AN UNIFYING FRAMEWORK FOR SPHERES AND PROJECTIVE SPACES

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Abstract. In the following article I shall introduce the concept of projective space and then introduce a general set of maps which will unify the study of spheres in any higher dimensions and projective spaces using the construction of Principle Bundles.

1. A General structure for spheres

Generally when we think of *spheres* we generally think of the sphere embedded in 3-dimensional real euclidean space given by the equation $x^2 + y^2 + z^2 = 1$ where x, y, z are labels for the orthogonal Cartesian axis. The most important assumption that goes unsaid is that the parameters x, y, z will take real numbers as their values. In this article we shall get not only out of the dimension constraint i.e we shall define spheres in any-dimension but also out of this constraint to define spheres over real-numbers.

We use the generic label $\mathbb{F}$ for either $\mathbb{R}$ (*The field of real numbers* ,$\mathbb{C}$ (*the field of complex numbers*) and $\mathbb{H}$ (*the ring of quaternions*)

I would like to caution the reader against the possibly misleading use of the symbol $\mathbb{F}$. The reader should notice that not all the things that $\mathbb{F}$ stands for are $fields$.

For each $n \in \mathbb{Z}^+$ let $\mathbb{F}^n = \{\xi = (\xi_1, \xi_2, .., \xi_n) | \xi^i \in \mathbb{F}, i = 1, 2, 3, .., n\}$

1.1. An algebraic structure for $\mathbb{F}^n$.

We define an *algebraic structure for* $\mathbb{F}^n$ by defining certain operations among the elements of$\mathbb{F}^n$ and $\mathbb{F}$ ::

Let ξ, ςbe two elements of$\mathbb{F}^n$ and $a \in \mathbb{F}$

We define the following operations among them::

- Addition between ξ and ς :

$$\xi + \varsigma = (\xi_1, \xi_2, .., \xi_n) + (\varsigma_1, \varsigma_2, .., \varsigma_n) = (\xi_1 + \varsigma_1, \xi_2 + \varsigma_2, .., \xi_n + \varsigma_n)$$

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- Right multiplication by a on elements of $\mathbb{F}^n$:

$$\xi a = (\xi_1, \xi_2, .., \xi_n).a = (\xi_1 a, \xi_2 a, .., \xi_n a).$$

We further postulate that $\mathbb{F}^n$ is an *an abelian group under addition.*

$$\forall \xi, \varsigma \in \mathbb{F}^n anda, b \in \mathbb{F}$$
$$(\xi + \varsigma).a = \xi a + \varsigma a$$
$$\xi.(a + b) = \xi a + \xi b$$
$$\xi(a.b) = (\xi a)b$$
$$\xi 1 = \xi$$

Note :

For $\mathbb{F} = \mathbb{H}$ ($\mathbb{H}$ is non-commutative), $\mathbb{F}$ is a division ring. So $\mathbb{F}^n$ *is a right-module over* $\mathbb{F}$ *or else a vector space .*

Defining a bilinear form

We define a bilinear form on $\mathbb{F}^n$ as ::

$$\langle ; \rangle : \mathbb{F}^n \times \mathbb{F}^n \mapsto \mathbb{F}$$
$$\langle \xi, \varsigma \rangle \mapsto \bar{\xi}_1 \varsigma_1 + \bar{\xi}_2 \varsigma_2 + ...\bar{x}i_n \varsigma_n$$

We note that *conjugate* makes sense for all the things that $\mathbb{F}$ can stand for.The reader can see my article tiled *Quaternions to* S^3 the definition of conjugates for quanternions.

Further we note that this *bilinear form* is *non degenerate* i.e $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

1.2. **A generalized definition for unitary matrices.** We define $U(n, \mathbb{F})$ be all $n \times n$ matrices with entries in $\mathbb{F}$ such that $A^{-1} = A^{\dagger}$.

So we have the following special cases where we can see how the generalized definition boils down to the well known cases.

$$U(n, \mathbb{R}) \equiv O(n) \text{ Orthogonal Matrices}$$
$$U(n, \mathbb{C}) \equiv U(n) \text{ Unitary Matrices}$$
$$U(n, \mathbb{H}) \equiv Sp(n) \text{Symplectic Matrices}$$

1.3. **A remark about the topology.** We give $U(n, \mathbb{F})$ a subspace topology thinking the matrix groups as *subsets* of the large ambient *Euclidean space* that is:

$$O(n) \subseteq \mathbb{R}^{n^2}, U(n) \subseteq \mathbb{R}^{2n^2}$$

2. Defining the sphere

Let S be a topological subspace of $\mathbb{F}^n$ given by

$$S = \{\xi \in \mathbb{F}^n | \langle \xi, \xi \rangle = 1\}$$

.

