Chennai Mathematical Institute
B.Sc Physics

Mathematical methods
Lecture 3: Complex functions

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1. Introduction to complex functions

- As we discussed in Lecture 1, a **function** is a rule which assigns to each element in a certain set called the **domain of definition of the function**, a **unique** element of another set **called the range of the function.** This notion is so fundamental that with the concept of a **set,** it forms the entire basis of mathematics.

- We shall find that functions involving complex numbers are of two kinds. If we have a rule which assigns to each integer n from any finite or infinite set of integers K a definite complex number z_n, such a function is called a **complex sequence.** We studied such “functions of an integer variable” in Lecture 2. Let us next consider what are called **complex functions of a real variable.**

- **Definition 3.1:** Let $t \in (a, b)$ be a real variable such that $a < t < b$, where $a < b$ are arbitrary real numbers. If for each t, there is assigned a unique complex number $f(t)$, we say that $f(t)$ is a **complex function of the real variable** t defined over the **open interval** (a, b).

- This definition says that to assign $f(t)$ means to define two **real** functions $u(t), v(t)$ such that $f(t) = u(t) + iv(t)$. Of course, for each t, there can only be **one** $u(t)$ and corresponding $v(t)$. This is requiring that functions mentioned are **single-valued.** Later we will allow for some other possibilities.

- We call t the **independent variable.** Strictly, $f(t)$ is the **value** of the function, but it is traditional to call it the function and set $z = f(t)$. We often call z the **dependent variable.**

1. Examples of complex functions

1. Consider, for real t, $-\infty < t < \infty$, $z = f(t) = c$, where c is a complex **number or constant.** This is called a **constant function.** This shows that many values of the **dependent variable** can be assigned to the same function value.

2. Functions need not be given by analytical formulae: using the same independent variable t defined above, let us define a function $\phi(t)$ in words: consider the decimal expansion of t. Define $u(t)$ to be that real number obtained using the **odd** digits of its decimal expansion after the decimal point. Let $v(t)$ be defined using only the even digits. Define $\phi(t) = u(t) + iv(t)$. Since the decimal expansion of any real number is unique with usual conventions about recurring 9's, this is also a unique assignment.

3. Let $0 \leq \theta \leq 2\pi$. This is called the **closed interval**, $[0, 2\pi]$. We define $e(\theta) = \cos\theta + i\sin\theta$. We have already met this expression in Lecture 1. It satisfies $|e(\theta)| = 1$ and the remarkable **functional equation**,

$$e(\theta_1 + \theta_2) = e(\theta_1)e(\theta_2)$$

A closely related function defined over $t \in (-\infty, \infty)$:

$$w(t) \quad = \quad (\frac{1-t^2}{1+t^2}) + i(\frac{2t}{1+t^2})$$

1. Functions of a complex variable

- It is possible to have functions which assign complex values to complex numbers.
- **Definition 3.2:** Let D be a subset of the complex plane and z is any point in D. If to each z is assigned a **unique** complex number, $w = f(z)$, we say that $f(z)$ is a function of a complex variable. In principle, we can also have **real functions** of a complex variable.
- To see what this definition entails, we consider several illustrative examples. Consider the function,

$$\mathbf{I}(z) \quad = \quad z$$

for any complex number z. This function is called the **identity** function and is defined for every complex number. A less trivial example is provided by the function,

$$w(z) \quad = \quad z^2$$

We may write, $z = x + iy; w = u + iv$. Using the rules of complex algebra, we find the relations,

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}$$

1. Picturing complex functions

- We may think of a **complex function of a real variable** as simple parametric representation of a curve, $x = u(t); y = v(t)$ in the complex plane. Thus, $e(\theta)$ describes the circumference of the unit circle anti-clockwise as θ varies from 0 to 2π. Such curves can be quite complicated with self-intersections and even separate pieces. The geometrical picture helps greatly in understanding the properties of the function.
- A (complex) function of a **complex variable** requires more mental gymnastics! Let us imagine **two separate complex planes**, the $z = x + iy$ plane and the $w = u + iv$ plane. If z belongs to D (a subset of the z plane, for example, the whole of it), for each z we have a point $u(x, y) + iv(x, y)$ in the w plane. The set of all such function values is the **range** R of the function and it is a sub-set of the w plane. The function is said to **map** the **domain** D into the **range** R.
- It is sometimes useful to think of a function of a complex variable as mapping the complex plane (or a sub-set) **into itself.** In either case, $w(z)$ is called the **image point** of the **object point** z. The language is drawn from optics. You should be aware that people also say that z is the **argument** of the function. One must be careful not to confuse this with the **other use** of this word to mean the **amplitude/phase** of a complex number!

