Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 5: Complex analysis: differentiable functions

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1. Differentiability

- So far we have looked at **continuous functions** of a complex variable defined in some region. Something amazing happens when we require that the functions of interest are more than merely continuous and are **differentiable.** What this concept means precisely is explained in the next definition.

- **Definition 5.1:** Let R be a compact, connected subset of the complex plane and let $f(z)$ be a (single-valued) function defined over it. The function $f(z)$ is said to be **differentiable** at a point $z_0 \in R$, if the limit,

$$\mathrm{Lim}_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad = \quad f'(z_0)$$

  exists as a complex number, $f'(z_0)$, where $z_0 + \Delta z$ is **any complex number** belonging to a sufficiently small neighbourhood of z_0 which is wholly contained in the **interior** of R. This number is called the **derivative or differential coefficient** of $f(z)$ at z_0. A function is said to be **analytic** in R if it has a **continuous derivative** $f'(z)$ at every **interior point** of R**.**

- **Equivalently**, at z_0, if we can find a number $f'(z_0)$ such that, the ratio,

$$\frac{|f(z_0 + \Delta z) - f(z_0) - f'(z_0)\Delta z|}{|\Delta z|} \to 0$$

  as $|\Delta z| \to 0$, then the function $f(z)$ is differentiable at z_0 with derivative, $f'(z_0)$.

1. Analytic functions: examples

- **Example 1:** Let $f(z) = c$, where c is a constant. Obviously, $f'(z) = 0$ everywhere. Thus constants are analytic at **every point** of the complex plane. We will see later that they are the **only such functions**!
  Example 2: Let $f(z) = kz$, where k is an arbitrary constant (real or complex). It is trivial to show that $f'(z) = k$ at all finite points of the complex plane.
  Example 3: Consider $f(z) = |z|$. This takes only real values. It is easily shown that $f'(0)$ does not exist. We will show later that it is not analytic anywhere.
  Eaxample 4: The class of powers, $f(z) = z^n$ for a positive integer n. We have already discussed the cases of $n = 0, 1$. For positive integral n, we see that,

$$\begin{aligned} f(z+\Delta z) - f(z) &= (z+\Delta z)^n - z^n \\ &= C_1^n z^{n-1}\Delta z + C_2^n z^{n-2}(\Delta z)^2 + .. + (\Delta z)^m \\ \mathrm{Lim}_{\Delta z \to 0} \frac{|f(z+\Delta z) - f(z) - nz^{n-1}\Delta z|}{|\Delta z|} &= 0 \end{aligned}$$

  Thus, in this case, $f'(z) = nz^{n-1}$, familar from ordinary calculus!
  Example 5: $f(z) = \frac{1}{z}$ defined for $|z| > 0$. Then, $f'(z) = -\frac{1}{z^2}$. The derivative does not exist at $z = 0$.

2. Complex differential calculus

- All the rules of ordinary calculus are applicable to analytic functions with (possibly) minor changes. The proof of the following Theorem is left to the Problem set.
- **Theorem 5.1:** Let f, g be analytic functions defined over a suitable region R. Let a, b be arbitrary complex constants. Then the following results hold.

$$\begin{aligned} (af + bg)' &= af' + bg' \\ (fg)' &= f'g + fg' \end{aligned}$$

- We often use the notation $f' = \frac{df}{dz}$. The next Theorem is the "Chain or function-of-a-function Rule". This is stated without proof.

  Theorem 5.2: Suppose $g(z)$ is analytic in R and takes it into a subset of S, the region of analyticity of f, then $h(z) = f(g(z))$ is defined in R. It is **analytic** in R with,

$$\frac{dh}{dz} = \left(\frac{df}{dg}\right)\frac{dg}{dz}$$

  Using these two rules alone we can show that all polynomials in z and ratios of polynomials (ie **rational functions**) are analytic wherever they are defined. Already, this gives us a lot of analytic functions to play with!

