Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 7: Complex analysis: holomorphic functions

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1. Deductions from Cauchy's Theorem

- **Definition 7.1:** An analytic function which is single-valued in a simply connected region of the complex plane is called **holomorphic**.
- Many of the functions we have seen (eg. polynomials, e^z, rational functions and analytic functions defined by absolutely convergent power series) are **holomorphic** in suitable regions. Many important functions (eg. solutions of the equations $w^2 = z; z\frac{df}{dz} = 1$) are **not** single-valued, although analytic. We will consider them later, after we have explored the holomorphic functions in some detail.
- One of the most remarkable consequences of Cauchy's formula is that if a function is analytic in a region, all its derivatives are analytic too! This is usually not the case with functions of a real variable. Thus, there are real functions $f(x)$ with everywhere continuous derivatives, but no second derivatives.
- Another consequence is a converse of Cauchy's integral theorem 6.1 due to **Morera.** This result is useful in proving some facts about infinite series of analytic functions.

2. Morera's Theorem

- **Theorem 7.1: (Morera)** Let $f(z)$ be single-valued and continuous in a region R. If $\oint_C f(z)dz$ is zero when C is any simple closed contour C entirely within R, then $f(z)$ is **holomorphic in** R.
- **Proof:** We already know that if the integral around arbitrary closed contours is zero, we may define, $F(z) = \int_{C(z_0,z)} f(u)du$, where C is an arbitrary contour lying within R and joining $z_0, z \in R$ and this is single-valued in R. Furthermore, we know that $F(z)$ is analytic and $\frac{dF}{dz} = f(z)$ in R. It follows therefore that $f(z)$ is itself analytic in R-QED.
- **Theorem 7.2:** Let $f_j(z); j = 1, ..$ be any sequence of holomorphic functions in R. Suppose that that the series, $\Sigma_{j=1}^{\infty} f_j(z)$ is **uniformly convergent** in R. Then, its sum, $F(z)$ is holomorphic in R and may be differentiated term-by-term: $F'(z) = \Sigma_{j=1}^{\infty} f'(z)$.
- **Proof:** Since the series converges uniformly, we may **integrate** term-by-term to obtain the integral of F around any closed contour from the sum of the integrals. Since the f_j are holomorphic by assumption, it follows that $F(z)$ satisfies the conditions of Morera's Theorem 7.1, and hence it is holomorphic and may therefore be differentiated term-by-term. Note that we infer a **differentiability** property here from an **integral** relation. This theorem has an important corollary called **Weierstrass' Double-series Theorem**.

3. Infinite series: Laurent-Taylor

- **Theorem 7.3: (Weierstrass):** If each term $f_j(z)$ in Theorem 7.2 is of the form of a convergent power series, $f_j = \Sigma_{n=0}^{\infty} a_{nj}(z - z_0)^n$, and if $F = \Sigma_{n=0}^{\infty} F_n(z - z_0)^n$, then, $F_n = \Sigma_{j=1}^{\infty} a_{nj}$. The proof is a simple deduction from Th. 7.2.
- The following theorem due to **Laurent** is extremely important and demonstrates the connection between power series and holomorphic functions.
- **Theorem 7.4: (Laurent-Taylor)** Let $f(z)$ be a holomorphic function in the annulus, $r_1 \leq |z| \leq r_2$. Then, $f(z)$ may be expanded in a **Laurent series**:

$$
\begin{aligned}
f(z) &= \Sigma_{n=0}^{\infty} a_n z^n + \Sigma_{n=1}^{\infty} \frac{b_n}{z^n} \qquad (1) \\
a_n &= \frac{1}{2\pi i} \oint_{|u|=r_2} \frac{f(u)du}{u^{n+1}} \\
b_n &= \frac{1}{2\pi i} \oint_{|u|=r_1} f(u) u^{n-1} du
\end{aligned}
$$