2. Limits of functions and continuity

- **Continuity** is the single most important concept in function theory. We will now do a quick **review** of properties of functions of a real variable. The basic definitions for **real** functions: if $x \in (-\infty, \infty)$ (ie the **domain** is the real line) and $f(x)$ is a real function of x (as in ordinary calculus), we find its **value** at any point $x_0 \in (-\infty, \infty)$ to be $f(x_0)$. Now if a sequence of points $y_1, y_2, ..$ is considered, and we are told that $\mathrm{Lim}_{n\to\infty} y_n = x_0$, what can we say about the **image sequence** $f(y_1), f(y_2), ..$? In **general, absolutely nothing!**
- If, however, we **know** that the function is **continuous** at x_0, both our geometrical intuition and the standard calculus definition tell us that,

$$\begin{aligned} \mathrm{Lim}_{n\to\infty} f(y_n) &= f(\mathrm{Lim}_{n\to\infty} y_n) \\ &= f(x_0) \end{aligned}$$

  This says that if a real function $f(x)$ **is continuous at a point** x_0 in its domain of definition, **every** sequence of points in its domain which converges to x_0 is mapped by the function to a **convergent sequence with limit** $f(x_0)$ in its range. In fact, the **converse defines continuity!**
- Thus, a function is **continuous at a point if and only if every convergent sequence in its domain is mapped to a convergent sequence in its range; furthermore, the image sequence must have as its limit the value of the function at the point of continuity.**

2. More about continuity

- Using limits of functions to test for continuity is easy. However, to show that a **given** function is continuous, the definition asks us to test **every possible sequence** converging to the point of continuity! This is very inconvenient.
- In practice, the following "$\epsilon - \delta$" definition is the most powerful one:
- **Definition 3.3:** A real function $f(x)$ defined in some interval, (a, b) is continuous at $x_0 \in (a, b)$ **if and only if,** given $\epsilon > 0$ arbitrarily small, we can find a number δ (in general dependent upon ϵ, x_0) such that for **all** $x \in (x_0 - \delta, x_0 + \delta); f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$.
- In otherwords, if the function is continuous if and only if we can be able to find a δ**-neighbourhood** of x_0 which is mapped entirely **into** any given ϵ**-neighbourhood** of $f(x_0)$.
- Applying this requires patience and practice! In this course, this level of rigour will not be expected, but you **must** clearly understand the definition and know how it is used.
- **Example 1:** The function $f(x) = c$, where c is any real constant is defined over the whole real axis. It is obviously continuous, at every point in its domain, according to both definitions.
- **Example 2:** Let $f(x) = x$; you can see that this too is clearly continuous everywhere!

2. Continuous functions: examples

- **Example 3:** The function defined by the rules: $f(x) = 1; x > 0$, $f(x) = -1; x < 0$ and $f(0) = 0$ is continuous at every point **except** $x = 0$. As x tends to zero through **positive values**, the function tends to 1. When x approaches zero through negative values it tends to -1. If x tends to zero in an arbitrary manner, the function does not approach any definite limit. Hence it is discontinuous at 0 (draw the function!).

- **Example 4:** If $f(x) = \frac{1}{x}; x \neq 0$ and $f(0) = 0$, again it is easy to see that f is discontinuous at zero.

- **Example 5:** $f(x) = \sin(1/x); x =\neq 0$ and $f(0) = 0$ is discontinuous at $x = 0$. Thus consider the sequence, $x_n = \frac{2}{\pi(4n+1)}; n = 1, 2, \ldots\ f(x_n) = 1$. Hence, as x tends to zero through these values, it cannot equal $f(0) = 0$. However, the function $g(x) = x\sin(1/x)$ **is** continuous at $x = 0$.

- **Example 6:** A function $f(x)$ defined everywhere on the real line has the **Lipschitz property** at a point x_0 if there exists a constant $K > 0$ such that, $|f(x_0) - f(x)| < K|x - x_0|$. It is easy to show that if a function has the Lipschitz property at x_0, it must be continuous there. The converse is not true in general, as there are continuous functions which are not Lipschitz at a point of continuity.

- **Example 7:**. The real function $f(x) = |x|$ is Lipschitz at $x = 0$ and hence continuous. The function $g(x) = |x|^{1/2}$ is continuous but not Lipschitz at $x = 0$.