3. Cauchy-Riemann equations

- Let $f(z) = u + iv$ be an analytic function defined over an appropriate region R. The fact that it is analytic imposes strong conditions on the **real** functions $u(x, y), v(x, y)$, where $z = x + iy$. The key observation is that the derivative $f'(z_0)$ must exist, as a single limiting value, **from whatever direction** we approach the point z_0. This is clarified by the following analysis of the situation:
  Let us calculate the derivative at $z_0 \in R$ by setting $\Delta z = \Delta x; \Delta y = 0$. Then,

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \frac{[u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)]}{\Delta x} \end{aligned}$$

  Taking the limit and using definitions of **partial derivatives** of real functions, we find, at $z = z_0$,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (1)$$

  Setting $\Delta z = i\Delta y; \Delta x = 0$, we again take limits to obtain,

$$\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad (2)$$

3. Analyticity and the C-R equations

- Comparing Eqs.(1) and (2), we must **at least satisfy**, for the existence of the derivative, the **Cauchy-Riemann** equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (4)$$

- **Remark:** This shows that the real and imaginary parts of an **analytic function** of a complex variable $x + iy = z$ cannot be **arbitrary continuous or even differentiable functions** but that the four partial derivatives, $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ **must satisfy the C-R equations.**

- What is more, we derived the C-R equations by considering the "approach" to the limit along the **coordinate directions.** It can be shown that if these **necessary conditions** are satisfied, the limit is the same, **no matter what the direction of approach is!.**

- It can be proved that the C-R relations are not only **necessary** for **analyticity**, but are also **sufficient** conditions. This means that **if** we define a function $f(z) = u(x, y) + iv(x, y)$ where u, v have continuous first partial derivatives obeying C-R in R, then $f(z)$ is an analytic function in R and $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

4. Properties of analytic functions

- Analytic functions have many interesting and important properties. We are going to study these systematically, beginning with the simplest.
- **Theorem 4.1:** Let R be a region of the complex plane and let $f(z) = u(x,y) + iv(x,y)$ be an analytic function defined over it. The level curves of the real functions, u, v form **mutually orthogonal families.** This means that if x, y is a point of intersection of $u(x,y) = a; v(x,y) = b$, where a, b are real numbers, then the normals at these points to the two curves are perpendicular to each other.
- **Proof:** The unit normal to $u(x,y) = a$ at (x,y) is the 2-vector, $\mathbf{n}_u = (\frac{u_x}{u_x^2+u_y^2)^{1/2}}, \frac{u_x}{(u_x^2+u_y^2)^{1/2}})$. Similarly, the curve, $u(x,y) = b$ has the normal vector, $\mathbf{n}_v = (\frac{v_x}{(v_x^2+v_y^2)^{1/2}}, \frac{v_x}{(v_x^2+v_y^2)^{1/2}})$. Using the C-R equations, we see that $\mathbf{n}_u.\mathbf{n}_v = \mathbf{0}$. This proves the result at any point where $f'(z) \neq 0$. At those zeros all the derivatives vanish and we still have the result.
- **Theorem 4.2:** The real and imaginary parts, u, v of the analytic function $f(z); (f = u + iv)$ satisfy the two-dimensional Laplace's equation in R, the region of analyticity of $f(z)$.
- **Proof:** From C-R equations, $u_x = v_y; u_y = -v_x$. Then $u_{xx} = v_{yx}; u_{yy} = -v_{xy}$. Since $v_{xy} = v_{yx}$, we must have that $\nabla^2 u = u_{xx} + u_{yy} = 0$. Similarly, it follows that $\nabla^2 v = v_{xx} + v_{yy} = 0$.

4. Analytic functions: examples

- **Ex. 1:** Consider the function, $E(z) = e^x(\cos y + i \sin y) = (e^x \cos y, e^y \sin y)$. Then,

$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= e^x \sin y \\ v_y &= e^x \cos y \end{aligned}$$

  Evidently the C-R equations are satisfied at every point (x, y) in the finite plane. Thus $E(z)$ must be an **analytic function** of z (by the sufficiency condition of the C-R equations). We know that it reduces to e^x on the real axis and to $e(y)$ on the imaginary axis. It is called the **complex exponential function** and will henceforth be simply denoted by e^z.

- **Ex. 2:** The function, $f(z) = (1 + z^2)^{-1}$ is defined for all z except at $z = \pm i$. It can clearly be differentiated at every z where it is defined and $f'(z) = -2z(1 + z^2)^{-2}$. Since it is purely real on the real axis, its imaginary part $v(x, y)$ is a **solution of Laplace's equation which vanishes on the real axis and tends to zero at infinity.** It solves the problem of electrostatics where we ask for the electric potential in the upper half space due to a "line charge" placed at $y = 1$ and a perfect conductor at $y = 0$!