Proof: Using the Cauchy integral formula, we have,

$$
f(z) = \oint_{|u|=r_2} \frac{f(u)du}{u - z} - \oint_{|u|=r_1} \frac{f(u)du}{u - z}
$$

3. Laurent's Theorem: proof (contd.)

- Within $|u| = r_2, |z| < r_2$. Hence, we may expand $\frac{1}{u-z}$ in the first integral in a geometric series which is absolutely and uniformly convergent in powers of $\frac{z}{u}$. Outside $|u| = r_1, |z| > r_1,$ so that we can similarly expand this function in the second integral in powers of $\frac{u}{z}$ obtaining the result:

$$
\begin{aligned}
f(z) &= \oint_{|u|=r_2} f(u) \left[\Sigma_{n=0}^{\infty} (\frac{z}{u})^n\right] \frac{du}{u} \\
&\quad + \oint_{|u|=r_1} f(u) \left[\Sigma_{n=0}^{\infty} (\frac{u}{z})^n\right] \frac{du}{z} \\
&= \Sigma_{n=0}^{\infty} z^n \oint_{|u|=r_2} f(u) \frac{du}{u^{n+1}} \\
&\quad + \Sigma_{n=1}^{\infty} \frac{1}{z^n} \oint_{|u|=r_1} f(u) u^{n-1} du
\end{aligned}
$$

This proves the Theorem. If $f(z)$ is analytic everywhere within $|z| = r_2, r_1 = 0$ and all the b_n's vanish. We then have **Taylor's Theorem for a function of a complex variable.** Of course, the theorems are valid **whatever** the centre of the two circles is. If the centre is at $z = a,$ we simply replace z in the series by $z - a$ and $\frac{1}{u} \to \frac{1}{u-a}$ in the integral formulas for the coefficients.

3. Laurent-Taylor Theorem: corollaries

- **Corollary 1:** The series of positive powers ("Taylor series") converges everywhere within the circle $|z| = r_2$ and represents a holomorphic function there. The series of negative powers converge everywhere outside the circle $|z| = r_1$ and represents a holomorphic function there. Furthermore, it tends to zero at infinity.
- **Corollary 2:** Thus, using Eq.(3), Lecture 6 (Cauchy's formula for $\frac{d^n f}{dz^n}$) , we may write, when $f(z)$ is **holomorphic everywhere** in a circle C of radius R and centre $z = a$,

$$\begin{aligned} f(z) &= a_0 + a_1(z-a) + a_2(z-a)^2 + .. \\ &= f(a) + \frac{f^{(1)}(a)}{1!}(z-a) + \frac{f^{(2)}(a)}{2!}(z-a)^2 + .. \\ \frac{f^{(n)}(a)}{n!} &= \frac{1}{2\pi i}\oint_C \frac{f(u)du}{(u-a)^{n+1}} = a_n \end{aligned}$$

  This **Taylor series** converges absolutely and uniformly for $|z-a| < R$.
- **Corollary 3: "Cauchy's inequality"** If $|f(z)| < M$ on C,

$$|f^{(n)}(a)| \leq n!\frac{M}{R^n}$$

These follow from the formulas for the coefficients. This inequality bounds the **derivatives** of $f(z)$ in terms of the bound on the **function**-a remarkable property of analytic functions.