This topological subspace S will be called a *Sphere*

We recall that in $\mathbb{R}^n$ the subspace defined by the coordinate expression $x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 = 1$ is denoted as S^{n-1} ($(n-1)$ indicates that the surface defined by the above constraint equation is a $n-1$ manifold). Its easy to see how the discrete 2 point set $\{1, -1\}$ is S^0 (also referred to as the abelian group under multiplication $\mathbb{Z}_2$) , our well-known *circle* is S^1 and what is commonly referred to as the sphere is S^2

Finally we note here *without proof* the following special cases of the general definition of spheres that we have given above ::

$$\text{If } \mathbb{F} = \mathbb{R} \text{ then } S \overset{Homeo}{\cong} S^{n-1}$$
$$\text{If } \mathbb{F} = \mathbb{C} \text{ then } S \overset{Homeo}{\cong} S^{2n-1}$$
$$\text{If } \mathbb{F} = \mathbb{H} \text{ then } S \overset{Homeo}{\cong} S^{4n-1}$$

3. Defining Projective Spaces

In this section we will be working on $\mathbb{F}^n$ where $n \geq 2$.

Let 0 be the *zero element* of $\mathbb{F}^n$.

Then on the topological subspace $\mathbb{F}^n \setminus \{0\}$ we define the following equivalence relation::

Let ξ and ς be two elements of $\mathbb{F}^n \setminus \{0\}$. Then we declare ξ and ς to be *equivalent* ($\xi \sim \varsigma$) if $\exists$ a non-zero$a \in \mathbb{F}$ such that $\varsigma = \xi a$.

We denote the *equivalence class* in which ξ lies as ::

$$[\xi] = \{\xi a | a \in \mathbb{F} \setminus \{0\}\}$$

If $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ then these *equivalence classes* are called *Complex (resp. Quaternionic) lines through the origin.* But note that they *do not* include the origin.If $\mathbb{F} = \mathbb{R}$ then these are our well known straight lines through the origin but without the origin.

Hence we define the $n-1$ *dimensional projective space over F* as ::

$\mathbb{FP}^{n-1} = [\xi] | \xi \in \mathbb{F}^n \setminus \{0\}$, where the equivalence class $[\xi]$ is defined above

At this point the $\mathbb{FP}^{n-1}$ is just a *set* of equivalence classes as defined above. We can endow it with a differential manifold structure.But in the following sections we will implicitly use the projective space with its topology so that we can consistently refer to continuous functions which have a projective-space as its range.

4. Spheres through projective spaces

Now we have defined all the required spaces i.e S , $\mathbb{F} \setminus \{0\}$,$\mathbb{FP}^{n-1}$ and the bilinear form on $\mathbb{F}^n$ has also been defined. Now we will define the following maps labelled as i , η , Q and P , whose domains and ranges are as follows::

$$S \overset{i}{\hookrightarrow} \mathbb{F} \setminus \{0\} \overset{\eta}{\longrightarrow} S \overset{P}{\longrightarrow} \mathbb{FP}^{n-1}$$
$$and$$
$$\mathbb{F} \setminus \{0\} \overset{Q}{\longrightarrow} \mathbb{FP}^{n-1}$$

The definitions for the maps are as follows::

$$i \text{ is the inclusion map of} S \text{ in } \mathbb{F} \setminus \{0\}$$
$$Q : \mathbb{F} \setminus \{0\} \longrightarrow \mathbb{FP}^{n-1}$$
$$\xi \mapsto [\xi]$$
$$and$$
$$P = Q|_S$$

5. Indicative proofs for the structure of the fibres for the 3 possibilities

As a result of the analysis of the last section we have the following basic structure::

$$S \xrightarrow{P} \mathbb{FP}^{n-1}$$

where $\mathbb{F}$ can be $\mathbb{H}, \mathbb{C}$ or $\mathbb{R}$ and

$$S = \{z \in \mathbb{F} | \langle z, z \rangle = 1\}$$

The fibre above the equivalence class represented by z_0 is given by

$$P^{-1}([z_0]) = S \cap \{z_0 a | a \in \mathbb{F} \setminus \{0\}$$

Equivalently

$$P^{-1}([z_0]) = \{z_0 a | a \in \mathbb{F} \text{ and } |a| = 1$$

In following 3 subsections I will give a sketch of the structure of the fibres for the 3 cases when $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$

5.1. $\mathbb{F} = \mathbb{R}$. When $\mathbb{F} = \mathbb{R}$ then $S \cong S^{n-1}$. Then $a \in \mathbb{R}$ and $|a| = 1$ implies $a = 1, -1$.