4. Analytic functions: more examples

- **Ex. 3:** Any pair of functions $u(x, y), v(x, y)$ which satisfy the C-R equations in a region are called **conjugate harmonic functions.** If $u(x, y)$ is a harmonic function, we can construct its **harmonic conjugate** $v(x, y)$. Let (x_0, y_0) be any point in the region where u is harmonic. Then,

$$v(x, y) = \int_{y_0}^{y} u_x(x, u) du + c$$

  Furthermore, $f = u + iv$ is an analytic function.

- **Ex. 4:** In 2-d Electrostatics, the electric field, $\mathbf{E} = (E_x, E_y) = -\nabla\phi$,where $\phi(x, y)$ is called the **electrostatic potential.** In a region **free of charges**, it must satisfy (since $\nabla.\mathbf{E} = 0$, there) the 2-d **Laplace's equation.** Thus $\phi(x, y)$ must be a **harmonic function.** The level curves of ϕ are called **equipotentials.** The function conjugate to ϕ is ψ. The analytic function, $f(z) = \phi + i\psi$ is called the **complex potential.** Evidently, the electric field in complex representation is just, $\mathcal{E} = -\frac{d\bar{f}}{dz}$. The electric field lines are therefore lines of constant ψ.

4. Analytic functions: examples (contd.)

- **Ex. 5:** It may come as a surprise, but many familiar functions are **not analytic**! Consider, $f(z) = \bar{z} = x - iy$. Obviously, the C-R equations cannot be satisfied, **even though** $x, -y$ are individually harmonic functions! Yet, as they are not conjugate, the function is not analytic at any point, although **differentiable infinitely many times** with respect to the real variables x, y! You can similarly prove that an arbitrary **real** harmonic function $u(x, y)$ cannot be an analytic function by itself unless it is a constant.

- **Ex. 6:** In 2-dimensional **inviscid hydrodynamics** the velocity vector $\mathbf{v} = (u, v)$ of an **incompressible fluid** must satisfy the **equation of continuity**: $\nabla.\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. If it is further assumed that the flow is **irrotational**, the velocity field is derivable from a **velocity potential**: thus, $\mathbf{v} = -\nabla\phi \rightarrow u = -\phi_x; v = -\phi_y$. On the other hand, the continuity equation (above) can be satisfied by introducing the **stream function**, such that $u = -\psi_y; v = \psi_x$. It follows that both ϕ and ψ must be **conjugate harmonic functions** in the flow domain, away from sources and sinks:

$$\phi_x = -u = \psi_y$$

$$\phi_y = -v = -\psi_x$$

Thus we have the important result that the **complex potential** $f(z) = \phi(x, y) + i\psi(x, y)$ is an analytic function. The complex representation of the velocity is $u + iv = -\overline{\frac{df}{dz}}$. If the flow is time-independent, the **pressure** in the fluid turns out to be proportional to $-|\frac{df}{dz}|^2$.

5. Analytic functions defined by series

- Consider the function defined by the **power series**:

$$w(z) \quad = \quad \Sigma_{n=0}^{\infty} w_n z^n$$

We suppose that for every n, the following inequalities are satisfied by the coefficients.

$$|w_n|^{1/n} \quad \leq \quad K$$

where $K > 0$ is some positive number. We may now apply Weierstrass' M-test and use the "comparison series", $\Sigma_{n=0}^{\infty} K^n |z|^n$ which surely converges for at least $|z| < \frac{1}{K}(= \rho)$. From this we infer that the series for $w(z)$ is absolutely and uniformly convergent in $0 \leq |z| < \rho$. We can only conclude from this that $w(z)$ is continuous.

- We "formally" differentiate each term of the series and obtain the differentiated infinite series,

$$w'(z) = \Sigma_{n=1}^{\infty} n w_n z^{n-1}$$

If we can prove that this is **also** uniformly convergent, it follows (from the previously stated theorems on such series) that it represents the derivative of $w(z)$. This is fortunately quite easy.

