remain single-valued, but in the neighbourhood of one of them, a complete circuit "transmutes" one branch into the other. Introducing a "straight cut" joining $z=a, z=b$, we can make $w_+(z) = (z-a)^{1/2}(z-b)^{1/2}$,$w_-(z) = -(z-a)^{1/2}(z-b)^{1/2}$ holomorphic in the cut plane. Many other valid cuts are also possible, and sometimes useful.

4. The logarithmic function

- We have seen that $f(z) = e^z$ is an entire function which generalises the real exponential function. For $x > 0$, we know that $x = e^{\ln x}$, where $g(x) = \ln x$ is the **natural logarithm.** We can ask what the "inverse function" of e^z is. This question has been treated in the Problems for the last Lecture, but we will now define an **analytic, multi-valued** inverse function to the complex exponential.
- If we consider the equation, $z = e^w$, we can solve it by setting $z = re^{i\theta}, w = u + iv$. Using the properties of the complex exponential, this is equivalent to the two **real** equations, $r = e^u$, $\theta = v$. The first is easily solved and its **unique solutions is** $u = \ln r$, as expected. However, we cannot simply set $v = \theta$, since $v = \theta + 2n\pi$ also solves the equation $z = e^w$, for any positive or negative integer n!
- Let us write the simplest solution in the form, where $z = re^{i\theta}; r > 0; \theta = \mathrm{Arg}(z)$:

$$w(z) \quad = \quad \ln r + i\mathrm{Arg}(z)$$

  If we start at any point in the complex plane and describe a closed contour which **does not contain the origin in its interior** we see that $\theta = \mathrm{Arg}(z)$ returns to its initial value. The function is **single-valued.** It is easy to show from the equation that the function is **unbounded** and therefore not analytic at $z = 0$. Elsewhere $\frac{dw}{dz} = \frac{1}{z}$ and the function is analytic.

4. The logarithm: other branches

- The other branches of the function are given by,

$$w_n(z) \quad = \quad \ln r + i\text{Arg}(z) + 2n\pi i$$

where $n = 0, \pm 1, ...$ This function has **infinitely many branches.** All of them satisfy, $z = e^w$. What is called the **Principal Branch** is defined by cutting the plane along the **negative real axis from the origin to infinity** and setting:

$$\ln z \quad = \quad \ln r + i\theta \qquad (-\pi \le \theta < \pi)$$

All the other branches differ from this by an integral multiple of $2\pi i$. Note that the principal branch reduces to the real function $\ln r$ along the real axis and suffers no discontinuity there in the complex plane.

- It is easy to demonstrate using Cauchy's Theorem that the principal branch, as defined above, can also be written in the form:

$$\ln z \quad = \quad \int_1^z \frac{du}{u}$$

where $z \neq 0$ and the integral is taken on any simple curve in the cut plane defined above, not passing through the origin which is a branch point of **infinite** order.

5. Riemann surfaces

- **Riemann** discovered a remarkable method of making multi-valued but analytic functions holomorphic. Let us go back to $w^2 = z$ and $w_{\pm}(z)$. In the **cut plane** we have seen that each function-branch is holomorphic. Riemann had the idea of introducing **two cut planes**, C_+, C_- placed on top of each other so that they have the origin (and also the point at infinity, the second branch point) in common. The **lower lip** of the plane C_+ is connected by an "invisible bridge" to the upper lip of C_-, and the lower lip of the latter to the upper lip of C_+. This is difficult to draw but not to imagine!
- Now, **Riemann** claims that **there is one and the same analytic function,** satisfying $w^2(z) = z$ as we wind around the origin in this weird "Riemann surface"!
- Starting at $x > 0; y = 0; z = x$, we follow the branch $w_+(z) = r^{1/2}e^{i\theta/2}$ along a curve (which may as well be the circle, $r = x$) **counter-clockwise** encircling the origin and arrive at $z = x + i(2\pi - \epsilon)$, on the "lower lip" of C_+, say. We will find that $w_+(z) \simeq x^{1/2}e^{i\pi} = -(x)^{1/2} = w_-(x)$, as expected.
- Now, following our peculiar linkages of the two cut planes, if we proceed further, we will move on to C_- and will be following $w_-(z)$ around (this is because, from the lower lip of C_+ we move **continuously** on to the **upper lip** of C_-. If we make a complete cicuit counter-clockwise, we arrive at the lower lip of C_- and find that $w_-(z) \simeq w_+(z)$, as it should be. By using two **sheets joined at the cuts, Riemann** has **united** both branches into a single holomorphic function **on the Riemann surface.**

