# Chennai Mathematical Institute
# B.Sc Physics

# Mathematical methods

## *Lecture 4: Complex analysis: functions of a complex variable*

A Thyagaraja

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# 1. Introduction: examples

- **Definition 4.1:** Let $R$ be a **connected region** of the complex plane. If for every $z \in R$ a **unique** complex number $w(z)$ is assigned, then $w(z)$ is called a **function of the complex variable** $z$ defined on the domain $R$.
- **Strictly speaking**, $w(z)$ is the **value** of the function at $z$. It is the **rule of assignment** which is called the function. It is traditional to denote a function by its values.
- **Examples:**
  1. If $R$ is the whole complex plane, the following are functions in the above sense: $w(z) = 1; w(z) = z; w(z) = \mathrm{Re}(z); w(z) = \mathrm{Im}(z); w(z) = \bar{z}$ Note that a real number is also regarded as a complex number!
  2. Functions need not be given by simple analytical formulae: thus, $w(z) = 1$ for $|z| > 1$ and $w(z) = -z$ for $|z| \leq 1$ is a perfectly well-defined function.
  3. Any polynomial in $z$ with complex coefficients: $w(z) = a_0 + a_1 z + a_2 z^2 + .. + a_n z^n$
  4. Often we may require $R$ to be a connected open set but **not simply connected**: thus consider $w(z) = 1/z; z \neq 0$. This function "blows up" near the origin but is well-defined everywhere else, including "infinity" (where it goes to zero). The region $|z| \neq 0$ is an open subset of the complex plane; it is obviously connected, but is not **simply connected** as paths connecting two points including the origin in their interior cannot be continuously deformed into each other without passing through the excluded point.

# 1. Introduction: continuous functions

- **Example 5.** The class of functions $w(z) = \frac{P_n(z)}{Q_m(z)}$ where $P_n(z), P_m(z)$ are polynomials of degree $n, m$ respectively; the region of definition of such **rational functions** is the whole complex plane except possibly the places (finite, as will be proved later) where the denominator vanishes.
  The most general types of functions of complex variables are not very useful in applications. We want them to be at least **continuous**. The following definition tells us how continuity is to be interpreted for functions of a complex variable.

- **Definition 4.2:** A function $f(z)$ defined in a region $R$ is said to be **continuous** at a point $z_0 \in R$ if and only if $\mathrm{Lim}_{z \to z_0} f(z) = f(z_0)$ when $z$ tends to $z_0$ **in any manner whatsoever**. Equivalently, given $\epsilon > 0$ arbitrarily small, if we can find a $\delta(\epsilon, z_0) > 0$ such that, $|f(z) - f(z_0)| < \epsilon$ for **all** $z$ satisfying, $|z - z_0| < \delta$, $f(z)$ is continuous at $z_0$. We say that a function is continuous in a set (which may be all of $R$) if it is continuous at **every point** of the set.

- It is important to remember that sequences in the complex plane "approach" their limit in many more ways than on the real axis. For a function to be **continuous**, it must firstly be **defined** at the point in question and every sequence $z_1, z_2, ..$ tending to the limit $z$ must be mapped into a convergent sequence, $f(z_1), f(z_2), ..$ with limit $f(z)$.

# 1. Continuous functions: examples

- **Examples:**
  1. $w(z) = c$, where $c$ is a constant is continuous, as is $w(z) = z$.
  2. The function $f(0) = 0; f(z) = 1/z, |z| > 0$ is defined everywhere and continuous at every point **except** at $z = 0$.
  3. $w(z) = |z|$ is continuous for all finite $z$.
  4. Consider the function, $w(z) = \frac{az+b}{cz+d}$ where $a, b, c, d$ are complex numbers such that $ad - bc \neq 0$. This is defined at all points of the complex plane (including $z = \infty$), except at $z = z_s; cz_s + d = 0$. It is continuous at every point where it is defined.
- We require functions of complex variables to have a **unique** value at every point they are defined. Later, we will encounter "multi-valued" functions which are really different **branches** of a function. They will require a separate discussion.
- Continuous functions of a complex variable defined over suitable regions possess many desirable properties, just like continuous functions (real or complex) of real variables. The next theorem is very important and should be learned. As usual, the proofs will be omitted as they depend upon quite subtle results of analysis.

# 2. Disks, limit points, open and closed sets

- In the complex plane, the analogue of **closed and bounded** intervals are **disks.** Recall that a disk centred at $c$ having radius $r$ is the set, $|z-c| \leq r$. I draw your attention to some basic geometric facts and nomenclature.

