A SHORT NOTE ON MOTIVATIONS TOWARDS TORSION FREE CONNECTIONS AND METRIC COMPATIBLE CONNECTIONS ON A RIEMANNIAN MANIFOLD THROUGH DIFFERENTIAL FORMS

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Abstract. In the following article we shall give a plausibility argument for the definitions of a Torsion free and Metric compatibility of a connection on a Riemannian manifold using the concept of differential forms on the Riemannian manifold with values in a vector bundle.

1. Some basic definitions

In this article we shall be working on a *rank n Vector Bundle over a Riemannian manifold M* whose *Total-space* $= E$ and the *Projection Map* being called π.The connection shall be a map ::

$$\nabla : \Gamma(E) \times \Gamma(TM) \mapsto \Gamma(E)$$

or alternatively

$$\nabla : \Gamma(E) \mapsto \Gamma(E) \otimes \Gamma(T^*M)$$

or if $\mu \in \Gamma(E)$ then

$$\nabla\mu \in \Gamma(E \otimes T^*M)$$

satifying the standard onditions of tensoriality in the entry from $\Gamma(E)$ and the entry from $\Gamma(TM)$ and the leibnitz like property of differentiation.Ofcourse it must be noted that unless E too is TM linearity in the entry from TM doesnt make sense.

To give the motivations for a **Torsion free connection** and **Metric connection** 2 major concepts will be required which I shall define here. They are ::

- Degree p differential forms over M with values in the vector bundle E.
- de-Rahm derivative of a vector bundle valued differential form as above.

1.1. Degree p differential forms with values in a rank n vector bundle E over a Riemannian manifold M.

Let $q \in M$ and E_q be the fiber in E over q.Let ω be a differential form of degree p over M with values in the vector-bundle E.Then ω_q (the value at the form at q) is an *alternating map* from ::

$$T_qM \times T_qM \times ..(p - times).. \times T_qM \longrightarrow E_q$$

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Or globally ::

$$\omega \in \Gamma(\wedge^p(T^*M) \otimes E)$$

1.1.1. **2 immediate observations**.

(1) When $E = \mathbb{R}$ then $\omega \in \Gamma(\wedge^p(T^*M) \otimes \mathbb{R})$ and such a section will be called a **scalar p-differential form**.In short it is also said in another form ::

$$\Omega^p(M) = \Gamma(\wedge^p M)$$

and

$$\omega \in \Omega^p(M)$$

(2) A 0-*differential form with values in* E is $\in \Gamma(E)$,that is a *Vector Field*.

1.2. **de-Rahm derivative of a vector-bundle valued differential form.** We note that the $De - Rahm$ derivative for differential forms is *derived* or is dependent on the *connection* constructed on the vector bundle and it will be denoted as d_∇ where ∇ is the connection on our vector bundle as defined at the start of the article.

Let ω be a *p-differential form* with values in E that is

$$\omega \in \Gamma(\wedge^p(T^*M) \otimes E)$$

Then d_∇ is a map as follows ::

$$d_\nabla : \Gamma(\wedge^p(T^*M) \otimes E) \longrightarrow \Gamma(\wedge^{p+1}(T^*M) \otimes E)$$

such that if $X_1, X_2, X_3 ... X_{p+1} \in \Gamma(E)$ then ::

$$d_\nabla \omega(X_1, X_2, X_3 ... X_{p+1}) = \sum_i (-1)^{i+1} \nabla_{X_i}(X_1, X_2, X_3 .., \hat{X}_i, ..X_{p+1}) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, .., \hat{X}_i, .., \hat{X}_j, .., X_{p+1})$$

where the vector fields with a caret on them are the ones to be omitted and the $[,]$ is the *Lie Bracket*.

1.2.1. **When is the de-Rahm derivative equal to the connnection?** We had said that the $de - Rahm$ *derivative* is dependent on the *connection*.Here we also note the case when the $de - Rahm$ *derivative* and the *connection* coincide.

If σ is a 0-*differential form with values in* E. Then $\sigma \in \Gamma(E)$ and $\wedge^1(T^*M) \otimes E = T^*M \otimes E$.So ::

$$d_\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$
$$\sigma \mapsto d_\nabla \sigma$$

If$X \in \Gamma(E)$ (a vector field) then by the definition of d_∇ we have ::

$$d_\nabla \sigma(X) = \nabla_X \sigma$$

Hence on 0-*differential form with values in* E the $De - Rahm$ derivative is the same as a connection.

2. 2 IMPORTANT LIFTS OF THE CONNECTION

There are various standard lifts of the de-Rahm derivative to various associated bundles of the vector bundle of interest.We here note 2 of them since they will be important for us in our analysis of the condition of metric compatibility. We must note that some of the lifts work when the $E = TM$ whereas some work in a more general setting.

(1) Lift ∇ to $TM \otimes TM$

Explicitly at every point q on M an element of $TM \otimes TM$ is a multi-linear map :

$$T_q^* M \times T_q^* M \longrightarrow \mathbb{R}$$

We note that basis elements of $\Gamma(TM \otimes TM)$ can be constructed by tensoring 2 elements from $\Gamma(TM)$.So defining ∇ on such a product will be sufficient to specify ∇ on $TM \otimes TM$.