5. Analytic functions and series

- Consider, $|nw_n|^{1/n} \leq n^{1/n}K$. It is obvious that $n < 2^n$, for $n \geq 1$. Hence, $|nw_n|^{1/n} < 2K$. It follows that the series for $w'(z)$ converges uniformly and absolutely for at least $0 \leq |z| < \rho/2$. We have shown that $w(z)$ is an analytic function in at least this neighbourhood of $z = 0$! By being a bit more careful with the argument, it can be shown that the derivative exists in the disk, $|z| < \rho$ where $w(z)$ itself is defined.
- The above result has very important implications. Indeed, we shall soon see that there is a very strong connection between the most general analytic functions and absolutely convergent power series. The significance of the "disk" where such series converge will also be clarified by the occasional failure of analytic functions to be analytic. The simplest example concerns the theory of the "complex exponential function".
- **The complex exponential function:** Consider the power series,

$$f(z) \quad = \quad \Sigma_{n=0}^{\infty} \frac{z^n}{n!}$$

Applying the **Ratio test** we see that $|\frac{f_{n+1}}{f_n}| = |\frac{z}{n+1}|$. This obviously implies that the series converges absolutely and uniformly for any finite $|z|$.

5. The complex exponential function

- If we differentiate the series term-by-term, we simply recover the **same series**! the function defined by our series satisfies the equation,

$$\frac{df}{dz} = f(z)$$

for all finite z. Thus $f(z)$ is **analytic in the entire finite complex plane.** It belongs to a class of functions called **entire functions.**

- There is more: we see that $f(0) = 1$. Since the series is absolutely and uniformly convergent over the plane, we can use **Cauchy's series multiplication theorem**: thus consider the product of the two series,

$$\begin{aligned} f(z)f(u) &= \left[1 + \frac{z}{1!} + \frac{z^2}{2!} + ..\right]\left[1 + \frac{u}{1!} + \frac{u^2}{2!} + ..\right] \\ &= 1 + \frac{z+u}{1!} + \frac{1}{2!}(z^2 + \frac{2!}{1!1!}zu + u^2) + \\ &\quad \frac{1}{3!}(z^3 + \frac{3!}{2!1!}z^2u + \frac{3!}{1!2!}zu^2 + u^3) + .. \\ &= 1 + \frac{(z+u)}{1!} + \frac{(z+u)^2}{2!} + \frac{(z+u)^3}{3!} + .. \\ &= f(z+u) \end{aligned}$$

5. The complex exponential: contd.

- We have thus shown that our power series defines a function which satisfies the characteristic differential and functional equation satisfied by the usual (real) exponential function. Using this property, we see that,

$$\begin{aligned} f(z) &= f(x+iy) \\ &= f(x)f(iy) \end{aligned}$$

where $f(x)$ is the **same power series** evaluated on the **real variable**! Furthermore, $f(iy)f(-iy) = f(0) = 1$.

- Since y is real we can verify that $f(\bar{iy}) = f(-iy)$. Thus it coincides with the function we called $e(y)$. Thus, $f(x) = e^x; f(iy) = \cos y + i \sin y$. It follows that our power series defines the **same** function denoted by e^z.
- Using these properties, we can obtain every known property of the exponential function. It represents an **analytic extension** of the ordinary real exponential function, e^x, to the whole of the finite complex plane.

6. Contour integrals of analytic functions

- We now come to the most important result concerning analytic functions. To state it, we consider a region R where $f(z) = u + iv$ is **single-valued and analytic.** Let a, b be any two points of R and $C : x(s) + iy(s)$ is a simple, smooth contour (no self intersections!) joining these terminii, and lying entirely within R. As usual, s is the arc-length (or any other convenient parameter) of the contour C. Since an analytic function is necessarily continuous, we can define the contour integral,

$$I\left[f(z) : C(a,b)\right] = \int_{C(a,b)} f(z)dz$$
$$= \int_{s(a)}^{s(b)} (u\frac{dx}{ds} - v\frac{dy}{ds})ds + i\int_{s(a)}^{s(b)} (u\frac{dy}{ds} + v\frac{dx}{ds})ds$$

  where the real integrals of the real and imaginary parts are explicitly written out.

- At first sight, it seems that such a contour integral must not only depend upon the **integrand**, $f(z)$, and its upper and lower limits, a, b, but also on the **contour** C. Indeed, we know many examples of continuous integrands for which this is indeed the case.

- However, **Cauchy** proved a remarkable theorem which says that so long as C_1, C_2 are **arbitrary, smooth** contours lying entirely within R and so long as f and the terminii a, b are fixed, the numerical value of the contour integrals over the two contours of an **analytic function** $f(z)$ are **exactly the same!**