4. Analytic functions: isolated singularities

- **Definition 7.2:** A point in the complex plane where $f(z)$ is **holomorphic** (ie analytic and single-valued) is called an **ordinary point.**
- **Remark:** If $z = a$ is an ordinary point of $f(z)$, there exists a circle C with centre $z = a$ and radius $\rho > 0$ such that $f(z)$ is single-valued and differentiable for $|z - a| < \rho$. Every function may be expanded in a **Taylor series** about an ordinary point, and the expansion is absolutely and uniformly convergent in the ρ-neighbourhood of the point. Note that analyticity always refers to an open set (here a disk), not just to a single point.
- **Definition 7.3:** A point in the complex plane where a function ceases to be analytic (ie its derivative does not exist) is called a **singular point or singularity** of the function. If $z = a$ is a singular point of $f(z)$ such that there is a disk of radius $\rho > 0$ where the function is holomorphic **except** at $z - a$ is called an **isolated singularity** of the function.
- **Examples:**
  1. Every point in the **finite** complex plane is an **ordinary point of** $f(z) = e^z$.
  2. The function $f(z) = \frac{1}{z}$ has an isolated singularity at $z = 0$.
  3. The function $f(z) = \frac{1}{1+z^2}$ has two isolated singularities at $z = \pm i$.

5. Laurent series at isolated singularities

- We can check the behaviour of $f(z)$ at the "point at infinity" by making the substitution, $u = \frac{1}{z}$ and considering the nature of the function at $u = 0$. Thus every non-constant polynomial has an isolated singularity at "infinity". Similarly, we see that the point at infinity is an **ordinary point** of $f(z) = \frac{1}{1+z^2}$.

- Laurent's Theorem has an important special case: the "inner circle" can actually shrink to a point, so that $z = a$ becomes an isolated singularity of the function. This is stated in the next result. Here C_r is **any circle** centred at $z = a$ with radius $0 < r < \rho$ and $M(r)$ is the maximum value of $|f(z)|$ on C_r.

- **Theorem 7.5 :** Let $f(z)$ be analytic (and single-valued) on and within a circle C of radius ρ centred about $z = a$ except at a. Then the function may be expanded in a Laurent series expansion valid for $0 < |z - a| < \rho$:

$$
\begin{aligned}
f(z) &= \Sigma_{n=-\infty}^{\infty} c_n (z-a)^n \\
c_n &= \frac{1}{2\pi i} \oint_{C_r} \frac{f(u)du}{(u-a)^{n+1}} \\
|c_n| &\leq \frac{M(r)}{r^n}
\end{aligned}
$$

  In general, the coefficients c_n cannot be expressed in terms of the derivatives of $f(z)$ at $z = a$, since it is not analytic there.

6. Poles and essential singularities

- There is some standard terminology associated with Laurent and Taylor expansions. Thus the **analytic part** of the expansion (series of non-negative powers of $z - a$) is called the **regular part** while the series of negative powers is defined as the **principal part.**

- If $f(z)$ has a Laurent expansion in $0 < |z - a| < \rho$, and its principal part vanishes identically (ie all the coefficients of negative powers are zero) then the function is said to have a **removable singularity** at $z = a$. It can be made analytic at $z = a$ by **defining** $f(a) = \text{Lim}_{z \to a} f(z)$. Its regular part then becomes its **Taylor series** which converges for $|z - a| < \rho$. A typical example is the function, $f(z) = \frac{e^z - 1}{z}$, defined for $|z| > 0$. Since the function is not even defined at $z = 0$, it is a singular point. However, the function has a definite Taylor expansion about the origin: $f(z) = 1 + \frac{z}{2!} + \ldots$ Thus **defining** $f(0) = \text{Lim}_{z \to 0} = 1$, we can **remove** the singularity! Removable singularities are of no interest.

- **Definition 7.3:** If $z = a$ is an isolated singularity of the holomorphic function $f(z)$ defined as in Laurent's Theorem 7.5, and if only a **finite number** of the coefficients c_n belonging to the **principal part** are **non-vanishing**, the singularity is called a **pole.** If an infinite number of c_n's are non-vanishing for $n < 0$, the singularity is called an **essential singularity.**

6. Poles, essential singularities: contd.

- **Ex. 1:** The function $e^{1/z}$ is holomorphic for $|z| > 0$. It has an essential singularity at $z = 0$.
  Ex. 2: The function $f(z) = \frac{1}{z^2-1}$ has poles at $z = \pm 1$.
  Ex. 3: The function $f(z) = e^{(z+1/z)}$ has essential singularities at $z = 0; z = \infty$.
  Ex. 4: The function $f(z) = \frac{e^z}{1-z}$ has a pole at $z = 1$ and an essential singularity at infinity.

