

BRAUER INVARIANT FOR A UNIVERSAL VECTOR BUNDLE

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ABSTRACT. Let X be a smooth complex projective curve with $\text{genus}(X) > 2$, and let \mathcal{M} be the moduli space parametrizing isomorphism classes of stable vector bundles E over X of rank r with $\bigwedge^r E = \xi$, where ξ is a fixed line bundle. We prove that the Brauer group $\text{Br}(\mathcal{M})$ is $\mathbb{Z}/n\mathbb{Z}$, where $n = \text{g.c.d.}(r, \text{degree}(\xi))$. Moreover, $\text{Br}(\mathcal{M})$ is generated by the class of the projective bundle over \mathcal{M} of relative dimension $r - 1$ obtained by restricting the universal projective bundle over $X \times \mathcal{M}$ to a point of X .

Résumé. Soit X une courbe projective lisse de genre $g(X) > 2$ et soit \mathcal{M} l'espace de modules paramétrant fibrés vectoriels E stables sur X de rang r et ayant déterminant $\bigwedge^r E = \xi$, où ξ est un fibré en droites donné. Nous montrons que le groupe de Brauer $\text{Br}(\mathcal{M})$ est égale à $\mathbb{Z}/n\mathbb{Z}$, où $n = \text{pgcd}(r, \text{deg } \xi)$. De plus $\text{Br}(\mathcal{M})$ est engendré par la classe du fibré projectif sur \mathcal{M} de dimension relative $r - 1$, obtenu par restriction du fibré projectif universel sur $X \times \mathcal{M}$ à un point de X .

1. BRAUER GROUP OF MODULI SPACE

Let X be a connected smooth projective curve defined over \mathbb{C} such that $g(X) := \text{genus}(X) \geq 2$. Fix an integer $r \geq 2$; if $g = 2$, then $r \geq 3$. Fix an algebraic line bundle ξ over X of degree d . Let $\mathcal{M}(r, \xi)$ denote the moduli space parametrizing all isomorphism classes of stable vector bundles E over X of rank r with $\bigwedge^r E \cong \xi$. For notational convenience the variety $\mathcal{M}(r, \xi)$ will simply be denoted by \mathcal{M} . There is a natural universal projective bundle over $X \times \mathcal{M}$ of relative dimension $r - 1$, which we will denote by \mathbb{P} . For any stable vector bundle $E \in \mathcal{M}$ and any point $x \in X$, the fiber of \mathbb{P} over $x \times \{E\}$ is canonically identified with $P(E_x)$, the variety of one dimensional subspaces of E_x . For any closed point $x \in X$, the restriction of \mathbb{P} to $\{x\} \times \mathcal{M}$ will be denoted by \mathbb{P}_x .

Let $\text{Br}(\mathcal{M})$ denote the *cohomological Brauer group* $H^2(\mathcal{M}_{\text{ét}}, \mathbb{G}_m)$ of \mathcal{M} . Let $\beta \in \text{Br}(\mathcal{M})$ be the class of the projective bundle \mathbb{P}_x . Recall that $\text{Pic}(\mathcal{M}) = \text{NS}(\mathcal{M}) = \mathbb{Z}$ by [4].

Remarks 1.1. (i) The class β is the class of the gerbe which gives the obstruction to the existence of a universal vector bundle. For every étale morphism $e : U \rightarrow \mathcal{M}$ let \mathcal{G}_U be the category of vector bundles E on $X \times U$ which are stable of rank r and determinant ξ on the fibers of $X \times U \rightarrow U$ such that $f_E = e$; the morphisms in \mathcal{G}_U are the vector bundle isomorphisms. These categories are the fibers of a fibered category which has the structure of a \mathbb{G}_m gerbe over the étale site of \mathcal{M} . We have an equivalence from \mathcal{G} to the gerbe $b(\mathbb{P}_x)$ defined in [6, V4.8].

(ii) Under our conventions, the cohomology class associated to an Azumaya algebra of degree d on a scheme coincides with the class associated to the Brauer-Severi variety parametrizing rank d right ideals; this agrees with [6] and differs from [1].