6. Other Riemann surfaces

- Using similar arguments, one can show that $w^3(z) = z$, which has three branches can be made holomorphic on a **single, three-sheeted Riemann surface** where the cuts along the positive real axis in the sheets can be linked appropriately. Naturally one has to wind around the origin three times, each time moving on to a different sheet and follow the function branches in turn before returning to the original sheet and value.
- It is equally easy to show that if $z = e^w$ is considered, this has infinitely many sheets all joined sequentially at the cuts along the negative real axis and one will never return to the principal branch however many times we wind around the origin.
- The beauty of the Riemann surface is, on it one may apply all the theorems on holomorphic functions. “Crossing cuts” is possible, provided we move to the appropriate Riemann sheet.
- It must be remembered that analytic function branches may behave differently at the “same point” z. They are of course in different sheets! Thus, the function, $f(z) = \frac{1}{1+\sqrt(z)}$ has a branch point at $z = 0$. One of its branches is analytic at $z = 1$ and has a perfectly valid Taylor series there. The other branch has a **pole** at that point. We can alternatively say that the function is analytic on one Riemann sheet but has a pole in the other. The cut in this case can be along the negative real axis.

6. Analytic continuation

- Cauchy's integral formula tells us that if a function $f(z)$ is holomorphic in a region, we can expand it in a **Taylor series** about any point $z = a$ in that region. This simple observation has profound consequences, some of which will be explored.
- We can ask what is the maximum extent of the disk in which the power series converges. It follows immediately from the proof of the integral formula that "radius of convergence" of the series must be **at least as far** as the distance from $z = a$ to the contour bounding the region of holomorphy.
- It may happen that the power series has an **infinite** radius of convergence: this means that the function is analytic at all finite points of the plane and is an **entire function** like $e^z, \sin z,$ etc. At every point within the circle of convergence, the function is analytic and all its derivatives may be calculated. The key point is that at every such point, the function and its derivatives can be expressed in terms of the values at $z = a$. Thus **all** the derivatives at **one point** suffice to determine the function at all points of the disk.
- The above discussion shows that if we know an entire function in a tiny disk in the complex plane, we can "continue" it to the whole plane using its power series.

7. Analytic continuation: principles

- Consider the function defined by the power series,

$$f(z) = 1 - z + z^2 - z^3 ..$$

This converges for $|z| < 1$. We know that it represents the function $\frac{1}{1+z}$ in this region. Evidently, we may define the function **everywhere** except at the pole $z = -1$ by the rational function.

- Thus, the formula is an **an analytic continuation** of the series from inside of the unit circle to all of the complex plane. In the same manner we may analytically continue any function of a real variable defined by a convergent power series into the complex plane by simply replacing x by z. Such a series will converge in some circle including the interval of convergence of the real series.

- The key point is this: given a power-series, we **know** that the function defined by it is holomorphic (single-valed and analytic) at every point **interior** to its disk of convergence, D_0. Thus, we can find the values of the function and all its derivatives at such a point. If we construct the Taylor-series about this new centre, the new disk of convergence, D_1, say may well extend to a region **beyond** D_0. If this is the case, we will have succeeded in “analytically continuing” the function to a region larger than the original disk of convergence.