- **Definition 7.4:** If $z = a$ is an isolated singularity of the holomorphic function $f(z)$ defined as in Laurent's Theorem 7.5, the coefficient c_{-1} is called the **residue** of the function at $z = a$. It is given by the formula,

$$c_{-1} \quad = \quad \frac{1}{2\pi i} \oint_{\Gamma} f(u)du$$

  where Γ is any simple closed contour lying entirely within $|z - a| = \rho$ and containing $z = a$ within it. In particular, we may take Γ to be a circle centred on the singularity with any radius less than ρ.

- The concept of the residue of a holomorphic function at an isolated singularity leads to an extremely powerful application of Cauchy's Theorem.

7. Cauchy’s Residue Theorem

- **Theorem 7.6:** Let $f(z)$ be holomorphic in a region R and let C be a simple closed contour in R which contains within its interior a **finite** number of isolated singular points, $a_i; i = 1, .., n$ at which the function has residues, r_i. Then,

$$\oint_C f(z)dz = 2\pi i \Sigma_{i=1}^n r_i \quad (2)$$

- **Proof:** This follows from applying Cauchy’s Theorem to the **multiply connected** region defined by the interior of the region enclosed by C and the exterior of "small" disks surrounding the n isolated singularities at $z = a_i$. Cauchy’s theorem states that $\oint_C f(z)dz - \Sigma_{i=1}^n \oint_{c_i} f(z)dz = 0$, where the c_i are the circumferences of the small circles centred at a_i. From Laurent’s Theorem these integrals are precisely respectively equal to $2\pi i r_i$, where r_i are the residues at these points. Thus **Cauchy’s Residue Theorem** is established.

- **Example 1:** The function $f(z) = \frac{1}{z}$ has a pole at $z = 0$ with residue $r = 1 \rightarrow \oint_C \frac{dz}{z} = 2\pi i$, where C is any simple closed contour around the origin. If the origin lies outside the region enclosed by C, by Cauchy’s Theorem, the integral vanishes!
  Example 2: The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$ where its residue $c_{-1} = 1$.

8. Behaviour near isolated singularities

- If the isolated singularity of $f(z)$ at $z=a$ happens to be a **pole**, the Laurent series around $z=a$ must have the form,

$$f(z) = \phi(z) + \frac{c_{-1}}{z-a} + .. + \frac{c_{-m}}{(z-a)^m}$$

where $\phi(z)$ is analytic at $z=a$ and $m>0$ is a positive integer. The pole is said to be of **order** m. Consider $(z-a)^m f(z) = F(z)$. Obviously, this function is analytic at the pole and $F(a) = c_{-m} \neq 0$. Clearly, we have the formula,

$$c_{-1} = \mathrm{Lim}_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

Laurent series near isolated singularities of a holomorphic functions can be used to discuss their behaviour. Suppose $f(z)$ has a pole of order m at $z=a$. Then its **principal part** may be written as,

$$f_p(z) = \frac{c_{-1}}{(z-a)} + .. + \frac{c_{-m}}{(z-a)^m}$$

where $c_{-m} \neq 0$. Evidently, by choosing z sufficiently close to the pole, we see that $|f(z)| \simeq \frac{|c_{-m}|}{|z-a|^m}$. This "blowing-up" behaviour is typical of **poles, in the vicinity of which an analytic function is single-valued, but unbounded.**

9. Uniqueness of Laurent-Taylor series

- The behaviour of an analytic function near an **essential singularity** which can be isolated or not is much more subtle. It can be shown quite simply (**Weierstrass' Theorem**) that in the neighbourhood of an essential singularity, an analytic function approaches **any given value** arbitrarily closely. Indeed, it was proved by **Picard** that near an essential singularity an analytic function **actually assumes every value with the possible exception of one.** For example, the function $f(z) = e^{1/z}$ has an essential singularity at the origin. It assumes every value in the neighbourhood of the origin except $f(z) = 0$.