Proposition 1.2. (a) *The group $\mathrm{Br}(\mathcal{M})$ is generated by β , (b) the variety \mathcal{M} is simply connected, and (c) the natural homomorphism $\mathrm{Pic}(\mathcal{M}) \rightarrow H^2(\mathcal{M}, \mathbb{Z})$ is an isomorphism.*

Proof. By [4], there is an open subset \mathbb{P}_ξ^s in a projective space together with a surjective morphism $f : \mathbb{P}_\xi^s \rightarrow \mathcal{M}$ satisfying the following condition: for any complex reduced variety Y and a vector bundle E on $X \times Y$, such that for all $y \in Y$, the restriction $E|_{X \times y}$ is a stable vector bundle of rank r with determinant ξ , the natural morphism $Y \rightarrow \mathcal{M}$ can be Zariski locally lifted to a morphism to \mathbb{P}_ξ^s . From [4, p. 89, Proposition 7.13] we have $\mathrm{Pic}(\mathbb{P}_\xi^s) = \mathbb{Z}$, since \mathbb{P}_ξ^s is the complement of a Zariski closed subset of codimension ≥ 2 in a projective space.

Since the projective bundle $\mathbb{P}_x \rightarrow \mathcal{M}$ pulled back to \mathbb{P}_x is associated to a vector bundle, we see that there is a vector bundle \mathcal{E} on $X \times \mathbb{P}_x$ such that for every $y \in \mathbb{P}_x$ the vector bundle $\mathcal{E}|_{X \times y}$ is stable of rank r and determinant ξ , and its isomorphism class corresponds to the image of y in \mathcal{M} . Hence there is a nonempty Zariski open subset U of \mathbb{P}_x and a commutative diagram of maps

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{P}_\xi^s \\ \downarrow & & \downarrow \\ \mathbb{P}_x & \longrightarrow & \mathcal{M} \end{array}$$

On the other hand, for any projective bundle $\tilde{\mathcal{P}}$ over a connected regular scheme Z , by [5, p. 193], there is an exact sequence

$$(1) \quad \mathbb{Z} \cdot \mathrm{cl}(\tilde{\mathcal{P}}) \longrightarrow \mathrm{Br}(Z) \longrightarrow \mathrm{Br}(\tilde{\mathcal{P}}) \longrightarrow 0.$$

Hence the above commutative diagram, together with the facts that $\mathrm{Br}(\mathbb{P}_\xi^s) = 0$ (since \mathbb{P}_ξ^s is the complement of a Zariski closed subset of codimension ≥ 2 in a projective space) and the pullback homomorphism $\mathrm{Br}(\mathbb{P}_x) \rightarrow \mathrm{Br}(U)$ is injective, prove part (a).

Part (a) implies that $\mathrm{Br}(\mathcal{M})$ is a finite group. Hence part (c) follows using [7, p. 145, (8.7)].

Part (b) follows by applying π_1 to the above commutative diagram and observing that \mathbb{P}_ξ^s is simply connected, the homomorphism $\pi_1(U) \rightarrow \pi_1(\mathbb{P}_x)$ is surjective and $\pi_1(\mathbb{P}_x) \simeq \pi_1(\mathcal{M})$. \square

Remark 1.3. From Proposition 1.2(c) it follows that $H^2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. Using the finiteness of $\mathrm{Br}(\mathcal{M})$ together with the Kummer sequence and the comparison of classical and étale cohomology, it follows (cf. [7, p. 146, (8.9)]) that $\mathrm{Br}(\mathcal{M})$ is identified with the torsion subgroup $H^3(\mathcal{M}, \mathbb{Z})_{\mathrm{tor}}$ of $H^3(\mathcal{M}, \mathbb{Z})$.

Let $\mathcal{O}_X(1)$ be a very ample line bundle on X . Then there exists an integer m_0 such that for every integer $m \geq m_0$ and every semistable vector bundle V of rank r and degree d on X , the vector bundle $V(m) = V \otimes \mathcal{O}_X(m)$ is generated by its global sections, and

furthermore, $h^1(V(m)) = 0$. Fix some $m \geq m_0$, and define $q = \dim H^0(X, V(m))$. Using the notation of [4], the universal bundle \mathbb{F} on $X \times R^s$ has a $\mathrm{GL}(q)$ linearization and the projectivization $P(\mathbb{F})$ descends to $X \times \mathcal{M}$, and the descended projective bundle is identified with \mathbb{P} ; see [4, pp. 61–62].

