Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 6: Complex analysis: Cauchy's Theorem

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1. Cauchy's Theorem

- **Theorem 6.1:** Let R be a simply connected open region of the complex plane and let $f(z) = u(x,y) + iv(x,y)$ be an (single-valued) **analytic function of** $z = x + iy$ in this region. Let C be any **simple, closed contour** lying entirely within R. Then, the following integral relation holds:

$$\begin{aligned} \oint_C f(z)dz &= \oint_{C(s)} \left[u\frac{dx}{ds} - v\frac{dy}{ds} \right] ds + i \oint_{C(s)} \left[u\frac{dy}{ds} + v\frac{dx}{ds} \right] ds \\ &= 0 \end{aligned} \qquad (1)$$

- **Remarks:** The theorem states that the contour integral of an analytic function taken around a closed contour lying entirely within a simply connected region of analyticity always vanishes. This means that **both** the real and imaginary parts of the integral must **separately** vanish. Virtually every important result in Complex Analysis depends, in one way or another, upon this fundamental theorem.

- The notation, $C(s)$ means the contour is parametrized by s, where $0 \le s \le L_C; C : (x(s), y(s)), L_C$ being the length of the curve. We always mean closed curves of **finite** total length in this context.

1. Cauchy's Theorem: special cases

- Before giving one of the simpler proofs of this theorem, let us consider two easy cases: if $f(z) = c$, where c is an arbitrary, complex constant, we evidently verify the theorem since, $\oint_{C(s)} \frac{dx}{ds} ds = \oint_{C(s)} \frac{dy}{ds} ds = 0$. This is evidently a consequence of the fact that for a closed contour, $x(0) = x(L_C); y(0) = y(L_C)$.
- Let us consider $f(z) = kz$, where k is an arbitrary complex constant. Now, $\oint_C f(z)dz$ can be written as, $\oint_C \frac{k}{2} \frac{d}{dz}(z^2) \frac{dz}{ds} ds$. It is evident that this too must vanish, since z^2 is periodic as we go round the contour and returns to its initial value when we traverse the whole contour. The same argument can be applied to **any polynomial in z in any finite sub region of the complex plane**.
- If $f(z)$ is expressed as a uniformly convergent **power series** in R, by using integration theorems, we can verify Cauchy's theorem for the function since the contour integral for each term of the series must vanish by the above argument. Since the series is assumed uniformly convergent, the result applies to its sum.
- We next consider some simple contours where we can verify the theorem rather easily.

2. Cauchy's Theorem: "proofs"

- A rigorous proof is beyond the scope of this course, but I shall indicate the lines such a proof might take. We need the idea of an "exact differential".
- **Definition 6.1:** Let R be a simply connected region of the x, y plane. Let $P(x, y), Q(x, y)$ be two real **continuously differentiable functions** in R. The expression, $P(x, y)dx + Q(x, y)dy$ is said to be an **exact differential** if there exists a single-valued real function $F(x, y)$ which is continuously twice differentiable in R such that, $P(x, y) = \frac{\partial F}{\partial x}; Q(x, y) = \frac{\partial F}{\partial y}$ in R, and hence,

$$P dx + Q dy = F_x dx + F_y dy = dF$$

- The following simple theorem is an immediate consequence of the above definition.
- **Theorem 6.2:** If $P(x, y)dx + Q(x, y)dy$ in R (as defined above) is an exact differential, then if $C(s)$ is an arbitrary simple, closed contour lying in R,

$$\begin{aligned} \oint_{C(s)} \left[P\frac{dx}{ds} + Q\frac{dy}{ds} \right] ds &= \oint_{C(s)} \frac{dF}{ds} ds \\ &= 0 \end{aligned}$$

2. Cauchy’s Theorem: exact differentials

- A 2-d vector field $(P(x,y), Q(x,y))$ associated with an exact differential is known in Physics as a “conservative field” and is said to be **derivable from the potential**, $F(x,y)$. Clearly, a **necessary condition** for this is,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

  But this means that we must have, if $Pdx + Qdy$ is an exact differential, the equality holds:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

  It is remarkable that this is also a **sufficient condition** for the expression to be an exact differential! This is a purely “real variable” theorem proved in advanced calculus/vector analysis. We shall simply assume it to be true and deduce Cauchy’s “complex integral theorem” from it.

- You should be aware that rigorously proving the sufficiency of the exact differential condition is really equivalent to proving Cauchy’s Theorem.

