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# Chennai Mathematical Institute

## B.Sc Physics

### Mathematical methods

#### *Lecture 2: Complex sequences and infinite series*

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# Sequences of complex numbers: definitio

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- I shall assume some familiarity with notions of elementary theory of sets like union, intersection, null set, set inclusion, set functions etc. I shall also assume knowledge of limits, sequences and series of **real numbers**. We can now extend these ideas to the **complex plane**. Many results are simple extensions but some are quite new and unique to the complex plane. We begin with sequences and their properties. We will often use the notation,  $[a_n]_{n=1}^{\infty} = [a_1, \dots, a_n, \dots]$  to denote a sequence. When the range of the index is understood, we may leave it out in the interests of brevity.
- **Definition 2.1:** An infinite sequence of complex numbers is a countable set (finite or infinite) of complex numbers,  $z_1, z_2, \dots, z_n, \dots$
- **Remarks:** The individual numbers of this ordered set of complex numbers are called the elements of the sequence. For example,  $z_5$  is the fifth element. If the total number of elements is finite, we have a finite sequence. Otherwise, the sequence is said to be infinite. We may occasionally also deal with two-way sequences:  $\dots, z_{-2}, z_{-1}, z_0, z_1, \dots$
- You will recall that a function is a rule which assigns a unique element of a set called its range to each element of a set called its domain. It is seen that a complex sequence is a function from the set of integers (domain)  $1, 2, \dots$  into the complex plane (range).
- **Examples:** 1)  $1, i, -1, -i$  is a four-element sequence. 2)  $1, z, z^2, z^3, \dots$  is an infinite sequence. 3)  $1, \frac{z}{1+z}, \frac{z^2}{1+z^2}, \dots$  is another infinite sequence.

# 1. Sequences: limits

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- In analysis we are mainly interested in infinite sequences. The key concept here is the idea of the limit of a sequence.
- **Definition 2.2:** An infinite sequence of complex numbers  $[z_n]_{n=1}^{\infty}$  is said to converge to a limit if there exists a complex number  $\lambda$  such that by taking  $n$  sufficiently large, we can make the “distance”  $|\lambda - z_n|$  between  $\lambda$  and  $z_n$  as small as we please.
- Note that this does not tell us how to find the limit of a sequence, only how we can find out if the sequence has a given complex number as its limit. We can picture the definition of a convergent sequence, ie one which converges, using the geometrical concept of neighbourhood.
- **Definition 2.3:** Given any complex number  $z$  and a real positive number  $\rho$ , we define the vicinity or  $\rho$ -neighbourhood of the given complex number to be the set of all those complex numbers  $w$  within the circle of centre  $z$  and radius  $\rho$ ; ie , all  $w$  such that  $|z - w| < \rho$ .
- According to this definition, the points on the circle,  $|z - w| = \rho$  do not belong to the neighbourhood in question!
- **Proposition 2.1:** If for every  $\rho > 0$ , all except possibly a finite number of elements of the infinite sequence,  $[a_n]$  belong to the  $\rho$ -neighbourhood of  $\lambda$ , then the sequence converges to  $\lambda$ .

# 1. Sequences: the meaning of convergence

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- The meaning of convergence is this: given any test of accuracy  $\rho > 0$  (however small), all, except possibly a finite number of elements of the sequence should lie within the circle of radius  $\rho$  with centre  $\lambda$ . Of course the “tighter” the test of accuracy, the smaller the  $\rho$  and more points of the sequence will fail to be in the circle.
- For every given measure of accuracy, all elements with index larger than some integer  $N$  which will depend upon  $\rho$  in general (naturally!) must indeed lie within the circle for us to conclude the sequence converges to  $\lambda$ .
- **Examples:**
  1. Consider  $[a_n]$  where  $a_n = c$  for  $n = 1, 2, \dots$ ,  $c$  being some complex number. Then, the sequence converges to  $c$  and we write,  $\text{Lim}_{n \rightarrow \infty} a_n = c$ .
  2. Let  $N = 10^{100}$ ; this number is larger than all the electrons in the cosmos! Let  $[b_n]$  be a complex sequence with the following properties:  $b_n = n^N i^n$ ; for  $n = 1, 2, \dots, N$ ; and  $b_n = c$ , for  $n > N$ , where  $c$  is a complex number. Show that  $\text{Lim}_{n \rightarrow \infty} b_n = c$ ! This shows that no matter how “badly behaved” a sequence is for a finite number of elements, its convergence depends only upon the “ultimate tail”.