8. Power-series: summary

- Before we discuss the details of the processes and results of analytic continuation, it is useful to recall some key results about **power series**:

  1. Every power series, $\Sigma_{n=0}^{\infty} a_n (z-a)^n$ has a **radius of convergence**, $0 \leq R \leq \infty$ such that it converges absolutely and uniformly for $|z-a| < R$. Such a **Taylor series** defines a **holomorphic function of the complex variable** z within its disk of convergence. Every power series $\Sigma_{n=0}^{\infty} a_n (z-a)^n = f(z)$ can be differentiated or integrated within its disk of convergence and $a_n = \frac{f^{(n)}(a)}{n!}; n = 0, 1, ...$ If $R = \infty$ the function is **entire.**
  2. If $\Sigma_{n=0}^{\infty} a_n (z-a)^n ; \Sigma_{n=0}^{\infty} b_n (z-a)^n$ have the same value in some neigbourhood of $z = a$ or even at an infinite set of points with limit point $z = a$, they are identical, namely, $a_n = b_n$ for all n. This is called the **uniqueness/identity** theorem.

- **Regular/ordinary points of an analytic function:** A point in the complex plane in the **neighbourhood** of which a function is both single-valued and differentiable is called a **regular/ordinary point** of the function. Thus, we have:

- **Definition 8.2:** A single-valued function which is differentiable at a point and in a neighbourhood of the point is said to be regular/holomorphic at the point.

- It is important to remember that **regularity implies not only analyticity at the point** but also in a neighbourhood of it.

9. Singularities of analytic functions

- If an analytic function is not regular at a point, the point is a **singularity** of the function. A limit point of regular points, which is not itself a regular point is a **singularity** of the function.
- **Classification of singularities:** Analytic functions may have the following types of singularities:
  1. **Isolated singularities (poles):** If a point $z = a$ is such that the function $f(z)$ is holomorphic in a disk surrounding the point, and there is an integer $n > 0$ such that $(z-a)^n f(z)$ is non-zero and analytic, it is a **pole** of nth order.
  2. **Isolated essential singularities:** If $z = a$ is an isolated singularity, but it is not a pole of finite order, it is an essential singularity.
  3. **Non-isolated essential singularities:** Some essential singularities can "crowd together" and may not be isolated. Examples will be given later.
  4. **Branch points:** A function may be bounded or even differentiable at $z = a$ but **may not be not single-valued** in a disk centred at the point. Thus circuits around the point lead to different branches (eg. $\sqrt{z}, \ln z$). Such functions can be made single-valued by introducing suitable cuts and Riemann surfaces. Such singularities are called branch points.

10. Radius of convergence of Taylor series

- We can ask, "why is not every analytic function entire?" In other words, what exactly determines the radius of convergence Taylor series of a function? Using Cauchy's theorem and the simplest properties of power series, the following theorem can be proved:

- **Theorem 8.1:** The radius of convergence of a power series of an analytic function is determined by the nearest singularity of the function from the centre of the disk of convergence. Thus, every function represented by the Taylor series must have at least one of its singularities on its circle of convergence.
  Examples: 1. $f(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3$.. converges for $|z| < 1$. The pole is at $z = -1$.
  2. $f(z) = \sin z = z - \frac{z^3}{3!} + ..$ converges for all finite z. It has an essential singularity at infinity.
  3. $f(z) = (1-z)^{-1/2} = 1 + \frac{z}{2} + \frac{1.3}{2!}(\frac{z}{2})^2$.. converges for $|z| < 1$. Here $z = 1$ is a branch point where the function is not even bounded.

- We can compute the radius of convergence of a power series, $\Sigma_{n=0}^{\infty} a_n z^n$, by studying the behaviour of the **real, non-negative sequence**, $|a_n|^{1/n}$: if $\mathrm{Lim}_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} < \infty$, the "Cauchy root test" tells us that the series converges absolutely and uniformly for, $|z| < R$ and diverges for $|z| > R$. Nothing can be said (in general) about what happens for $|z| = R$.