3. Proof A of Cauchy's Theorem

- If $f(z) = u(x, y) + iv(x, y)$ is analytic in R, then the differentials,

$$u(x, y)dx - v(x, y)dy$$
$$v(x, y)dx + u(x, y)dy$$

are **both exact.** This follows by applying the sufficient condition which requires the relations, $u_y = -v_x; v_y = u_x$. These **are the Cauchy-Riemann equations**, which themselves result from the assumed analyticity of $f(z)$ in R. Then, it follows from Theorem 6.2 that,

$$\oint_{C(s)} \left[u\frac{dx}{ds} - v\frac{dy}{ds} \right] ds = 0$$
$$\oint_{C} \left[v\frac{dx}{ds} + u\frac{dy}{ds} \right] ds = 0$$

These are equivalent to Cauchy's Integral Theorem, 6.1- Q.E.D.

- I shall present a more general approach based on **Green's Theorem** in 2-d.

3. Proof B: via Green's formula

- **Theorem 6.3 (Green's formula):** If $(P(x,y), Q(x,y))$ is a 2-d vector field defined and continuously differentiable in R, **Green's Theorem** (in the plane, also related to **Stokes' Theorem** of vector analysis) states that, for any simple closed contour $C(s)$ in R enclosing (by virtue of Jordan's Theorem!) a domain D, the following integral formula which links the contour integral with the double integral over the enclosed domain holds:

$$\oint_{C(s)} \left[P\frac{dx}{ds} + Q\frac{dy}{ds}\right] ds \quad = \quad \int\int_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

- **Proof B of Cauchy's Theorem 6.1:** We apply the above formula to the real and imaginary parts in Eq.(1) of Theorem 6.1:

$$\oint_{C(s)} \left[u\frac{dx}{ds} - v\frac{dy}{ds}\right] ds \quad = \quad -\int\int_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dxdy$$

$$\oint_{C(s)} \left[u\frac{dy}{ds} + v\frac{dx}{ds}\right] ds \quad = \quad \int\int_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy$$

Evidently, thanks to the C-R equations, the integrands of both the double integrals vanish identically. We have thus deduced Cauchy's Theorem as a corollary of Green's Theorem.

4. Corollaries and extensions

- We have restricted the region to **simply connected** ones. What about regions which have "holes" as shown in the Figure in the next page? To see how this is handled, consider the region R which is simply connected and contains within it the unit disk. We now consider the region D which is what remains in R after removing the interior of the unit disk. If $f(z)$ is given to be analytic in D, what can we say about contour integrals in D?
- The answer depends upon whether the contour **encircles** the unit circle U (which is one of the boundaries of D) or not. If C does not contain U within it, Cauchy's integral of f round C vanishes. In general, if C encloses the unit circle, the Cauchy integral will not vanish, as shown by the following example.
- **Example:** Consider the function, $f(z) = 1/z$ in the region, $D = |z| > 0$, the whole complex plane with the origin removed. This function is analytic at every point in D which is **not simply connected.** Clearly, it is not analytic at $z = 0$. Consider $C(\rho)$, a circle with centre $z = 0$ and radius $\rho > 0$. This lies entirely within D but "encircles" the origin. We find, **independently of the value of** ρ the result:

$$\begin{aligned} \oint_{C(\rho)} \frac{dz}{z} &= \int_0^{2\pi} (\frac{1}{\rho e^{i\theta}}) \rho e^{i\theta} i d\theta \\ &= 2\pi i \end{aligned}$$

