

# QUATERNIONS TO $S^3$

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ABSTRACT. In the following article I shall give a brief introduction to the concept of *Quaternions* and would show how the study of  $S^3$  can be approached through it.

## 1. A DIFFERENT STRUCTURE FOR $\mathbb{R}^4$

In this section we define a multiplicative structure on  $\mathbb{R}^4$ . To do this we write the elements of  $\mathbb{R}^4$  as  $2 \times 2$  matrices with entries in  $\mathbb{C}$ . The set of matrices is defined as

$$\left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$$

$$\alpha = y_0 + iy_1, \beta = y_2 + iy_3; y_0, y_1, y_2, y_3 \in \mathbb{R}$$

Expressed as above we name  $\mathbb{R}^4$  as  $R^4$ .

We can observe that ::

$$y = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} = y_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

We name the matrices as ::

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Hence the set of vectors  $\{I, i, j, k\}$  form a basis for  $\mathbb{R}$ . There exists a natural isomorphism from  $R^4 \rightarrow \mathbb{R}^4$  ::

$$(y_0 I + y_1 i + y_2 j + y_3 k) \mapsto (y_0, y_1, y_2, y_3)$$

Under the metric  $\langle A, B \rangle = \text{Tr}(A^\dagger B)$  ( $A^\dagger$  is the conjugate-transpose of the matrix  $A$ ) we see that the above basis is orthogonal.

$$\Rightarrow |y|^2 = y_0^2 + y_1^2 + y_2^2 + y_3^2 = \det \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

## 2. SOME DECLARATIONS AND OBSERVATIONS

- We declare that a subset  $U$  of  $R^4$  is open iff its image under the natural isomorphism as shown above is open in  $\mathbb{R}^4$ .
- $R^4$  is closed under sums, real scalar multiplications, matrix multiplications and has inverse for each of its *non-zero* elements.
- Under matrix multiplication the *non-zero* elements of  $R^4$  form a *non-abelian* group.

## 3. THE COMMUTATIONS RELATIONS

The commutation relations among the basis elements  $\{I, i, j, k\}$  are as follows::

- $i^2 = j^2 = k^2 = -I$
- $ij = -ji = k$
- $jk = -kj = i$
- $ki = -ik = j$

## 4. THE ABSTRACTION

Thus what we have given above is a concrete representation of the *Algebra of Quaternions*  $H$  defined as the *4 dimensional real vector space* on which is defined a multiplication  $(x, y) \mapsto xy$  which satisfies the following distributive laws for all  $x, y, z \in H$  and  $a \in \mathbb{R} ::$

$$\begin{aligned}(xy)z &= x(yz) \\ x(y+z) &= xy + yz \\ (x+y)z &= xz + yz \\ a(xy) &= (ax)y = x(ay)\end{aligned}$$

and in which there exists a distinguished basis  $\{I, i, j, k\}$  which satisfies ::

$$\begin{aligned}i^2 &= j^2 = k^2 = -I \\ ij &= -ji = k, jk = -kj = i, ki = -ik = j \\ Ix &= xI = x\end{aligned}$$

where

$$x = x_0I + x_1i + x_2j + x_3k, x_0, x_1, x_2, x_3 \in \mathbb{R}$$

In this algebra we define the following for any  $x$  as defined above ::

- $Re(x) = x_0$
- $Im(x) = x_1i + x_2j + x_3k$
- Imaginary quaternions =  $\{x_1i + x_2j + x_3k | x_0, x_1, x_2, x_3 \in \mathbb{R}\}$
- $\bar{x}$  =conjugate of  $x = x_0I - (x_1i + x_2j + x_3k)$

## 5. SOME OBSERVATIONS

- If  $x$  is thought of as an element of  $\mathbb{R}$  then  $\bar{x} = x^\dagger$ .
- if  $y \in H$  then  $y^{-1} = \frac{\bar{y}}{|y|^2}$ .
- The subset of  $H$  of the form  $x = x_0I + x_1i$  is also closed under multiplication i.e it is a sub-algebra of  $H$  and  $i^2 = -I$ . This subalgebra is naturally isomorphic to the usual algebra of  $\mathbb{C}$

6.  $SU(2)$  RESIDES INSIDE  $\mathbb{R}^4$ 

Let  $x \in H$  and  $|x| = 1$  i.e the *unit quaternions*. We observe that this is closed under multiplication. We have the following further observations::

- $|xy| = |x||y|$
- $|x| = 1$   
 $\Rightarrow |x^{-1}| = \left| \frac{\bar{x}}{|x|^2} \right| = |\bar{x}| = |x| = 1$

Based on the above observations we can call the set  $\{|x| = 1, x \in H\}$  as the *the group of unit quaternions*

So in the matrix model  $R^4$  of  $H$  we have for the above group ::

$$|x|^2 = \det(x) = 1$$

$$x = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \text{ and } x^{-1} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

Since  $\det(x) = 1 \Rightarrow x^{-1} = x^\dagger$ , what is usually called the group  $SU(2)$

$$\Rightarrow SU(2) \subseteq R^4$$

**So the unit quaternions in  $H$  are precisely  $SU(2)$**

## 7. THE CONCLUSION

For the moment we accept without explanation that the unit radius sphere in 4 – *dimensional* space i.e  $S^3$  ( or is it obvious :- ) is *homeomorphic* to  $SU(2)$ . Therefore from the above discussion we conclude that::

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{\text{natuarally isomorphic}} & R^4 \text{ or } H \\ S^3 & \xrightarrow{\text{homeomorphically onto}} & \text{some subspace of } \mathbb{R}^4 \end{array}$$

We observe that homeomorphisms preserve norm and hence we can write ::

$$S^3 \xrightarrow{\text{homeo}} \cong SU(2) \subseteq R^4 \text{ or } H$$