Chennai Mathematical Institute
B.Sc Physics

Mathematical methods

Lecture 1: Introduction to complex algebra

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1. Real numbers

- The set of all **real** numbers (hereafter denoted by R) form a **commutative ordered field**. This means that if a, b are any two real numbers, their **sum**, and **product** are defined and are real numbers. A **commutative field** is subject to the **Laws of Algebra**:

$$
\begin{aligned}
a+b &= b+a && \text{“additive} - \text{commutativity''} \\
ab &= ba && \text{“multiplicative} - \text{commutativity''} \\
a+(b+c) &= (a+b)+c && \text{“additive} - \text{associativity''} \\
a(bc) &= (ab)c && \text{“multiplicative} - \text{associativity''} \\
a(b+c) &= ab+ac && \text{“distributivity''} \\
a+0 &= a && \text{“additive} - \text{identity''} \\
a+(-a) &= 0 && \text{“additive} - \text{inverse''} \\
a.1 &= a && \text{“nultiplicative} - \text{identity''} \\
a.(a^{-1}) &= 1 \quad (a \neq 0) && \text{“multiplicative} - \text{inverse''}
\end{aligned}
$$

1. Real numbers: contd.

- If any two real numbers a, b are given, we can **compare them.** Thus, either $a > b$ **or** $a < b$, **or** $a = b$. These are the only mutually exclusive alternatives. Furthermore, the "ordering relation" is a "total order" which is **transitive**: If $a > b$ **and** $b > c$, then $a > c$. Furthermore, $a = b$ if and only if $a \geq b$ and $b \geq a$ are simultaneously true.
- This makes the real numbers an **ordered** field. Furthermore, it is **complete**, which means that every monotonic increasing sequence of real numbers either tends to $+\infty$ or **converges** to a bounded real **limit.** Similarly, every monotonic decreasing sequence of real numbers must either tend to $-\infty$ or to a finite real number.
- The set of all **rational numbers** form an ordered field, but is **not** complete. This means that the limit of a sequence of rational numbers need not be a rational number. **Cauchy and Dedekind** showed that the real number field can be contructed by **completing** the rational number field.
- The set of real numbers is **not an algebraically closed field.** Thus, there are algebraic equations which have no **real solutions**; e.g $x^2 + 1 = 0$.

2. Complex numbers: Gauss' construction

- **Problem:** Can we "extend algebraically" the real field in some way so that it is a **sub-field** of some larger field which **is algebraically closed**?
- **Answer: "Yes, but...!" (Gauss)** We can indeed construct such a field but will find that it **cannot be an ordered field.** Thus, in general, two elements of that field cannot be compared. We'll look in detail at **Gauss' construction.**
- **Definition 1.1**: We designate by (a, b) an **ordered pair** of real numbers, a and b. For future reference, we shall call such a pair a **complex number**. The first element a is called the **real part** and the second element b the **imaginary part** of the complex number $z \equiv (a, b)$.
- **Definition 1.2**: The special pair, $(0, 0)$ will be called the **complex zero** and will be denoted (for now) by $\mathbf{0}$, as in vector analysis in the plane.
- **Definition 1.3**: Two complex numbers, (a, b) and (c, d) are said to be **equal if and only if** $a = c; b = d$.

2. Complex numbers: algebraic rules

- **Definition 1.4**: If $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ are any two complex numbers, we define their **sum/difference** by the vector or component-wise addition rule,

$$z_1 \pm z_2 \quad = \quad (x_1 \pm x_2, y_1 \pm y_2)$$

- **Remark:** It is obvious that the addition thus defined is both **commutative and associative.** Furthermore, the complex zero is the **additive identity.** Thus our "complex numbers" already form an additive **Abelian group.** In fact this is none other than the algebra of vectors in a two-dimensional Euclidean plane-it was Gauss' great discovery that such 2-vectors can be multiplied also.