4. Cauchy’s Theorem in “annular regions”

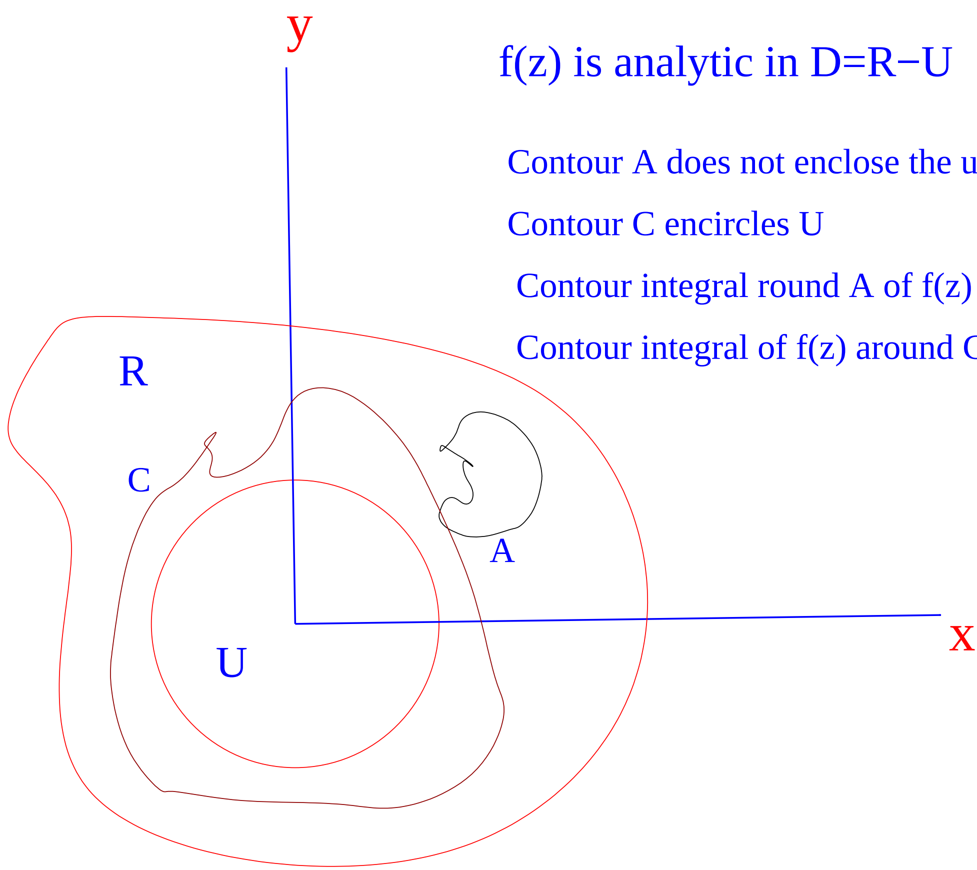
f(z) is analytic in D=R−U

Contour A does not enclose the unit disk U

Contour C encircles U

Contour integral round A of f(z) vanishes

Contour integral of f(z) around C may not



Cauchy’s Theorem in multiply connected regions

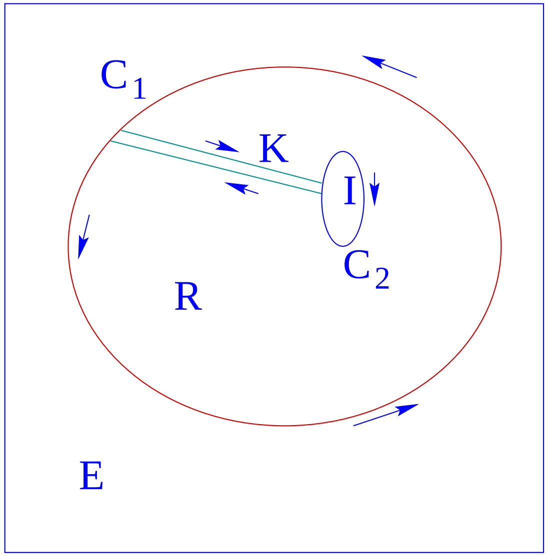
4. Corollaries and extensions-2

- The rule then is: if the contour C contained wholly within a multiply connected region R bounds a **simply connected** subregion, the Cauchy integral will vanish, but not **necessarily otherwise.** In the previous example, $f(z) = 1/z$ fails to be analytic at $z = 0$.
- If, for example, we had taken $f(z) = z$, which **is** analytic at the "excluded point", the integral would have vanished.
- We should also be careful about inferring analyticity of functions simply from the fact that **particular** contour integrals of them happen to vanish. Here is an example: consider $f(z) = \frac{1}{z^2}$ in $|z| > 0$. Clearly, this is **not** analytic at the origin. Let us evaluate its contour integral on the circle, $C(\rho)$:

$$
\begin{aligned}
\oint_{C(\rho)} \frac{dz}{z^2} &= \int_0^{2\pi} \frac{\rho i d\theta e^{i\theta}}{\rho^2 e^{2i\theta}} \\
&= \frac{i}{\rho} \int_0^{2\pi} (\cos\theta - i\sin\theta) d\theta \\
&= 0
\end{aligned}
$$

In this case, the vanishing of the contour integrals, indeed around any closed contour in the region excluding the origin does not allow us to conclude anything about the analyticity of the function **at the origin.**

4. Cauchy's Theorem in "annular regions"



D-domain of analyticity of f(z) =E+R

C 1-outer closed contour

C 2-inner closed contour

R-annular region between contours

I-interior of set bounded by inner contour,
f(z) is not necessarily analytic in I!