- **Definition 4.3:** If $S$ is **any set** of complex numbers, a **limit point** $z_L$ of the set is a point (which need not belong to the set) such that **every disk centred at** $z_L$ contains at least one point of $S$ **distinct from** $z_L$ itself.

- If $z_L$ is a **limit point** of a set $S$, we can find an infinite sequence of distinct points $z_1, z_2, ...$ belonging to $S$ such that $\mathrm{Lim}_{n\to\infty} z_n = z_L$. Recall that if $G$ is an **open set**, around **every point** in $G$ we can find at least one neighbourhood (circle with some radius $\rho > 0$ ) such that every point of it belongs to $G$.

- It is now obvious that every point on an open set is a **limit point** of the set in the above sense. However, not all **limit points** of an open set belong to it! For example, $|z| < 1$ is the "open unit disk". The points, $|z| = 1$ do **not** belong to it. However, as you can see by simple geometry that every such "boundary point" is a limit point of the unit disk.

- The **complement** of an open set is closed and vice versa. Unlike an open set, a **closed set** is one which **contains all its limit points.** A closed set with the property that every point of it is also a limit point is called a **perfect set.** These are results of topology which you can use, whenever required.

# 2. Disks, limit points, open and closed sets

- The preceding discussion motivates the following:
  **Definition 4.4:** A closed and bounded set of points of the complex plane is called a **compact set.** By definition, if $S$ is a compact set, it is a subset of a sufficiently large **disk** centred at the origin and it contains all its limit points.

- Every point of a compact set $S$ which has a neighbourhood contained entirely in $S$ is called an **interior point** of $S$. The set of all interior points of $S$ is called the **interior** of the set. It is necessarily open. All the points of $S$ which do not belong to its interior are called its **boundary points.**

  **Examples:**

  1. The unit disk, $U = (z : |z| \leq 1)$ is compact (in fact it is a perfect set).
  2. The entire complex plane $C$ is closed, but is not bounded, so it cannot be compact.
  3. A finite union of compact sets is compact.
  4. A triangle bounded by three line segements is compact (we include points enclosed by the perimeter and those belonging to the perimeter).
  5. A **polygon** is a set bounded by a finite number of straight-line segements called **sides** joining its **corner points.** The union of the sides of a polygon is a non-self intersecting closed curve (**perimeter**) which divides the plane (according to Jordan's theorem) into an unbounded open set and a bounded, simply connected open set **interior.** The **polygon** is the union of the perimeter and the interior. It is a perfect compact set.

# 3. Properties of continuous functions

- **Theorem 4.1:** Let $R$ be a **compact set** of the complex plane and let $f(z)$ be a continuous function of the complex variable defined over $R$. Then,
  1. There exists a finite constant such $K$ that, $|f(z)| \leq K$; in other words, the function must be bounded **uniformly** over $R$.
  2. $f(z)$ is **uniformly continuous** in $R$: this means, given $\epsilon > 0$, we can find a $\delta$ **which depends only on** $\epsilon$ such that $|f(z) - f(u)| < \epsilon$, whenever $|z - u| < \delta$ for arbitrary $z, u \in R$.
  3. If $a$ is any constant, $af(z)$ is continuous in $R$.
  4. If $g$ is another continuous function defined over $R$, the functions, $f + g; fg$ are also continuous.
  5. If $g$ does not vanish in $R$ in addition to being continuous, $f/g$ is continuous in $R$.
  6. If a function $f(z)$ is defined on the finite complex plane, we can discuss its continuity at the "point at infinity" by considering the the continuity near $u = 0$ of the function $f(\frac{1}{u})$.
  7. A function $f(z)$ is continuous in $R$ if and only if its real and imaginary parts are continuous functions of the real variables $x, y$ where $z = x + iy$. In particular, if $f(z)$ is continuous, $|f(z)|$ is also continuous. The converse is not true in general.
- This theorem is the complex variable analogue of Theorem 3.3. It is a rich source of continuous functions of complex variables. Note that there must be at least one point $z = z_M$ such that $|f(z_M)| = K$. In other words the function must **attain** its bound at some point in $R$.