- **Definition 1.5**: If z_1, z_2 are any two complex numbers, we define their **product** to be the complex number, $z_1 * z_2 = (X, Y) = Z$:

$$\begin{aligned} z_1 * z_2 &= [(x_1 x_2 - y_1 y_2), (x_1 y_2 + y_1 x_2)] \\ &= (X, Y) \end{aligned}$$

2. Complex numbers: magnitude

- **Definition 1.6**: The **length or magnitude** of a complex number $z = (x, y)$ is defined to be the **real, non-negative number**, $|z| = (x^2 + y^2)^{1/2}$.
- **Remark:** The rules are now complete! You immediately realise that every complex number can be represented by a 2-vector in the **complex plane**. The x-axis is called the **real axis**, whilst the y-axis (for largely historical reasons) is called the **imaginary axis**.
- The magnitude of the complex number is obviously the ordinary length of the 2-vector representing it. The **angle** (measured anti-clockwise) which it makes with the **positive real axis** (using the usual conventions of trigonometry) is called **the argument/phase/amplitude** of the complex number. I will use these terms interchangeably. It is denoted by $\mathrm{Arg}(z)$.
- This geometrical representation is called the **Argand-Wessel diagram**. From simple trigonometry, we see that $z = (x, y) = (|z| \cos\theta, |z| \sin\theta)$, where $\theta = \mathrm{Arg}(z)$.

2. Complex numbers: basic properties

- I shall now present the basic properties of our complex number system in a series of simple propositions. It is an excellent excercise (see Problem set 1) for you to **prove** them, using only the information provided thus far.

- **Theorem 1.1:** The "Laws of Algebra" enumerated in Section 1 for real numbers are valid for **complex numbers** provided the following provisos apply: the symbols for addition and multiplication are to be interpreted according to definitions 1.4 and 1.5 of this section. Zero is to be understood as $\mathbf{0} = (0, 0)$. "1" is to be understood as the **complex number** $\mathbf{1} = (1, 0)$. Furthermore, all real numbers x can be represented by the complex numbers, $(x, 0)$ (this is saying that the complex numbers of this form behave exactly like real numbers!).

- The only point which requires calculation is proving the existence of a "multiplicative inverse" to a complex number, $z = (x, y)$. You should show that **provided** a complex number is not $\mathbf{0}$, it has an inverse:

$$z^{-1} \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

Complex numbers: polar form, conjugac

- We have seen that any complex number $z = (x, y)$ can be written in the **Modulus-Amplitude or Magnitude-Argument or Polar form:**

$$
\begin{aligned}
z &= |z|(\cos\theta, \sin\theta) \\
|z| &= (x^2 + y^2)^{1/2} \\
\theta &= \tan^{-1}(y/x)
\end{aligned}
$$

  The complex number $e(\theta)$ **defined** by $e(\theta) = (\cos\theta, \sin\theta)$ is said to be **unimodular**, namely has modulus **unity.** It therefore represents a 2-vector with its base at $(0, 0)$ and tip on the **unit circle.** It makes an angle θ with the positive real (x) axis. In the Problem Set 1, several interesting and important properties of complex numbers are obtained from the above basic definitions.

- If $z = (x, y)$ is any complex number, $\bar{z} = (x, -y) = |z|e(-\theta)$ is called its **conjugate.** Note that the conjugate of the conjugate is the original number! $\bar{\bar{z}} = z$. **Real numbers** are represented by complex numbers with zero imaginary parts. They are **self-conjugate.**

2. Complex numbers: key relations

- **Theorem 2.1:** Let, $z_1 = (x_1, y_1); z_2 = (x_2, y_2)$ be arbitrary complex numbers. Then, the following relations hold:

$$
\begin{aligned}
z_1 z_2 &= |z_1||z_2|e(\theta_1)e(\theta_2) \\
&= |z_1||z_2|e(\theta_1 + \theta_2) \\
|z_1 z_2| &= |z_1||z_2| \\
z_1 \bar{z}_1 &= |z_1|^2 \\
(x_1 x_2 + y_2 y_2)^2 &\leq (x_1^2 + y_1^2)(x_2^2 + y_2^2) \\
|z_1 + z_2| &\leq |z_1| + |z_2| \\
|z_1 - z_2| &\geq |(|z_1| - |z_2|)|
\end{aligned}
$$