K-"cross cut"

- **Corollary:** If R is an **annular** region bounded by two simple closed contours C_1, C_2 and the function $f(z)$ is analytic on these contours and within R , irrespective of whether or not it is analytic in the region ("hole") enclosed by C_2, then:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

- All closed contour integrals are usually taken **counter-clockwise** so that the region enclosed always lies to the **left** of the tangent vector at any point on the contour.

4. Path independence of contour integrals

- **Theorem 6.4:** If $f(z)$ is analytic in a region R (not necessarily simply connected) and C_1, C_2 are two non-self-intersecting contours in R connecting the terminii z_1, z_2 (also in R) such that they enclose a simply connected subregion D where f is analytic everywhere, the following equation holds:

$$\int_{C_1(z_1;z_2)} f(z)dz = \int_{C_2(z_1;z_2)} f(z)dz$$

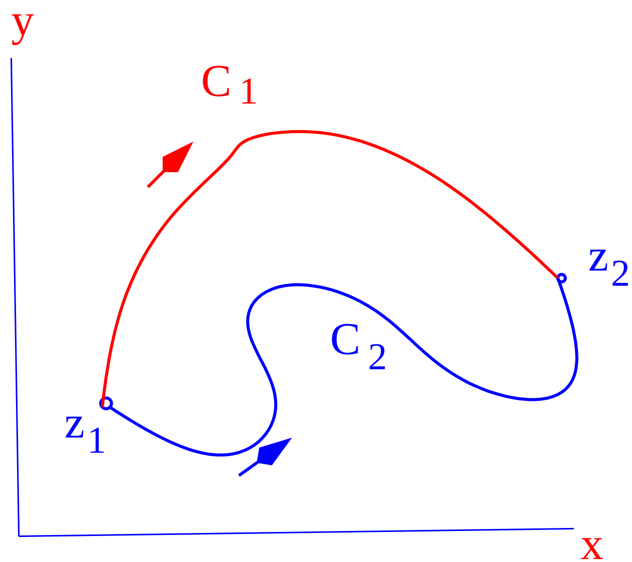
- **Proof:** Clearly, the simply connected region D is bounded by $C_1 + C_2$, the latter being traversed from z_2 to z_1; denote this by $-C_2$. Then, by Cauchy's theorem we have,

$$\oint_{C_1 - C_2} f(z)dz = 0 \rightarrow \text{the result.}$$

- This means that where a function $f(z)$ is analytic we may integrate it between any two points, choosing **any convenient contour** joining the points and get the **same result** subject to the important condition is that when we "deform the contour", we stay within the (simply connected) region of analyticity.

- Thus, for a fixed "lower limit" z_1, a contour integral of an analytic function in a simply connected region defines a **single-valued function of the upper limit** z_2!

4. Integrating analytic functions



Illustrating path independence

f(z) is analytic in whole region R

- This deduction from Cauchy's Theorem enables us to **define** analytic functions using definite integrals by integrating a given analytic function from some initial point z_0 to a point z along any suitable smooth curve joining them, lying entirely within the region of analyticity. If this region is **not simply connected**, the integral will exist but may not be single-valued.

5. Integrals of analytic functions

- In Lecture 3 I stated the real variable result: the "Fundamental theorem of integral calculus" of **Newton and Leibniz** which asserts that $\frac{d}{dx}(\int_a^x g(u)du) = g(x)$. The following theorem states that the line integral of an analytic function $f(z)$ defined in the previous Section, gives s single-valued function which is actually analytic, with derivative, $f(z)$.
- **Theorem 6.5:** If $f(z)$ is analytic in R and $F(z) = \int_a^z f(u)du$ taken on any convenient contour in the simply connected region R, then $F(z)$ is analytic in R, $F(a) = 0$, and satisfies,