# 3. Properties of continuous functions

- **Proposition 4.1:** 1. All polynomials in $z$ are continuous at every point in the finite complex plane.
  2. The function defined for every finite $z = x + iy$ by,

$$e(z) \quad = \quad e^x \left[\cos y + i \sin y\right]$$

  generalizes the **real exponential function**, $e^t$ and the **Euler-De Moivre function** $e(\theta)$ ($\theta$ real) defined previously. It is continuous at every $z$.
  3. The function $g(z) = 1/z$ defined for $0 < |z| < \infty$ is continuous at every point. We may consider its behaviour at "infinity" by setting $u = 1/z$ and consider the function, $g(u) = u$. Strictly the original function is not defined at $u = 0$; if we define it to be zero, we see that $g(u)$ is continuous at $u = 0$. Hence $g(z)$ is continuous at infinity.
  4. If $f(z)$ is a **rational function**, it is continuous at every finite point except at the **zeros** of the denominator. If the substitution $u = 1/z$ is made and the function is suitably defined at $u = 0$, we may examine its continuity at infinity. All polynomials are unbounded near infinity and so cannot be continuous there. Hence, a rational function can be continuous at infinity only if the degree of the numerator does not exceed that of the denominator polynomial.

# 4. Sequences and series of functions

- We next consider some well-known limit processes with functions of a complex variable $z$ defined over a compact region $R$. If you don't like abstract thinking, you may take the compact region of definition to be some disk of finite radius $\rho$ and centre $c$ in the complex plane. The following definition describes **uniform convergence of a sequence of functions.**
- **Definition 4.5:** A sequence of functions $[u_n(z)]$ defined over $R$ is said to be **uniformly convergent** in $R$ to a limit function $f(z)$, if for any given $\epsilon > 0$, we can find an $N(\epsilon)$ such that $|f(z) - u_n(z)| < \epsilon$ for all $n > N(\epsilon)$ **and for all** $z \in R$.
- **Theorem 4.2:** The limit of a uniformly convergent sequence of continuous functions defined over a compact set $R$ is a continuous function. If a series of continuous functions converges uniformly in $R$, its sum is continuous. (To be learned and used without proof).
- **Theorem 4.3 "Weierstrass' M-test for absolute and uniform convergence":** Let $F(z) = \Sigma_{n=1}^{\infty} u_n(z)$ be a series of continuous functions $u_n(z)$ defined over a closed and bounded (ie compact) region $R$. Suppose further that $|u_n(z)| < M_n$ where $M_n$ are positive constants for $z \in R$ and each $n$. Then, the series is **absolutely and uniformly convergent** in $R$ if the **dominant series of positive terms,** $\Sigma M_n$ **converges.** (To be used without proof).

# 4. Sequences and series: applications

1. The **Geometric series**

$$G(z) \quad = \quad \Sigma_{n=0}^{\infty} z^n$$

converges absolutely and uniformly in $0 \leq |z| \leq r < 1$ and therefore defines a continuous function. Obviously, $|z| \leq r \rightarrow |z^n| \leq r^n$. Using Weierstrass' M-test and the fact that the real-valued geometric series converges for $r < 1$, we obtain the result. Of course, we already know that the series converges to $G(z) = \frac{1}{1-z}$ in the set concerned.

2. The **exponential function** $E(z)$ defined by the power series for every finite $z$ by the **power series**

$$E(z) \quad = \quad 1 + \Sigma_{n=1}^{\infty} \frac{z^n}{n!}$$

Converges absolutely and uniformly in any disk with centre $z = 0$ and radius $\rho$. It therefore represents a continuous function in any compact subset of the complex plane. In any such disk, $|z| \leq \rho$, using the M-test, we see that the **dominant series** is simply the **real exponential series**, $e^{\rho} \geq |E(z)|$. Theorem 4.3 gives the required result. You may use this to prove that the series for $E(-z), E(iz), E(-iz)$ are all uniformly and absolutely convergent convergent in such disks, and therefore represent continuous functions in the finite complex plane.

# 5. Contours in the complex plane

- You already know that **continuous functions** of a real variable can integrated over any finite interval. We wish to study the analogue of such integrals for continuous functions of a **complex variable.**
- **Definition 4.6:** A **rectifiable contour** in the complex plane is a **complex function** $z(t)$ of a **real variable** $t$ defined over a bounded, closed interval $t \in [a, b]$, and which has a finite total length. The end points or terminii are $z(a)$ and $z(b)$ respectively.
- Remarks: The curve is called **piece-wise smooth** if it has a continuous tangent except for a **finite** number of **corner points.** The curve is called **closed** if the terminii coincide. It is called **simple** if it has no self-intersections.
- **Examples:**
  1. The perimeter of a triangle or a regular polygon is a piece-wise smooth, simple, closed, rectifiable contour.
  2. The circumference of any circle or ellipse is a smooth, closed, rectifiable contour.
  3. The semi-circle formed by the upper half of the unit circle and the real axis is a simple, closed, recifiable, piece-wise smooth contour.
  4. The straight line segments joining $z = 0; z = 1$ and the latter to $z = i$ form a **piece-wise linear, non-closed contour.** The terminii are $z = 0, z = i$ respectively.