# 1. Sequences: examples

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1. Let  $[a_n], [b_n]$  be a pair of real, convergent sequences. They define a unique complex sequence,  $z_n = a_n + ib_n$ . Then, if  $\text{Lim}_{n \rightarrow \infty} a_n = \alpha; \text{Lim}_{n \rightarrow \infty} b_n = \beta$ , where  $\alpha, \beta$  are the respective (real) limits, the complex sequence,  $[z_n = a_n + ib_n]$  converges to the complex number,  $\alpha + i\beta$  and conversely, if the complex sequence is convergent, its real and imaginary parts are also convergent.

**Proof:** Consider  $z_n - (\alpha + i\beta) = (a_n - \alpha) + i(b_n - \beta)$ . Using the “triangle inequality”,  $|z - (\alpha + i\beta)| \leq |(a_n - \alpha)| + |(b_n - \beta)|$ . Since the real sequences converge (by hypothesis) the RHS can be made arbitrarily small by choosing  $n$  sufficiently large. This shows that the complex sequence must converge. To prove the converse, we merely note that,  $|z - (\alpha + i\beta)|^2 = (a_n - \alpha)^2 + (b_n - \beta)^2$  from Pythagoras’ Theorem. If the complex sequence converges, since  $|z - (\alpha + i\beta)|$  is greater than  $|a_n - \alpha|$  and  $|b_n - \beta|$  separately, the two real sequences must converge.

2. If  $|z| < 1$ , the infinite sequence,  $[z^n]_{n=1}^{\infty}$  converges to 0, for  $|z^n| = |z|^n$ . Now,  $[|z|^n]$  is a real, positive sequence which obviously converges to zero for  $|z| < 1$ .
3. The complex sequence  $[z_n]$  converges to zero if and only if the real sequence,  $[|z_n|]$  also converges to zero. This clearly follows from Example 1, above.

# 2. Sequences: divergence, boundedness

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- **Definition 2.4:** An infinite sequence which has no finite complex number as its limit is called **divergent**.
- As an example of a divergent sequence consider the sequence defined by the elements,  $z_n = i^n$ . The elements always take on the four values  $1, -1, i, -i$ . Clearly there is no complex number around which these numbers “cluster”. They occur arbitrarily far into the sequence. Thus, the sequence is divergent.
- Another type of divergence arises when the elements do not simply oscillate as in the above example but actually “blow up” by having their moduli grow without limit. A typical example is provided by the sequence,  $z_n = c^n$ , where  $c$  is a complex number with modulus greater than unity. Evidently,  $|z|^n$  becomes arbitrarily large as  $n$  increases and therefore there can be no finite limit.
- **Definition 2.5:** An infinite sequence  $[z_n]$  with the property that  $\lim_{n \rightarrow \infty} \frac{1}{|z_n|} = 0$  is said to **tend to infinity**.
- **Definition 2.6:** An infinite sequence  $[z_n]$  such that there exists a finite positive number  $M$  satisfying,  $|z_n| < M$  for all  $n$  is called a **bounded sequence**. If for any given positive number  $K$  however large, we can find some  $m$  such that  $|z_m| > K$ , the sequence is called **unbounded**.

# 2. Sequences: properties

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1. The sequence  $[z_n]$  is bounded or unbounded, according as the real sequence  $[|z_n|]$  is bounded or unbounded.
2. A complex sequence which tends to infinity is necessarily unbounded, but the converse may not be true.
3. The complex sequence defined by  $[a_n = (1 + \frac{z}{n})^n]_{n=1}^{\infty}$  is bounded.