$$\frac{dF}{dz} = f(z)$$

Proof: Clearly we have,

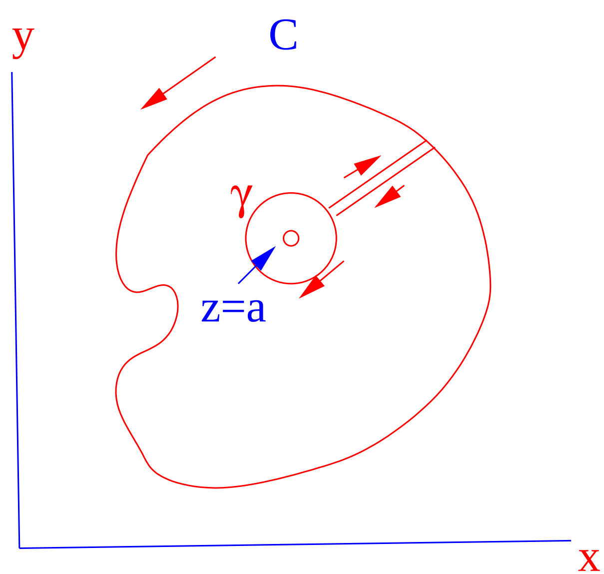
$$\begin{aligned}\frac{F(z+\Delta z) - F(z)}{\Delta z} &= \frac{\int_a^{z+\Delta z} f(u)du - \int_a^z f(u)du}{\Delta z} \\ &= \frac{\int_z^{z+\Delta z} f(u)du}{\Delta z} \\ &= f(z) + \int_z^{z+\Delta z} \frac{(f(u) - f(z))}{\Delta z} du\end{aligned}$$

As $\Delta z \to 0$, the integral on the RHS tends to zero (since $f(z)$ is analytic and therefore continuous), and the result follows.

6. Cauchy’s integral formula

- We now give a very powerful representation for analytic functions based on the properties of $1/z$ and Cauchy’s Theorem 6.1.
- **Theorem 6.6:** Let C be a simple closed contour within a region R where a function $f(z)$ is analytic and let a be an **interior point** of the simply connected sub-region bounded by C. Then the following **integral formula** holds:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} \tag{2}$$



7. Proof of Cauchy’s formula

- **Proof:** The function, $\frac{f(z)}{z-a}$ is an analytic function of z within C, except at a. Let us suppose that $\gamma(a;\rho)$ is a circle of arbitrarily small radius, ρ with centre a. From Cauchy’s theorem and the analyticity of the function in the annular region bounded by $C, \gamma(a;\rho)$, we have the identity,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{\gamma(a;\rho)} \frac{f(z)dz}{z-a}$$

From the fact that $f(z)$ is analytic, and therefore continuous, we may, given $\epsilon > 0$ arbitrarily small, assume ρ to be sufficiently small as to imply that, $|f(a)-f(z)| < \epsilon$ on $\gamma(a;\rho)$.Then,

$$\oint_C \frac{f(z)dz}{z-a} = f(a)\oint_{\gamma(a;\rho)} \frac{dz}{z-a} + \oint_{\gamma(a;\rho)} \frac{f(z)-f(a)}{z-a}dz$$

This first term, as I have previously shown (see Section 4), equals $2\pi i f(a)$. The second is in absolute value less than, $\frac{\epsilon}{\rho}2\pi\rho = 2\pi\epsilon$. Hence,

$$|\oint_C \frac{f(\zeta)d\zeta}{\zeta-a} - 2\pi i f(a)|| < 2\pi\epsilon$$

Since the LHS is independent of ϵ, it must vanish. We have therefore established Cauchy’s integral formula.

8. Consequences of Cauchy's formula

- Cauchy's formula Eq.(2) is very remarkable because it says that knowing the values of an analytic function $f(z)$ on the contour C enables us to calculate its value at **every interior point** a! In other words, **the values of an analytic function in the interior of a simply connected region bounded by a simple closed contour on which it is analytic are determined by its values on the boundary.**
- We change the notation slightly and use z in place of a and u in the itegrand, as it is after all a "dummy" integration variable. The next Theorem shows that an analytic function is "infinitely differentiable"!
- **Theorem 6.7:** Under the conditions when the Cauchy integral representation (ie Eq.(2)) is valid, $f(z)$ may be differentiated arbitrarily many times at an **interior point** z and all its derivatives are analytic in R. In fact, we have the formula $n = 1, 2, ..$:

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \oint_C \frac{f(u)du}{(u-z)^{n+1}} \qquad (3)$$

- **Proof:** This is by induction. Starting with $f(z) = \frac{1}{2\pi i} \oint \frac{f(u)du}{(u-z)}$ (namely, Eq.(2) re-written), differentiate both sides with respect to z. This can be easily justified, because the integrand is continuously differentiable with respect to z, which does **not** lie on C.