Let $\mu_1, \mu_2 \in \Gamma(TM)$ and $Z \in T_q M$. Then $\mu_1 \otimes \mu_2 \in \Gamma(TM \otimes TM)$.Note that in the following definition we do not distinguish between the connection on $TM \otimes TM$ and the one on TM, the context should make it clear.We define the connection on $TM \otimes TM$ at the point q as ::

$$\nabla_Z(\mu_1 \otimes \mu_2) = \nabla_Z \mu_1 \otimes \mu_2 + \mu_1 \otimes \nabla_Z \mu_2$$

(2) Lift ∇ to E^*

Let ∇ be the connection on E , and $X \in T_q M$, $\sigma \in \Gamma(E^*)$, $v \in E_q$. Further $\tilde{v} \in \Gamma(E)$ such that $\tilde{v}_q = v$.

Let $\langle , \rangle$ be the bilinear pairing between the dual of a vector-space and the vector-space given by the natural evaluation of the homomorphisms on the vectors.

Then we want that the connection on E^* should respect this bilinear pairing on every fibre. Which is equivalent to demanding the following equation to hold ::

$$X\langle \sigma, \tilde{v} \rangle = \langle \nabla_X \sigma, v \rangle + \langle \sigma, \nabla_X \tilde{v} \rangle$$

Hence *globally* we want that if $\tau \in \Gamma(E)$ and $\omega \in \Gamma(E^*)$ and $X \in T_q M$ then the connection on E^* should satisfy the following equation (the context should make it clear as to which ∇ is on which vector-bundle) ::

$$X\langle \omega, \tau \rangle = \langle \nabla_X \omega, \tau \rangle + \langle \omega, \nabla_X \tau \rangle$$

3. Torsion free connection

From this section we shall assume that E is the *Tangent Bundle*, that is $E = TM$.

Let I be the *vector valued differential* 1-*form with values in* TM.Let $q \in M$ and let I be defined by the following identity map ::

$$I : T_qM \longrightarrow T_qM$$
$$v \mapsto v$$

Let ∇ be the connection on TM then ::

$$d_\nabla I \in \Gamma(\wedge^2(T^*M) \otimes TM)$$

So by definition of the $De - Rahm$ derivative we have for vector fields X and Y ::

$$d_\nabla(I)(X,Y) = (-1)^{1+1}\nabla_X I(Y) + (-1)^{2+1}\nabla_Y I(X) + (-1)^{1+2} I([X,Y])$$
$$\Longrightarrow d_\nabla I(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Now we feel that this particular $0 - differential\ form$,I is *like* a constant function which maps every tangent vector in every tangent space to itself. So we demand that the *derivative* of this map be $= 0$.

Hence the above demand amounts to the torsion free condition on the connection ∇ that ::

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

4. Metric compatible connection

Let g be the metric on the the Riemannian manifold M. Here we are looking at the TM over M.Then we realize that $g \in \Gamma(T^*M \otimes T^*M)$.

We define the natural bilinear pairing between $T^*M \otimes T^*M$ and $TM \otimes TM$ through the following evaluation where $\mu_i \in \Gamma(TM)$ for $i = 1, 2, 3, 4$ ::

$$\langle \mu_1^* \otimes \mu_2^*, \mu_3 \otimes \mu_4 \rangle = \mu_1^*(\mu_3)\mu_2^*(\mu_4)$$

Let $g = g_{ij}dx^i \otimes dx^j$ where $dx^i \in \Gamma(T^*M)$ such that on every fibre of T^*M it gives the dual basis to the fibre of TM at the same point.

Let $X, Y, Z \in T_qM$(context should make it clear where we are using the same symbol to label its local extension to a vector field). Then by using the 2 lifts of the connections as explained in the section 2 we have ::

$$Z\langle g_{ij}dx^i \otimes dx^j, X \otimes Y\rangle = \langle \nabla_Z(g_{ij}dx^i \otimes dx^j), X \otimes Y\rangle + \langle g_{ij}dx^i \otimes dx^j, \nabla_Z(X \otimes Y)\rangle$$

$$= \langle \nabla_Z(g_{ij}dx^i \otimes dx^j), X \otimes Y\rangle + \langle g_{ij}dx^i \otimes dx^j, \nabla_Z X \otimes Y\rangle + \langle g_{ij}dx^i \otimes dx^j, X \otimes \nabla_Z Y\rangle$$

So on rearranging we get ::

$$\langle \nabla_Z(g_{ij}dx^i \otimes dx^j), X \otimes Y\rangle = Z\langle g_{ij}dx^i \otimes dx^j, X \otimes Y\rangle - g_{ij}dx^i(\nabla_Z X)dx^j(Y) - g_{ij}dx^i(X)dx^j(\nabla_Z Y)$$

We can now identify the terms on the right hand side as evaluation of the metric on various pairs of vectors in the space T_qM and hence we get ::

$$\langle \nabla_Z(g_{ij}dx^i \otimes dx^j), X \otimes Y\rangle = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

The left hand side of the above equation is clearly the evaluation of $\nabla_Z g$ on an element of $T_qM \otimes T_qM$.

Hence if we demand that in some sense the *derivative* of the metric is 0 with respect to any arbitrary vector in any tangent space is zero , the effectively we are demanding that $\nabla_Z g$ is identically 0 for any Z. In other words $\nabla_Z g$ if identically 0 will evaluate any $X \otimes Y$ to 0.

Hence the above demand translates to the equation given below to be satisfied by the metric g and the connection ∇ on TM for any arbitrary vector fields (or vectors , as the context will imply) $X, Y or Z$ ::

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Hence the condition of metric compatibility of the connection on TM

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