- **Uniqueness of the Laurent expansion:** If we have obtained, in any manner the formula,

$$f(z) = \Sigma_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad (R' < |z-a| < R)$$

The series is necessarily identical with its Laurent series. Thus, we consider a circle $\Gamma : |z-a| = \rho; R' < \rho < R$. From uniform convergence, the Laurent coefficients are:

$$\begin{aligned} d_n &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)dz}{(z-a)^{n+1}} \\ &= \Sigma_{m=-\infty}^{\infty} \frac{c_m}{2\pi i} \oint_\Gamma \frac{(z-a)^m dz}{(z-a)^{n+1}} \\ &= c_n \qquad \mathbf{Q.E.D} \end{aligned}$$

10. Liouville's Theorem

- It might be imagined that there might be **bounded** functions which are analytic **everywhere.** The following theorem due to Liouville shows that only **constants** can be analytic everywhere and be bounded.

- **Theorem 7.7: (Liouville)** If $f(z)$ is analytic at every point of the complex plane and is also bounded, it must be a constant.

- **Proof:** We have, from Cauchy's integral formula,

$$\begin{aligned} |f'(z)| &= \left|\frac{1}{2\pi i}\oint_{C_\rho}\frac{f(u)du}{(u-z)^2}\right| \\ &\leq \frac{M}{\rho}(1-\frac{|z|}{\rho})^{-2} \end{aligned}$$

  where C_ρ is a circle centred at the origin, with an arbitrarily large radius $\rho > |z|$. By hypothesis, we have, for all z, $|f(z)| < M$. Making $\rho \to \infty$, we see that $f'(z) = 0$, for any z. Thus $f(z)$ must be a constant.

- This theorem can be used to prove the **Fundamental Theorem of Algebra** which states that every non-constant polynomial $P(z)$ has at least one **zero** $z = a$ such that $P(a) = 0$.

11. Liouville's Theorem: applications

- **Theorem 7.8:** If $P(z)$ is a non-constant polynomial, there exists a complex number a such that $P(a) = 0$.
- **Proof:** Suppose there is no such a, so that $P(z) \neq 0$ for any z. Consider the rational function, $g(z) = \frac{1}{P(z)}$. Since $P(z)$ never vanishes, $g(z)$ must be bounded. It is non-constant and clearly differentiable everywhere, since $g'(z) = -\frac{P'(z)}{P^2(z)}$, which is also finite everywhere. Hence, by Liouville's Theorem $g(z)$ must be constant, which contradicts the fact that $P(z)$ is assumed to be non-constant! Hence, $P(z)$ must have at least one zero. Later we will give another proof which will show that if the degree of the polynomial is n, then it must have n zeros (including multiple zeros).
- **Definition 7.5:** Let $f(z)$ be holomorphic in R. If $z = a$ is a point such that $f(a) = 0$, it is a **zero** of the function. Suppose the **Taylor series** of $f(z)$ near $z = a$ is of the form,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + ..$$

  then, $z = a$ is said to be a zero of order $m \geq 1$. At such a zero, $f(a) = f^1(a) = .. = f^{m-1}(a) = 0$ and, $f^m(a) \neq 0$.

12. Zeros of analytic functions

- **Theorem 7.9:** Zeros of analytic functions are **isolated points.** Thus if $f(z)$ is holomorphic and not identically zero around $z = a$, then, there is a disk $D : 0 \leq |z - a| < \rho; \rho > 0$ such that $f(z) \neq 0$, except possibly at $z = a$.