So

$$P^{-1}([x]) = x, -x \cong S^0$$

In other words in the fibres the *anti-podal points* of S^{n-1} has been *identified*. We claim without proof that this has the structure of a **Principle Bundle** and will be represented as::

$$S^0 \hookrightarrow S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$$

5.2. $\mathbb{F} = \mathbb{C}$. When $\mathbb{F} = \mathbb{C}$ then $S \cong S^{2n-1}$. Then $a \in \mathbb{C}$ and $|a| = 1$ implies $a \in S^1$.

So

$$P^{-1}([z_0]) = \{(z_{01}a, z_{02}a, ..., z_{0n}a) | a \in S^1\} \cong S^1$$

We defer the continuity argument to section 5.2.1 and claim that this also has the structure of a **Principle Bundle** and will be represented as::

$$S^1 \hookrightarrow S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$$

5.2.1. *The issue of continuity.* To see the continuity of the map P let us see that the map Q is continuous since each its coordinates are polynomials in the domain variables and hence the requirement is satisfied since P is a restriction of Q to a topological subspace(i.e $S^(2n-1)$) of its domain and hence continuous.

We identify $\mathbb{C}^n$ to $\mathbb{R}^{2n}$. So,

$$z \in \mathbb{C}^n = (z_1, z_2, ..., z_n) = (x_1 + iy_1, x_2 + iy_2, ..., x_n + iy_n)$$

In $\mathbb{C}^n \setminus \{0\}$ there exists a z such that $z_j \neq 0$.Further in S^{2n-1} there exists a z_0 such that $z_{0j} \neq 0$.Let that z_{0j} be equal to $\alpha + i\beta$.

Then the map $Q : \mathbb{C}^n \longrightarrow \mathbb{C}$ maps $z \longrightarrow \frac{z_j}{z_{0j}} = (\frac{\alpha x_j + \beta y_j}{\alpha^2+\beta^2}, \frac{\alpha y_j - \beta x_j}{\alpha^2+\beta^2})$.

Hence it is easy to see that,

$$Q|_{P^{-1}[z_0]}(z_{01}a, z_{02}a, ..., z_{0n}a) \longmapsto \frac{z_{0j}a}{z_{0j}} = a$$

and hence continuous.

5.3. $\mathbb{F} = \mathbb{H}$. When $\mathbb{F} = \mathbb{H}$ then $S \cong S^{4n-1}$. Then $a \in \mathbb{H}$ and $|a| = 1$ implies $a \in S^3$ (In my earlier writing on **Quaternions** this was shown

Here the construction of the maps Q and P are similar to the one in the case of $\mathbb{C}$ and we claim without proof that here too the structure is of a **Principle bundle** and it will be represented as::

$$S^3 \hookrightarrow S^{4n-1} \longrightarrow \mathbb{HP}^{n-1}$$

6. A final comment

So we see that in a very beautiful way we have been able to tie together the structure of the higher-dimensional spheres ,projective spaces ,real numbers ,complex numbers and quaternions through the construction of the following 3 **Principle Bundles** ::

$$S^0 \hookrightarrow S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$$

$$S^1 \hookrightarrow S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$$

$$S^3 \hookrightarrow S^{4n-1} \longrightarrow \mathbb{HP}^{n-1}$$

We note without proof that ::

$$\mathbb{RP}^1 \cong S^1$$

$$\mathbb{CP}^1 \cong S^2$$

$$\mathbb{HP}^1 \cong S^4$$

Hence we have the following beautiful principle bundles where the fibre , the base-space and the total-space are all spheres ::

$$S^0 \hookrightarrow S^1 \longrightarrow S^1$$

$$S^1 \hookrightarrow S^3 \longrightarrow S^2$$

$$S^3 \hookrightarrow S^7 \longrightarrow S^4$$

We note for record that over *Octonians* we can get another such Principle Bundle whose base-space,total-space and fibre are all spheres and that is

$$S^7 \hookrightarrow S^{15} \longrightarrow S^8$$

.In this context we note that Adam's theorem states that these are the only 4 such Principle Bundles with such a property.

In a following article I shall show a detailed analysis of the structure of S^3 as a simple case of *Heegard Splitting* using the *Clifford Torus* and the construction of the principle bundle $S^1 \hookrightarrow S^3 \longrightarrow S^2$