**Proof:** Consider,

$$\begin{aligned} |(1 + \frac{z}{n})^n| &= |(1 + \frac{z}{n})|^n \\ &\leq |(1 + \frac{|z|}{n})|^n \end{aligned}$$

You know that for any real, positive  $x$ ,  $1 + x < e^x$  (I hope you do!). Substituting, we obtain the bound for the LHS:  $|(1 + \frac{z}{n})^n| < (e^{|z|/n})^n = e^{|z|}$ . Hence the result holds.

4. The sequence  $[2^n i^n]$  oscillates unboundedly.

# 3. Algebra of convergent sequences

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## Theorem 2.1:

1. Every convergent complex sequence has a **unique limit**. Furthermore, if  $\text{Lim}_{n \rightarrow \infty} z_n = c = a + ib$ , then,  $\text{Lim}_{n \rightarrow \infty} \text{Re}(z_n) = a$ ,  $\text{Lim}_{n \rightarrow \infty} \text{Im}(z_n) = b$ ,  $\text{Lim}_{n \rightarrow \infty} |z_n| = |c| (= (a^2 + b^2)^{1/2})$ .
2. If  $[z_n]_{n=0}^{\infty}$  is any complex, convergent sequence, it is **bounded**. (The converse is not necessarily true!)
3. If  $[z_n]_{n=0}^{\infty}$  is any complex sequence converging to the **finite limit**  $z$ , and  $c$  is any complex number, the sequence  $[cz_n]_{n=0}^{\infty}$  converges to  $cz$ .
4. If  $[a_n], [b_n]_{n=0}^{\infty}$  are complex sequences convergent to finite complex numbers  $a, b$  respectively, their **sum** defined by,  $[a_n + b_n]_{n=0}^{\infty}$  and **elementary product** defined by  $[a_n b_n]_{n=0}^{\infty}$  are also convergent, respectively to  $a + b$  and  $ab$ . Thus the sum and product of convergent sequences converge to the sum and product of the limits.
5. If  $[a_n], [b_n]_{n=0}^{\infty}$  are convergent complex sequences with limits  $a, b$  and  $b \neq 0$ , there is a finite integer  $m$  and a positive real number  $l$  such that  $|b_n| > l > 0$  for all  $n > m$  and the infinite sequence,  $[\frac{a_n}{b_n}]_{n=m+1}^{\infty}$  converges to  $\frac{a}{b}$ .
6. A complex sequence  $[a_n]$  with the property that  $|a_n|^{1/n} \leq r < 1$ , converges to zero.



# 3. Discussion of Theorem 2.1

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I will merely indicate the key ideas of the proof. You are expected to complete them! The problems explain some details.

1.) Left as an exercise.

2.) Since we know that the limit of the sequence exists (call it)  $c$  we know that  $|c - z_n|$  can be made as small as we please by choosing  $n$  sufficiently large. This means for all large enough  $n$ ,  $|c - z_n| < 1$  say. Using the **triangle inequality** we see that this implies,  $|z_n| = |z_n - c + c| \leq |c| + 1$  for all sufficiently large  $n$ . This shows that  $z_n$  is a bounded sequence.

3.) Consider the sequence  $[kz_n]$ . Now,  $|kz_n - kc| < |k||z_n - c|$ . We know that  $|z_n - c|$  can be made as small as we please by choosing  $n$  large enough. This means that we can make the LHS also arbitrarily small, since  $|k|$  is a fixed, finite, positive number. This proves the result, which is sometimes written as,

$$k(\text{Lim}_{n \rightarrow \infty} z_n) = \text{Lim}_{n \rightarrow \infty} (kz_n) = kc$$

# 3. Discussion of Theorem 2.1: contd.

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4.) Consider  $a_n + b_n - (a + b) = (a_n - a) + (b_n - b)$ . Using the triangle inequality,  $|a_n + b_n - (a + b)| \leq |(a_n - a)| + |(b_n - b)|$ . By choosing  $n$  sufficiently large both terms on the right can be made arbitrarily small:

$$\text{Lim}_{n \rightarrow \infty} (a_n + b_n) = \text{Lim}_{n \rightarrow \infty} a_n + \text{Lim}_{n \rightarrow \infty} b_n$$