2. Properties of real continuous functions

- **Theorem 3.1:** Let $f(x)$ be a real function defined on an interval $D = (a, b)$. If $a < x_0 < b$ is a point of continuity of the function, the function is **bounded in a neighbourhood of x_0 included in D.**
  Proof: Given $\epsilon > 0$, we know that a δ can be found such that for **all** $a < x_0 - \delta, < x < x_0 + \delta < b$, $|f(x) - f(x_0)| < \epsilon$. Hence, $|f(x)| < |f(x_0)| + \epsilon$, by triangle inequality.

- We are usually interested in functions which are continuous not merely at a single point but in an interval. These come in four types: $[a, b]$ (**closed interval** including both end points), (a, b) (**open** interval excluding both endpoints) and semi-closed intervals, $[a, b)$, $(a, b]$. A function continuous at every point in an interval is said to be continuous over the interval. Particularly important types of interval are **compact** intervals which are both closed and **bounded**. This means that their end points are finite numbers. The following theorem may be assumed and used without proof.

- **Theorem 3.2:** A real function which is defined and continuous over a compact interval $[a, b]$ is bounded in the interval; namely, two finite numbers m, M ("minimum" and "maximum") exist such that $m \leq f(x) \leq M$. There are points of the interval where $f(x)$ actually equals m and M (ie "attains its bounds"). If $v \in [m, M]$ is any number, there exists a y in the interval such that $f(y) = v$ ("intermediate value theorem").

2. Continuous functions: algebra

- **Theorem 3.3:** Let $f(x), g(x)$ be a real functions which are defined and continuous over the interval $D = [a, b]$. Then, if α, β are arbitrary real numbers, the following results hold:

  1. The functions $s(x) = \alpha f(x) + \beta g(x); p(x) = f(x)g(x)$ are continuous over D.
  2. The function $h(x) = |f(x)|$ is continuous over D.
  3. The functions $f_+(x) = \mathrm{Max}[f(x), 0] \geq 0, f_-(x) = -\mathrm{Min}[f(x), 0] \leq 0$ are continuous over D and $h(x) = f_+(x) + f_-(x); f(x) = f_+(x) - f_+(x)$.
  4. If $g(x) \neq 0$ at any point, $q(x) = \frac{f(x)}{g(x)}$ is defined and continuous over D.
  5. If $f(x)$ maps D onto $R_f = [m, M]$ and $v(x)$ is defined and continuous over R_f, the "function-of a function", $w(x)$ defined by $w(x) = f(g(x))$ is defined and continuous over D.

  Proof: This is surprisingly simple to prove and left as an exercise! (Hint: use the "algebra of limits" and apply the sequential definition of continuity at a point.)

- We can thus "manufacture" continuous functions from just constants and $f(x) = x$ in profusion. For example, positive integral powers of a real variable x are all continuous functions over every compact interval, as are positive square and n-th roots of $|x|$.

- "Polynomials" are functions of the form, $P(x) = a_0 + a_1 x + .. + a_n x^n$ for some integer $n \geq 0$ and the a's are real constants. All such polynomials are continuous over every compact interval.

2. Real functions: differentiability

- **Definition 3.4:** Let $f(x)$ be a real-valed function of a real variable, defined and continuous over (a, b). It is **differentiable** at a point x_0 if the limit,

$$\mathrm{Lim}_{\epsilon \to 0} \frac{f(x_0 + \epsilon) - f(x_0)}{\epsilon} \quad = \quad f'(x_0) = \frac{df}{dx}$$

  exists as a finite number. The number $f'(x_0)$ is called the **derivative or differential coefficient** of f at $x = x_0$.

- The function is said to be differentiable over the interval if it is differentiable at every point of the interval. It is said to be **continuously differentiable** if the derivative, $f'(x)$ is a continuous function. If the function is differentiable except at a finite number of points, it is called **piece-wise** differentiable.

- I will assume from now on all the results from both Differential and Integral calculus and vector analysis. The following theorems are stated without proof.

- **Theorem 3.4:** If a function is differentiable over a compact interval, it is continuous over that interval.

- The converse is not always true. There are continuous functions which have no finite derivative anywhere! The stated results help us to create large and useful classes of functions from polynomials and algebraic operations.