1. General method of analytic continuation

- Suppose D_1 and D_2 are two regions and $f_1(z)$ is holomorphic in D_1 and $f_2(z)$ in D_2. Suppose $D_{1,2}$ have a common region D where $f_1(z) = f_2(z)$. Then, f_1 represents an analytic continuation of f_2 to D_1 and vice versa. We thus have one single analytic function, $f(z)$ in $D_1 + D_2$, with $f = f_1; z \in D_1$ and $f = f_2; z \in D_2$.
- Now, suppose $f(z)$ is holomorphic in a region R bounded by a simple contour C. We may calculate the function and all its derivatives at **any** interior point $z = a$. The Taylor series of $f(z)$ about $z = a$ must converge in a circle of radius r, where r is at least the minimum distance of a to the boundary C. If it were smaller, we could draw a larger circle on which the function would still be analytic and using Cauchy's integral formula, prove that the power series must converge in the larger circle. It is interesting that in many cases, even if the function is not entire, the power series may converge in a circle **going outside** R. After all, since R was "given", it may be that the function is actually holomorphic in some larger region and the singularity of the function nearest to $z = a$ may lie outside R.
- If the disk of convergence extends beyond R, we will have **analytically continued** the function outside it.

12. Circle-chain continuation

- **Definition 8.3:** If $z = a$ is a **regular point** of an analytic function, the value of the function and all its derivatives at $z = a$ is called a **function element**. From it we create a disk $D_1(a)$ of convergence of the Taylor series of the function.
- We next calculate the Taylor series of the function. If the series converges in a larger region, the function has been analytically continued to the larger region. In this way, construct a **chain** of circles which **maximally extend** the original region.
- It is easily proved from the identity theorem of power series that the analytic continuation of a function is **unique** and does not depend upon the path or chains of circles used to continue it, **provided the paths do not enclose a singularity of the function.**
- The following statements can be rigorously proved: if R is a region of holomorphy and a function element is known at one interior point of it, one can **uniquely continue the function analytically to the whole of** R. If the function is entire, it can be continued to the whole plane. If the function has isolated singularities, it can be continued by the chain of circles method everywhere (except of course at the singularities).
- If the function has **branch points**, analytic continuation yields, in principle, **all** the branches of the function which may be made holomorphic on its **Riemann surface.**

13. Results on analytic continuation

- If a function can be continued by any method whatever, it can also be continued by the general method of power series and the result will be the same. However, the power series method merely demonstrates the existence of continuations, and is not very practical.
- There are **lacunary functions** which are restricted by their very nature to a sub-domain of the complex plane (eg. the interior of the unit circle) so that they have a **natural boundary** made up of essential singularities beyond which it is impossible to continue the function. Examples of such functions are well-known, but are apparently not of interest in applications.
- **Theorem 8.2:** Let two regions R_1, R_2 be **adjacent**: they share a common portion of their boundaries. Let $f_1(z)$ be holomorphic in R_1 and $f_2(z)$ in R_2. The two functions are assumed to be continuous up to the common boundary and are equal along it. The combined function is holomorphic in $R_1 + R_2$.
- **Proof:** If C is any closed curve entirely contained within R_1 (or R_2), it is clear that the integral of the combined function around the curve will vanish. If the curve intersects the common boundary, we can always introduce "cross-cuts" on either side of the common portion intercepted by the curve. Since the two functions are equal along the portion intercepted, the whole integral splits into two integrals, each of which is separately zero. Then, **Morera's Theorem** gives the required result.

14. The reflection principle

- **Theorem 8.3: ("Principle of reflection")** Suppose $f(z)$ is holomorphic in a region R intersected by the real axis. Let $f(z)$ be real on the real axis. Then, $f(z)$ takes conjugate values for conjugate values of z.

- **Proof:** We consider a point $z = a$ interior to R on the real axis. We know that $f(z)$ has a Taylor expansion about $z = a$ valid in some neighbourhood interior to R. Since $f(z)$ is real on the real axis, its value and and those of its derivatives at a must be real. Thus all the Taylor coefficients are real. If we substitute $\bar{z} - a$, in the series, we see that the function must take conjugate values. Thus the result is proved in the disk of convergence. By analytic continuation we can extend it to all of the region, since the power series at conjugate points will have conjugate values.