Now,  $|a_n b_n - ab| = |(a_n - a)b_n + a(b_n - b)| \leq |(a_n - a)||b_n| + |a|(b_n - b)|$ . Convergent sequences are bounded, hence RHS can be made arbitrarily small by choosing  $n$  large enough:

$$\text{Lim}_{n \rightarrow \infty} (a_n b_n) = (\text{Lim}_{n \rightarrow \infty} a_n)(\text{Lim}_{n \rightarrow \infty} b_n)$$

5.) Now from (1),  $\text{Lim}_{n \rightarrow \infty} |b_n| = |b| > 0$ . It is easily seen geometrically that there is an  $m$  such that  $||b| - |b_n|| < |b|/2$  for all  $n > m$ . Therefore  $|b_n| > |b|/2 > 0$  for all such  $n$ . Hence  $a_n/b_n$  is bounded for  $n > m$ . Consider  $(\frac{a}{b} - \frac{a_n}{b_n}) = \frac{(ab_n - a_n b)}{bb_n}$ . This can be made as small as we please by choosing  $n$  to be sufficiently large:

$$\text{Lim}_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{\text{Lim}_{n \rightarrow \infty} a_n}{\text{Lim}_{n \rightarrow \infty} b_n}$$

6.) Now,  $|a_n|^{1/n} < r < 1$ . This implies,  $|a_n| < r^n$ . Since  $r < 1$  as  $n \rightarrow \infty$ ,  $a_n \rightarrow 0$ ,

# 3. The Cauchy convergence criterion

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- We have defined convergence of sequences as if we knew in advance what the limit is. Cauchy proved a general criterion for the convergence of any sequence which does not require the prior knowledge of the limit.
- **Theorem 2.2: ("The Cauchy convergence criterion")** Let  $[u_n]$  be a an infinite sequence of complex numbers. Then, a **necessary and sufficient condition** for the convergence of this sequence is that, for any given positive number  $\epsilon$  (which can be arbitrarily small) there must exist an integer  $N(\epsilon)$  such that,  $|u_n - u_m| < \epsilon$  for **all**  $m, n > N(\epsilon)$ .
- I shall prove that the criterion is **necessary**. It is beyond the scope of this course to prove it is also **sufficient**. Thus, suppose,  $\text{Lim}_{n \rightarrow \infty} u_n = u$ . Since  $|u_n - u_m| = |u_n - u + u - u_m| \leq |u_n - u| + |u_m - u|$ , we see that choosing  $N(\epsilon)$  such that  $|u_n - u| < \epsilon/2$  for **all**  $n > N(\epsilon)$ , we can satisfy the Cauchy criterion. Hence all **convergent sequences** satisfy the criterion.
- In the theory of **convergent real sequences** the same criterion occurs. There, the sufficiency proof depends upon subtle properties of the real number system. If we take these for granted, we may apply the real-variable Cauchy criterion to the real and imaginary parts separately and derive the sufficiency of criterion for complex sequences.

# 4. Infinite series of complex numbers

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- **Definition 2.7:** An infinite sequence  $s_1, \dots, s_n, \dots$  of complex numbers  $s_n$  is called an **infinite series** if it can be written in the form,

$$s_n = \sum_{j=1}^n u_j$$

where  $u_j$  is called the  $j$ -th term of the series. If the sequence  $[s_n]$  is convergent to a finite limit we say that the series is **convergent**. Otherwise, it is called a **divergent series**. The limiting value of the sequence is called the **sum** of the infinite series. This is written as,  
 $s = \sum_{n=1}^{\infty} u_n$ .