- **Proof:** Without loss of generality, we may assume the zero is at the origin: $a = 0$. Then,

$$f(z) = a_0 + a_1 z + a_2 z^2 + .. \qquad (\mathbf{Taylor - series})$$

It converges in some disk, $0 \leq |z| < r; r > 0$. If **not all the coefficients** are zero, there must be a first non-zero one, $a_m; m \geq 1$, say. Then,

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + .., (|z| < r)$$

Now, if $0 < r_1 < r$, the series converges for $|z| \leq r_1$ and $|a_n| r_1^n \to 0$; hence it is bounded, by K say. Then we have,

$$\begin{aligned} |f(z)| &\geq |z|^m \left[|a_m| - \frac{K|z|}{r_1^{m+1}} - \frac{K|z|^2}{r_1^{m+2}} - .. \right] \\ &\geq |z|^m \left[|a_m| - \frac{K|z|}{r_1^m (r_1 - |z|)} \right] \end{aligned}$$

Choosing $\rho > 0$ sufficiently small, and using, $|z| < \rho < r_1$, we can make the RHS positive in $0 < |z| < \rho$. Hence the zero must be isolated.

13. Argument principle of Cauchy

- **Alternate form of Th. 7.9:** If $f(z)$ is holomorphic in a region R and $z_1, z_2, z_3, ..$ are a set of points having a limit point z_* in R, and $f(z_i) = 0$ at every z_i, then $f(z) = 0$ for all z in R.
- **Corollary 1 (Th. 7.9):** If $f(z)$ is holomorphic in a region and vainshes in any sub region or along any arc of a continuous curve or at an infinity of points with a limit point in the region, it must vanish **identically.**

  Corollary 2 (Th. 7.9): If two analytic functions are equal at an infinity of points in their common region of analyticity, they must be equal throughout the region.
- We can use the Residue Theorem of Cauchy to “count” the number of zeros and poles of any holomorphic function in a compact region. This is called the “argument Principle” and is a very important result: we find that a **contour integral** can count the zeros and poles!
- **Theorem 7.10:** Let $f(z)$ be holomorphic on and within a simple closed contour C, apart from a finite number of poles, and $f(z)$ is not zero on the contour, and let N_Z be the number of **zeros** and N_P the number of **poles** inside the countour, counted with the appropriate **multiplicities.** Then:

$$N_Z - N_P = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz \qquad (3)$$

14. Proof of the argument principle

- **Proof:** We apply the **Residue Theorem** to the holomorphic function $\frac{f'(z)}{f(z)}$ which has **poles** at the zeros of $f(z)$ and also at those of $f'(z)$ which are located at the **poles** of f itself. Thus,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz \quad = \quad \Sigma_i R_i + \Sigma_j S_j$$

where R_i are the residues of $\frac{f'}{f}$ at the zeros of f and S_j are the residues at the poles of f. Note that both are finite sums. If $z = z_i$ is a **zero** of f of order m, $f(z) = a_m(z - z_i)^m + .., f' = ma_m(z - z_i)^{m-1}$, in the immediate neigbourhood of the point. Hence, $R_i = m$. In the case of z_j being a **pole** of order n $f(z) = a_{-n}(z - z_j^{-n} + .., f'(z) = -na_{-n}(z - z_j)^{-n-1}$ in its neighbourhood and $S_j = -n$. Applying this to the finite sets of zeros and poles, we get the required result.

- **Cor:** Let $P_n(z) = z^n + b_1 z^{n-1} + b_2 z^{n-2} + .. + b_n$. Then $P_n(z)$ has precisely n complex zeros (including multiplicity).
  Proof: Consider $\frac{P'(z)}{P(z)}$ integrated on a circle of very large radius. The integral in the argument principle equals n for a sufficiently large circle (show this explicitly!). Since $P'(z)$ is analytic for all z (why?), $N_P = 0$. Hence, $N_Z = n$.This gives a bit more information on the zeros of polynomials than Liouville's Theorem.