For a $\mathrm{GL}(q)$ -linearized line bundle L on R^s , by $e(L)$ we will denote the integer that satisfies the following condition: the center $\mathbb{C}^* \subset \mathrm{GL}(q)$ acts on L_y , $y \in R^s$, by the character $t \mapsto t^{e(L)}$.

Proposition 1.4. [4, p. 75, Proposition 5.1]. *Set $n = \mathrm{g.c.d.}(r, d)$. Let p be an integer. Then there exists a $\mathrm{GL}(q)$ -linearized line bundle L on R^s such that $e(L) = p$ if and only if p is a multiple of n .*

Lemma 1.5. [1, p. 203, Proposition 4.4(ii)]. *For an everywhere nonzero vector bundle V on a scheme, consider the exterior power representation $\mathrm{PGL}(V) \rightarrow \mathrm{PGL}(\bigwedge^m(V))$, $0 \leq m \leq \mathrm{rank}(V)$. Let β_V be the Brauer class of the projective bundle associated to a principal $\mathrm{PGL}(V)$ -bundle for the standard action on $P(V)$. Then the Brauer class of the associated $\mathrm{PGL}(\bigwedge^m V)$ -bundle is given by $m \cdot \beta_V$.*

Proposition 1.6. *If β is the Brauer class of \mathbb{P}_x in $\mathrm{Br}(\mathcal{M})$, then $n \cdot \beta = 0$, where n is as in Proposition 1.4.*

Proof. We will denote by $\bigwedge^n \mathbb{P}$ the projective bundle over $X \times \mathcal{M}$ associated to the $\mathrm{PGL}(r, \mathbb{C})$ -bundle \mathbb{P} for the natural action of $\mathrm{PGL}(r, \mathbb{C})$ on $P(\bigwedge^n \mathbb{C}^r)$. The restriction of $\bigwedge^n \mathbb{P}$ to $\{x\} \times \mathcal{M}$ will be denoted by $\bigwedge^n \mathbb{P}_x$.

In view of Lemma 1.5, we need to show that $\bigwedge^n \mathbb{P}_x$ is Zariski locally trivial on \mathcal{M} . It is enough to show that $\bigwedge^n \mathbb{P}$ is Zariski locally trivial on $X \times \mathcal{M}$. This is equivalent to showing that the equivariant vector bundle $\bigwedge^n \mathbb{F}$ on $X \times R^s$ descends to a vector bundle after being tensored by a suitable equivariant line bundle. Now by Proposition 1.4, there exists an equivariant line bundle L on R^s such that $e(L) = n$. Further, \mathbb{C}^* acts on $\bigwedge^n \mathbb{F}$ by $t \mapsto t^n$. Hence, the vector bundle $(\bigwedge^n \mathbb{F}) \otimes (p_{R^s})^*(L^*)$ has a trivial action of $\mathbb{C}^* \hookrightarrow \mathrm{GL}(q)$, and consequently it descends to $X \times \mathcal{M}$. This implies that $n \cdot \beta = 0$. \square

Proposition 1.7. *Let $0 < m < n$. Then $m \cdot \beta \neq 0$.*

Proof. Suppose that $m \cdot \beta = 0$. We will get a contradiction. By Lemma 1.5, if $m \cdot \beta = 0$, it follows that $\bigwedge^m \mathbb{P}_x$ is the projectivization of some vector bundle V on $x \times \mathcal{M} = \mathcal{M}$. This implies the equivariant vector bundle $\bigwedge^m \mathbb{F}_x$ on R^s must be the pullback of V tensored with an equivariant line bundle on R^s . Hence there is an equivariant line bundle L on R^s with $e(L) = m$. Since $0 < m < n$ this is a contradiction (see Proposition 1.4). \square

We have proved the following theorem.

Theorem 1.8. *The Brauer group $\mathrm{Br}(\mathcal{M}) = \mathbb{Z}/n\mathbb{Z}$. The Brauer group is generated by the Brauer class of \mathbb{P}_x .*

The above theorem remains valid for $g = r = 2$ using the explicit descriptions of \mathcal{M} in these cases [8].