2. Integrability of continuous functions

- **Theorem 3.5: (Cauchy)** The integral of a continuous function $f(x)$ over an interval $[a, b]$ is the limit of the sum,

$$\mathrm{Lim}_{n\to\infty}\Sigma_{r=0}^{n} f(a + \frac{r(b-a)}{n})(\frac{b-a}{n}) \quad = \quad \int_a^b f(x)dx$$

- As you know this has a well-known interpretation in terms of the area under the curve $y = f(x)$. The following result (due to **Newton and Leibniz**) is very important.

- **Theorem 3.6:** If $f(x)$ is continuously differentiable over $[a, b]$ with derivative $f'(x)$, then,

$$f(x) - f(a) \quad = \quad \int_a^x f'(t)dt$$

  Thus, integration and differentiation are inverse processes.

- A final item in our review of the properties of functions of real variables, we will consider some important concepts relating to sequences and series of functions. I will state the key definitions and theorems with which you should be familiar (proofs will not be required although worth reading up!)

2. Sequences and series of real functions

- **Definition 3.5:** An infinite sequence of **real functions** $[f_n(x)]$ defined on (a, b) is said to tend to a function $g(x)$ at every point x of the interval, if the following holds:

$$\mathrm{Lim}_{n\to\infty} f_n(x) \quad = \quad g(x)$$

Example 1: The sequence $[x^n]$ where $-\infty < x < \infty$. For $|x| < 1$, the limit function exists, and, $g(x) = 0$; If $x = 1$, $g(x) = 1$. If $x > 1$, the sequence is unbounded (ie does not converge) and the limit does not exist. If $x = -1$, again, the sequence does not converge but "oscillates finitely". If $x < -1$, it oscillates unboundedly.

Example 2: The sequence $\frac{1-x^{n+1}}{1-x} = 1 + x + ... + x^n$ converges to $g(x) = \frac{1}{1-x}$, for $|x| < 1$. The limit function is continuous (and even differentiable to all orders!) at every point of the **open interval**, $(-1, 1)$.

Example 3: The sequence $\frac{x^n}{1+x^n}$ for $x \in [0, \infty)$ converges to $g(x)$, where $g(x) = 0$ for $0 \leq x < 1$, $g(1) = 1/2$ and $g(x) = 1$ for $x > 1$. In this case, the limit exists at every point of the set but the limit function is discontinuous!

- Recall that for convergence at a point x, we mean that for any $\epsilon > 0$, we can find an integer N, which will depend upon **both** ϵ, x, such that $|g(x) - f_n(x)| < \epsilon$ for $n > N$. There may be cases where N depends only upon ϵ, **independently** of x. This type of convergence has a special significance and name.

2. Uniform convergence

- **Definition 3.6:** Let $[a, b]$ be a compact interval. A sequence $f_n(x)$ of real functions defined over this interval is said to **converge uniformly** to a limit $g(x)$ if and only if, for any given $\epsilon > 0$, we can find an $N(\epsilon)$ such that for all $n > N(\epsilon)$, $|f_n(x) - g(x)| < \epsilon$, for **every point of the interval.**
- The following theorem indicates the very great importance of this concept.
- **Theorem 3.7:** Let $f_n(x)$ be a series of real functions defined and continuous on the closed and bounded) interval, $[a, b]$. Suppose this sequence converges **uniformly** over this interval to a function $g(x)$. Then,
  1. The limit function $g(x)$ is continuous over the interval. ("a uniformly convergent sequence of continuous functions has a continuous limit").
  2. If $a \leq c < d \leq b$ (ie, $[c, d]$ is a **sub-interval**), the sequence of **integrals** $\int_c^d f_n(x)dx$ converges to the integral, $\int_c^d g(x)dx$.
  3. If the given sequence of functions are **continuously differentiable** in the interval and if the sequence of derivatives, $f_n'(x)$ converges uniformly in the interval to a limit function $h(x)$, then, $h(x) = g'(x)$.

2. Uniformly convergent series of functions

- Theorem 3.7 allows us to "interchange" limiting operations as follows :

$$\mathrm{Lim}_{n\to\infty}\int_c^d f_n(x)dx = \int_c^d (\mathrm{Lim}_{n\to\infty} f_n(x))dx$$

$$\mathrm{Lim}_{n\to\infty}\frac{d}{dx}f_n(x) = \frac{d}{dx}\left[\mathrm{Lim}_{n\to\infty} f_n(x)\right]$$

- If we consider the real sequence, $f_n(x) = \frac{e^{nx}-e^{-nx}}{e^{nx}+e^{-nx}}$ (ie of hyperbolic tangents), it is easy to show that it converges (as $n \to \infty$) at every x to a function which is discontinuous at $x = 0$ but everywhere else to either 1 or -1. It does not converge **uniformly** in any closed interval surrounding the origin, as otherwise the limit function would be continuous there! This sequence arises naturally in the theory of **shock waves**.