- The element  $s_n$  is usually referred to as the  **$n$ -th partial sum of the series**. A divergent series has no determinate "sum".
- **Examples:**
  1. Obviously any infinite sequence can be interpreted as an "infinite series". We merely define the  $n$ -th term in terms of the partial sums:  $u_n = s_n - s_{n-1}; s_0 = 0$ .
  2. Consider  $s_n = 1 + z + z^2 + \dots + z^n; z$  a complex number. This is called the **geometric series with common ratio  $z$** . Evidently, we have by the usual algebra,

$$s_n = \frac{1 - z^{n+1}}{1 - z}$$

# 4. Infinite series: examples

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## Examples:

3. We see that if  $z < 1$ ,  $\text{Lim}_{n \rightarrow \infty} s_n = \frac{1}{1-z} = s$ . When  $|z| \geq 1$ , the series is divergent.
4. A series in which  $u_j$  is identically zero for  $j > l$ , where  $l$  is a fixed integer is called a **finite series**. Obviously any finite series is automatically convergent.
5. You have already encountered infinite series with **real terms**. Of course **all complex infinite series are equivalent to a pair of real infinite series**. Thus if  $u_j = a_j + ib_j$  and  $A_n = \sum_{j=1}^n a_j$ ;  $B_n = \sum_{j=1}^n b_j$ . From our work with sequences, we know that the complex series converges if and only if the two real series do. All the tests of convergence for real series can be applied to the complex series by testing (if necessary) the real and imaginary parts.
6. The series defined by the  $j$ -th term  $u_j = \frac{1}{2^j} + i(-1)^j$  is **divergent** (why?).

# 4. Infinite series: basic theorems

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## ● Theorem 2.3:

1. If  $s_n$  are the partial sums of a **convergent series** of complex terms, there exists a real positive number  $K$  such that  $|s_n| < K$  for all  $n$ .
2. If  $s_n = \sum_{j=1}^n u_j$  are the partial sums of a convergent series,  $\lim_{n \rightarrow \infty} u_n = 0$  ("the  $n$ -th term of a convergent series tends to zero").
3. If  $s_n = \sum_{j=1}^n u_j$  are the partial sums of a series and  $M_n = \sum_{j=1}^n |u_j|$  are the partial sums of its **real, dominant series**, the series converges if the dominant series does.

## ● Proof: 1. Follows from Theorem 2.1.

2. We have seen that  $u_{n+1} = s_{n+1} - s_n$ . Hence,

$|u_{n+1}| = |s_{n+1} - s_n| \leq |s_{n+1} - s| + |s_n - s|$ . As  $n \rightarrow \infty$ , we see that the RHS tends to zero leading to the result.

3. We apply the **Cauchy criterion for convergence**. Let,  $n \leq m$ :

$$\begin{aligned} |s_n - s_m| &= |u_{n+1} + \dots + u_m| \\ &\leq |u_{n+1}| + \dots + |u_m| \\ &\leq M_m - M_n \end{aligned}$$

# 4. Infinite series: absolute convergence

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- 3. (contd.) Since we are given that the **positive, monotonic increasing** sequence of partial sums  $M_n$  is convergent, applying the Cauchy criterion to the convergent **dominant series** implies (via the Cauchy criterion for the complex series!) the theorem. This is a **very important result!**
- An infinite series whose real dominant series converges is called an **absolutely convergent series**. There are series which converge but not absolutely. They are called **conditionally convergent series**. Absolutely convergent series have many "nice" properties, as we shall learn, which conditionally convergent ones do not have.
- This theorem often helps us to determine if a given series is convergent or not. If the  $n$ -th term of an infinite series **does not** tend to zero (as  $n \rightarrow \infty$ ) we can confidently assert that the series must be **divergent**, since the latter is a **necessary condition** for convergence according to Theorem 2.3. Unfortunately, the converse is not true! Thus there are **divergent series** the  $n$ -th terms of which **do** tend to zero. Let us consider an interesting "counter-example":

$$s_n = \sum_{j=1}^n \sqrt{j+1} - \sqrt{j} = \sqrt{n+1} - 1$$

# 4. Infinite series: absolute convergence

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- (Contd.) Clearly the series diverges (ie does not tend to any finite limit). However, the  $n$ -th term is,

$$\begin{aligned}u_n &= \sqrt{(n+1)} - \sqrt{(n)} \\ &= \frac{n+1-n}{\sqrt{(n+1)} + \sqrt{(n)}} \\ &= \frac{1}{\sqrt{(n+1)} + \sqrt{(n)}}\end{aligned}$$

Clearly the  $n$ -th term tends to zero as  $n \rightarrow \infty$ . Hence, simply checking that the  $n$ -th term of a series tends to zero with  $n$  is not sufficient to conclude the convergence of a series.