These are proved in the Problem set 1. You should learn these results as they are repeatedly used! The first two constitute the "Multiplication rules" for complex arithmetic: **"the modulus of the product equals the product of the moduli of the factors; the amplitude of the product is the sum of the amplitudes of the factors".**

2. Complex numbers: deductions (1)

- The third states that multiplying a complex number by its conjugate is always a real, positive number (except for $\mathbf{0}$) equal to the square of its modulus. The fourth is called **the Cauchy-Schwarz inequality** and applies to any four **real numbers** x_1, x_2, y_1, y_2. The fifth is called the **triangle inequality**. The last is a slight variant of the same.
- It is obvious from the rules that the **additive inverse** of $z = (x, y)$ is $-z = (-x, -y)$.
- The **multiplicative inverse** of z is written as $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{e(-\theta)}{|z|}$.
- **Complex division:** $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e(\theta_1 - \theta_2)$
- The complex number defined by the equation, $\mathbf{i} = (0, 1)$ is called the **imaginary unit**. It and its conjugate, $-\mathbf{i}$ satisfy the quadratic equation-which no real number can satisfy,

$$z^2 + \mathbf{1} = \mathbf{0} \tag{1}$$

2. Complex numbers: deductions (2)

- Every complex number $z = (x, y)$ can be written **uniquely** in the form,

$$\begin{aligned} z &= x\mathbf{1} + y\mathbf{i} \\ &= x + \mathbf{i}y \\ x &= \mathrm{Re}(z) \\ y &= \mathrm{Im}(z) \end{aligned}$$

  Henceforth, without incurring confusion, I shall drop the bold-faces on $\mathbf{1}, \mathbf{0}, \mathbf{i}$ and simply use, $1, 0, i$ respectively.

- **Proposition I**: If z_1, z_2 are any two complex numbers, the equation $z_1.z_2 = 0$ can only be true if one or both the factors vanish.

- **Proof**: Follows easily from Theorem 2.1 (convince yourself of this!).

- **Proposition II**: Every equation/identity in which only finite algebraic operations are used and which applies for real variables, also applies to complex variables. This is also a simple consequence of Th. 2.1

2. Complex numbers: deductions (3)

- **Proposition 3: "Lagrange's identity"**: Suppose a_k, b_k, $k = 1, 2, ..n$ are arbitrary complex numbers. The following identity holds:

$$|\Sigma_{k=1}^{n} a_k b_k|^2 + \Sigma_{1\leq k<j\leq n}|a_k \bar{b}_j - a_j \bar{b}_k|^2 \quad = \quad (\Sigma_{k=1}^{n}|a_k|^2)(\Sigma_{k=1}^{n}|b_k|^2)$$

Proof: Consider,

$$\begin{aligned} \Sigma_{1\leq k<j\leq n}|a_k \bar{b}_j - a_j \bar{b}_k|^2 &= \Sigma_{1\leq k<j\leq n}(a_k \bar{b}_j - a_j \bar{b}_k)(\bar{a}_k b_j - \bar{a}_j b_k) \\ &= \Sigma_{1\leq k<j\leq n}(|a_k|^2|b_j|^2 + |a_j|^2|b_k|^2) \\ &\quad -\Sigma_{1\leq k<j\leq n}(a_k \bar{a}_j b_k \bar{b}_j + a_j \bar{b}_k \bar{a}_k b_j) \\ |\Sigma_{k=1}^{n} a_k b_k|^2 &= \Sigma_{k=1}^{n}\Sigma_{j=1}^{n} a_k b_k \bar{a}_j \bar{b}_j \\ &= \Sigma_{k=1}^{n}|a_k|^2|b_k|^2 \\ &\quad +\Sigma_{1\leq k<j\leq n}(a_k \bar{a}_j b_k \bar{b}_j + a_j \bar{b}_k \bar{a}_k b_j) \end{aligned}$$

adding the two identities, we get the stated result.

- An important corollary: **The Cauchy-Schwarz inequality for complex numbers,**

$$|\Sigma_{k=1}^{n} a_k b_k|^2 \leq (\Sigma_{k=1}^{n}|a_k|^2)(\Sigma_{j=1}^{n}|b_j|^2) \tag{2}$$