# 5. Geometry of contours

- I remind you of some simple properties of curves in the plane. If we are given two smooth real functions of a variable $t \in [0,1], x(t), y(t),$ a smooth plane curve $K$ is **parametrically** defined by the point $(x(t), y(t))$.
- The **tangent** to $K$ at a point $t$ is obtained from the 2-vector: $(\frac{dx}{dt}, \frac{dy}{dt})$ This gives the **direction** of the tangent at $(x(t), y(t))$. For smooth curves, this is a **continuous** function of the **curve parameter** $t$. The **unit tangent vector** $\mathbf{t}$ is given by the formula:

$$\mathbf{t} \quad = \quad \frac{dx}{ds}\mathbf{e}_x + \frac{dy}{ds}\mathbf{e}_y$$

where $\mathbf{e}_x, \mathbf{e}_y$ are unit vectors along the $x, y$ axes and $ds = \left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2\right]^{1/2} dt,$ is the arc-length differential (distance between the points, $(x(t), y(t))$ and $(x(t+dt), y(t+dt))$ on the curve).

- The unit **normal vector** to the curve $K$ at a point $t$, denoted by $\mathbf{n}$ is orthogonal to $\mathbf{t}$ and has the formula:

$$\mathbf{n} \quad = \quad -\frac{dy}{ds}\mathbf{e}_x + \frac{dx}{ds}\mathbf{e}_y$$

# 5. Geometry of contours: contd.

- Remarks: A **smooth curve** is always **rectifiable.** Thus, its **length** $s(t_1, t_2)$ between any two points with parameters $t_1, t_2$ is given by the formula,

$$s(t_1,t_2) = \int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \left[ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 \right]^{1/2} dt$$

- At a **corner point** of the curve, the tangent vector can have two distinct values, depending upon which side of the point we approach it. At all other **ordinary points** we find that $\mathbf{t}$ has a finite derivative. Since $\mathbf{t}.\mathbf{t} = 1$ and $\mathbf{t}.\mathbf{n} = 0$, we may write,

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}$$
$$\kappa = -\frac{dy}{ds}\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2y}{ds^2}$$

where $\kappa$ is the **curvature** of $K$ at the point $t$. If $\theta$ is the angle made by $\mathbf{t}$ with the $x$-axis, $\frac{d\theta}{ds} = \kappa$. The curvature is infinite at corner points. Since $s(0,t)$ is a smooth, monotonic increasing function of $t$, we may use $s$ as the “curve parameter”, if we wish!

# 5. Contours: complex representation

- We can translate the results into complex notation: $\frac{ds}{dt} = |\frac{dz}{dt}|$

$$
\begin{aligned}
s(t_1, t_2) &= \int_{t_1}^{t_2} |dz| \\
&= \int_{t_1}^{t_2} |\frac{dz}{dt}| dt
\end{aligned}
$$

- The tangent and normal unit vectors at a point (parametrized by $s$) on the curve are represented by **complex numbers on the unit circle:**

$$
\begin{aligned}
\mathbf{t} &= \frac{dz}{dt} / |\frac{dz}{dt}| = \frac{dz}{ds} \\
\mathbf{n} &= i\frac{dz}{dt} / |\frac{dz}{dt}| = i\frac{dz}{ds}
\end{aligned}
$$

Since, $\frac{dz}{ds}\frac{d\bar{z}}{ds} = 1$ and $\frac{dz}{ds} = \cos\theta(s) + i\sin\theta(s)$, where $\theta(s)$ is the angle made by $\mathbf{t}$ with the real axis, $\frac{d^2z}{ds^2} = \kappa i \frac{dz}{ds}$, which implies that

$$
\kappa = \frac{d\theta}{ds}
$$

# 5. Contour integrals

- **Definition 4.7:** Let $K$ be a **rectifiable, piece-wise smooth, simple contour** with the equation, $z(t) = x(t) + iy(t); t \in [0,1]$ and terminii $z(0) = a; z(1) = b$ (not necessarily closed). Let $R$ be a **compact region** containing $K$ in its interior. Let $f(z) = u + iv$ be a **continuous** function defined over $R$. Then, the **contour integral** of $f(z)$ taken between two points, $z(t_1), z(t_2)$ on $K$ is defined by,

$$\begin{aligned}\int_{K(z_1,z_2)} f(z)dz &= \int_{t_1}^{t_2} f(z(t))\frac{dz}{dt}dt \\ &= \int_{t_1}^{t_2} [u(x(t),y(t)) + iv(x(t),y(t))]\,(\frac{dx}{dt} + i\frac{dy}{dt}) \\ &= \int_{t_1}^{t_2}(u\frac{dx}{dt} - v\frac{dy}{dt})dt + i\int_{t_1}^{t_2}(u\frac{dy}{dt} + v\frac{dx}{dt})dt\end{aligned}$$