- **Absolute convergence** guarantees convergence according to the Th. 2.3. However, there are series which converge but not absolutely. In the problem set you will find an exercise which explains a typical case in which this happens. Of course this is well-known even for infinite series using reals; complex numbers do not bring in anything new. In most applications (there are always exceptions!) we will be dealing mainly with absolutely convergent series.



# 5. Infinite series: some key concepts

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- **Theorem 2.4:** If  $s_n, t_n$  are partial sums of convergent/absolutely convergent series (to sums  $s, t$ ),  $as_n + bt_n$  is convergent/absolutely convergent (to sum  $as + bt$ ) where  $a, b$  are arbitrary complex numbers.
- **Proof:** For ordinary convergence, this follows from the properties of the corresponding Theorem 2.1 for sequences. Let  $u_n, v_n$  be the  $n$ -th terms of  $s_n, t_n$  respectively. The absolute convergence of these series implies that  $S_n = \sum_{j=1}^n |u_j|; T_n = \sum_{j=1}^n |v_j|$  are real dominants converging to sums  $S, T$  say. Obviously,  $S_n \leq S; T_n \leq T$ . The real dominant of  $as_n + bt_n = \sum_{j=1}^n |au_n + bv_n| \leq |a|S_n + |b|T_n \leq |a|S + |b|T$ . Now, this dominant is a series of **positive** terms. Its partial sums form a monotonic increasing sequence which is bounded above. From the basic principles of real number system, it must be a convergent sequence. Hence the result follows.
- **Definition 2.8:** Consider a sequence (real or complex)  $u_1, u_2, \dots, u_n, \dots$ . A **rearrangement** of this sequence is another sequence,  $v_1, v_2, \dots, v_n, \dots$  with the following property. Each  $v_j$  is simply a certain  $u_k$  so that every term of the second sequence is a term from the first, and every term of the first sequence occurs once in the second.

# 5. Infinite series: re-arrangements

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- We know from the laws of Algebra that if we have a **finite sequence**, the **sum** is unchanged no matter what permutation/re-arrangement of the terms. It was shown by **Riemann** in a beautiful theorem that if an infinite series has its terms re-arranged in a certain way, it can be made to converge to any number at all, **so long as the infinite series is only conditionally convergent!**
- This means the **commutative law of addition of algebra** is **not** generally valid for conditionally convergent series.
- However, **Cauchy** showed that if an infinite series is **absolutely convergent** no matter what re-arrangement we choose of its terms, its sum is **invariant, as if the series were finite!** Thus, as far as re-arrangements of a series are concerned, absolutely convergent series behave "well" whilst conditionally convergent (or divergent) series behave "badly". This is stated in the following theorem which you can assume freely without proof.
- **Theorem 2.5: An absolutely convergent series may be summed in any re-arrangement to the same sum.**

# 5. Infinite series: the Cauchy product

- The **Cauchy product** (not to be confused with products introduced for sequences!) of two absolutely convergent series is explained in the following Theorem. This important concept (crucial to many applications) is again stated without proof.
- **Theorem 2.6:** Let  $s_n = \sum_{j=0}^n u_j$ ;  $t_n = \sum_{j=0}^n v_j$  be any two absolutely convergent series converging to  $S, T$  respectively. Then their **Cauchy product**  $W_n = \sum_{k=0}^n w_k$  is defined by Cauchy product rules:

$$\begin{aligned}w_0 &= u_0 v_0 \\w_1 &= u_0 v_1 + u_1 v_0 \\&\dots = \dots \\w_k &= \sum_{j=0}^k u_j v_{k-j}\end{aligned}$$

Then,

$$\begin{aligned}\lim_{n \rightarrow \infty} W_n &= \sum_{k=0}^{\infty} w_k = \left( \sum_{j=0}^{\infty} u_j \right) \left( \sum_{j=0}^{\infty} v_j \right) \\&= ST\end{aligned}$$

The "Cauchy product series" converges absolutely. The reason for this particular rule for  $w_k$  will be explained later.