- **Theorem 3.8:** If $s_n = \Sigma_{j=1}^{n} u_j(x)$ is the n-th partial sum of real continuous functions $u_j(x)$ defined on $[a, b]$ and the series (ie $s_n(x)$) converges **uniformly** on the interval, the limit $s(x)$ is continuous. It may be integrated term-by-term (over any sub-interval $[c, d]$) and the resulting series converges uniformly to the integral of $s(x)$ over the corresponding sub-interval. Theorem 3.6.3 can be applied to justify term-by-term differentiation of uniformly convergent series.

3. Smooth curves in the complex plane

- We can immediately extend all the previous results to **complex functions of a real variable.** Indeed if $z = f(t)$, then we can always write, $z = f(t) = u(t) + iv(t)$. We simply investigate the **real** functions u, v to make statements about f.

- **Definition 3.7:** If $t \in [0, 1]$ and $x(t), y(t)$ are real, twice continuously differentiable functions of t, the complex function, $z(t) = x(t) + iy(t)$ represents a **smooth** curve in the complex plane. The complex number, $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$ is called the **tangent vector** at the point $z(t)$. We say that the function maps the closed interval $[0, 1]$ onto the **curve** $z(t)$ in the complex plane. The curve is **closed** if $z(0) = z(1)$; ie the function is **periodic.**

- If $s(t)$ is the **arc length** of the curve measured from $z(0)$, it is clear that, $\frac{ds}{dt} = |\frac{dz}{dt}| = (\frac{dz}{dt}\frac{d\bar{z}}{dt})^{1/2}$. Smooth curves as defined have **finite lengths** given by the usual formula:

$$s(1) \quad = \quad \int_0^1 |\frac{dz}{dt}|dt$$

Smooth curves have continuously turning tangents at every point t and have a unique curvature at each point. If two distinct values of the **parameter** t are mapped into distinct points (the mapping is “one-one”) in the complex plane, the curve has no **self-intersections.**

4. Complex topology: key notions

- We now move on to the real “meat” of the subject: **complex functions of a complex variable**. The following definitions are intended to convey the essential ideas of **point set topology**. The first generalizes the idea of an **open interval on the real axis** and is key to analysis with functions of a complex variable.
- **Definition 3.8:** An **open disk** $D(c; \rho)$ centred at c with radius ρ is the set of points z such that, $|z - c| < \rho$. Its **closure** is the set D_* of points z such that $|z - c| \leq \rho$, and its **boundary** is the circle, $|z - c| = \rho$. The **complement** D_*^c of the closure of the disk is its **exterior**, and is made up of z such that $|z - c| > \rho$.
- **Definition 3.9:** An **open set** S of the complex plane is a set of points z such that at every $z \in S$ we can find a $\rho(z) > 0$ such that the **open disk**, $D(z, \rho(z))$ is **a sub-set** of S. Thus, at every point z in S, there exists an open disk with **all** of its points included in S. The complement of an open set is called a **closed** set. The **null set and whole complex plane** are, by definition open sets.
- **Definition 3.10:** An open set S with the property that any two points are the end points of a continuous non-self intersecting curve C lying entirely within S is called a **connected set**. If any two such connecting curves can be continuously deformed into each other, the set is said to be **simply connected**.

4. Complex topology: examples

- Let us consider some simple examples to understand these concepts
- Example 1: Every open disk is an open set.
  Example 2: The exterior of any open disk is also an open set
  Example 3: The boundary of any open disk is not an open set.
  Example 4: An arbitrary **union** of open sets is an open set.
- **Definition 3.11:** If an open set has the property that any two of its points can be connected by a straight-line segment of which they are end points, and every point of the line belongs to the set, it is called **convex**.
- It can be shown that **continuous, closed curves which have no self-intersections** divide the whole complex plane into two simply connected open sets called the “interior” and the “exterior”. These are separated by set of boundary points (ie, the curve itself). This intuitively obvious observation is a non-trivial result of topology called **the Jordan Curve Theorem.** We will simply assume it without proof!
- **Definition 3.11:** A connected open set of the complex plane is called a **Region.** Unless stated otherwise, we shall always consider complex functions defined over regions. The entire complex plane itself is a region. Regions need not always be **simply connected** or **convex**, although they can be in many cases.