  The contour can also be parametrized by the arc length $s(t)$ along it measured from $a$.

- Note that the **complex contour integral** is a complex **number** equivalent to two real **line integrals** you are familiar with from advanced calculus. These can always be proved to exist using **Theorem 3.5.** Note that the value of the integral depends not only upon the integrand and the limits, but also on the **curve** $K$. The following theorem states some basic properties of the contour integral.

# 5. Contour integrals: key properties

- **Theorem 4.4:** Let $K$ be a **rectifiable, piece-wise smooth, simple contour** with the equation, $z(t) = x(t) + iy(t); t \in [0,1]$ and terminii $z(0) = a; z(1) = b$ (not necessarily closed). Let $R$ be a **compact region** containing $K$ in its interior. Let $f(z) = u + iv$ be a **continuous** function defined over $R$. Let $0 \leq t_1 < t_2 < t_3 \leq 1$, where $z_2 = z(t_2)$ is a **corner or ordinary point** between the **ordinary points**, $z_1 = z(t_1), z_3 = z(t_3)$. Then, if $c$ is any complex constant and $g(z)$ is any continuous function defined over $R$,

$$\int_{K(z_1,z_3)} f dz = \int_{K(z_1,z_2)} f dz + \int_{K(z_2,z_3)} f dz \quad (1)$$

$$\int_{K(z_1,z_3)} f dz = -\int_{K(z_3,z_1)} f dz \quad (2)$$

$$\int_{K(z_1,z_3)} cf(z) dz = c\int_{K(z_1,z_3)} f dz \quad (3)$$

$$\int_{K(z_1,z_3)} (f(z) + g(z)) dz = \int_{K(z_1,z_3)} f dz + \int_{K(z_1,z_3)} g dz \quad (4)$$

$$\left|\int_{K(z_1,z_3)} f dz\right| \leq \int_{K(z_1,z_3)} |f|\left|\frac{dz}{dt}\right| dt \quad (5)$$

# 5. Contour integrals: proof of Theorem 4.4

- **Proof:** These results are simple consequences of well-known properties of real integrals and Definition 4.7. It is important to understand that in contour integration, the **sense** in which the contour is described (ie from one point to another in the direction of increasing curve parameter) is important.
- If a contour is **closed** we can describe it **positively** in the sense of **counter-clockwise** direction. This means that the **interior** of the region bounded by the contour lies to the **left** of the tangent vector as the curve is described. The opposite direction is **negative**.
- Part 5 follows from the corresponding inequality for sums and taking of suitable limits (this will appear in the Problem set as will several illustrative examples). It is a very important result and will be used frequently, as will the theorem itself.
- The definitions and theorems given thus far deal with contours which are defined as rectifiable curves contained within compact sets. Using limits we can extend many results, under suitable conditions, to contours which “go to infinity”. This will be done later, when required.

# 6. Contour integrals of series

- We have seen in Theorem 3.8 that integrating a uniformly convergent series of real functions over finite limits leads to simple results about the interchanging of limits (ie "integral of the sum equals the sum of integrals"). We can now state a corresponding theorem (without proof!) relating to **contour integration** dealing with absolutely and uniformly convergent series of functions of a complex variable defined over a compact set relative to rectifiable contours.

- **Theorem 4.5:** Let $R, K$ be defined as in Theorem 4.4. Let $s(z) = \Sigma_{n=1}^{\infty} u_n(z)$ be an absolutely and uniformly convergent series of continuous functions $u_n(z)$ defined over $R$. Then,

$$\int_{K(z_1,z_2)} [\mathrm{Lim}_{n\to\infty} \Sigma_{k=1}^{n} u_k(z)]\, dz \quad = \quad \mathrm{Lim}_{n\to\infty} (\Sigma_{k=1}^{n} \int_{K(z_1,z_2)} u_k(z) dz)$$

where the series on the right converges absolutely and uniformly for any pair of points $z_1, z_2$ on $K$.