2. BRAUER GROUP AND STABILITY

Let \overline{M} be an irreducible normal complex projective variety of positive dimension. Fix a very ample line bundle ζ on \overline{M} . A nonempty Zariski open subset U of \overline{M} will be called *big* if the complement $\overline{M} \setminus U$ is of codimension at least two. The smooth locus of \overline{M} , which is a big open subset, will be denoted by M . For any torsionfree coherent sheaf F defined on a big open subset $U \subset M$, the *degree* of F is defined to be the degree of F restricted to the general complete intersection curve obtained by intersecting hyperplanes on \overline{M} from the complete linear system $|\zeta|$.

Let G be a complex reductive group. A principal G -bundle E_G defined over a big open subset $U \subset M$ is called *stable* if for all triples of the form (U', P, σ) , where $U' \subset U$ is a big open subset of M , $P \subset G$ is a proper maximal parabolic subgroup, and $\sigma : U' \rightarrow E_G/P$ is a reduction of structure group of $E_G|_{U'}$ to P , the inequality $\text{degree}(\sigma^*T_{\text{rel}}) > 0$ holds, where T_{rel} is the relative tangent bundle for the natural projection $E_G/P \rightarrow U$; see [9], [2], [3]. A principal bundle defined over a big open subset is called a *rational principal bundle* (see [9], [2]).

Now take $G = \text{PGL}(n, \mathbb{C})$. Let $E_{\text{PGL}(n, \mathbb{C})}$ be a principal $\text{PGL}(n, \mathbb{C})$ -bundle over a big open subset $U \subset M$. The projective bundle over U , of relative dimension $n - 1$, associated to $E_{\text{PGL}(n, \mathbb{C})}$ for the natural action of $\text{PGL}(n, \mathbb{C})$ on the projective space of lines in \mathbb{C}^n will be denoted by E .

Lemma 2.1. *If the order of E in $\text{Br}(M)$ is n , then the principal $\text{PGL}(n, \mathbb{C})$ -bundle $E_{\text{PGL}(n, \mathbb{C})}$ is stable. In fact, E does not admit any reduction of structure group to any proper parabolic subgroup of $\text{PGL}(n, \mathbb{C})$ over any big open subset of M .*

Proof. Any maximal parabolic subgroup of $\text{PGL}(n, \mathbb{C})$ preserves a proper linear subspace of $\mathbb{C}\mathbb{P}^{n-1}$. So a reduction of structure group of $E_{\text{PGL}(n, \mathbb{C})}$ to a maximal parabolic subgroup is given by a linear subbundle of E . Let

$$(2) \quad \mathbb{L}_{U'} \subset E|_{U'}$$

be a linear subbundle over U' of relative dimension d , where $d \in [0, n - 2]$.

Let E_d be the projective bundle over U associated to $E_{\text{PGL}(n, \mathbb{C})}$ for the natural action of $\text{PGL}(n, \mathbb{C})$ on the projective space $P(\wedge^{d+1} \mathbb{C}^n)$ of lines in $\wedge^{d+1} \mathbb{C}^n$. Using the natural embedding of the Grassmannian $\text{Gr}(d, n)$ in $P(\wedge^{d+1} \mathbb{C}^n)$ we see that the Grassmann bundle over U parametrizing d dimensional linear subspaces in the fibers of the projective bundle E is embedded in E_d . Therefore, the subbundle $\mathbb{L}_{U'}$ in (2) gives a section of E_d over U' .

Note that $\text{Br}(U') = \text{Br}(M) = \text{Br}(U)$ as U and U' are both big open subsets of M . Since the order of the class of E in $\text{Br}(U')$ is n , and $d < n - 1$, from Lemma 1.5 we know that the class of E_d in $\text{Br}(U')$ is nonzero. This is in contradiction with the fact that we have a

section of E_d over U' . Therefore, a subbundle as in (2) cannot exist. This completes the proof of the lemma. \square

Recall that under the assumptions of Section 1, \mathcal{M} is the smooth locus of a normal projective variety. From Theorem 1.8 and Lemma 2.1 it follows that the projective bundle \mathbb{P}_x over \mathcal{M} is stable provided the degree d is a multiple of the rank r .

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