$\to$ Hopf was studying continuous maps between spheres. $\quad i^2 = 1\}$

$\to$ $S^1 = \{ e^{i\theta} : \theta \in \mathbb{R} \}$ $\qquad , x_2 + i y_2)$

S^1 is closed under multiplication

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$$(e^{i\theta})^{-1} = e^{i(-\theta)}$$

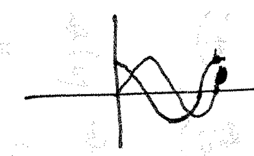
$1\}.$

an abelian group

topological theory $\to$ they are restriction of smooth maps for $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ + $\mathbb{C}-0 \to \mathbb{C}-0$.

$S^1 \cong U(1)$

S^1 is a Lie group.



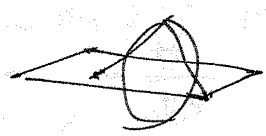
$\cos(\frac{\phi}{2})$, $x_2 = \sin(\frac{\phi}{2})$

$\to$ S^2

$U_S = S^2 - (0,0,1)$, $U_N = S^2 - (0,0,-1)$

$\phi_S : U_S \to \mathbb{R}^2$

$$\phi_S(p) = \left(\frac{p_1}{1-p_3}, \frac{p^2}{1-p^3} \right)$$



ϕ_S is continuous, 1to1 & has an inverse.

$\phi_S^{-1} : \mathbb{R}^2 \to U_S$

$$\phi_S^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$$

$$= \left(\frac{z+\bar{z}}{z\bar{z}+1}, \frac{z-\bar{z}}{i(z\bar{z}+1)}, \frac{z\bar{z}-1}{z\bar{z}+1} \right)$$

ϕ_S is a homeomorphism.

ϕ_S is also a ~~diffeomorphism~~ C^∞ for $\mathbb{R}^2 \to \mathbb{R}^3$

ϕ_S is a restriction to U_S ~~of~~ of a (diffeomorphism) (C^∞ both + C^∞ inverse).

identifying x-y plane with $\mathbb{C}$ + adjoin to it a pt at infinity $\mathbb{C} \cup \{\infty\}$ {1 point compactification}

$\phi_S^* : S^2 \to \mathbb{C}^*$ // So S^2 is the $\underline{1}$-point compactification of $\mathbb{C}$ (1,4)

$\hookrightarrow$ Rieman Sphere in other geos

If $p \in U_S \cap U_N$ $\quad \phi_S(p) = z, \ \phi_N(p) = \frac{1}{\bar{z}}$

if $p \in U_S \cap U_N$

$$\phi_S(p) = z = \frac{x}{\ } + i\,\frac{y}{\ }$$

$$\phi_N(p) = \frac{1}{x-iy} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{y}{x^2+y^2}\right)$$

$$\phi_S(p) - \phi_N(p) = \left(x - \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$$

$$\nabla f = \left(x\left(1-\frac{1}{x^2+y^2}\right),\ y\left(1-\frac{1}{x^2+y^2}\right)\right)$$

$$\frac{\partial f}{\partial x} = x - \frac{x}{x^2+y^2}, \quad \frac{\partial f}{\partial y} = y - \frac{y}{x^2+y^2}$$

$$\frac{1}{2}\int \frac{2x}{x^2+y^2}\,dx = \frac{1}{2}\ln(x^2+y^2) + g(y)$$

$$f = \frac{x^2}{2} - \frac{1}{2}\ln(x^2+y^2) + g(y)$$

$$\frac{\partial f}{\partial y} = -\frac{y}{x^2+y^2} + \frac{dg}{dy} = y - \frac{y}{x^2+y^2}$$

$$\Rightarrow \quad g = \frac{y^2}{2} + c$$

$$f = \frac{1}{2}\left\{(x^2+y^2) - \ln(x^2+y^2)\right\} + k$$

$$\boxed{\phi_S(p) = \phi_N(p) + \nabla f}$$

a ... differ by a ~~change of gauge~~.

* Consider the,

$K_1 =$

$\cos \frac{\ }{2}$

2) $\frac{\pi}{4}$

i) $\frac{\phi}{2} = \frac{\pi}{4}$

ii) $\frac{\phi}{2} = \frac{\pi}{2}$

S^3 :- $\{ (p_1, p_2, p_3, p_4) \mid p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1 \}$

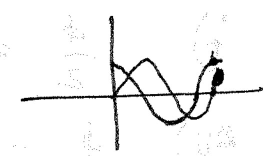
we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ as follow.

$$(x_1, y_1, x_2, y_2) \longleftrightarrow (x_1 + iy_1, x_2 + iy_2)$$

$$\therefore S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}.$$

$\longrightarrow$ we parametrize S^3 as follows:

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$



$r_1^2 + r_2^2 = 1$ & $r_1, r_2 \geq 0$ but not simultaneously $= 0$

$\rightarrow \exists$ some $\phi \in [0, \pi]$ s.t $r_1 = \cos(\frac{\phi}{2})$, $r_2 = \sin(\frac{\phi}{2})$

$$S^3 = \{ (\cos\tfrac{\phi}{2} e^{i\theta_1}, \sin\tfrac{\phi}{2} e^{i\theta_2}) \mid 0 \leq \phi \leq \pi, \theta_1, \theta_2 \in \mathbb{R} \}$$

By some stereographic projection we can thy of S^3 too as 1 pt compactification of R^3 i.e $(R^3)^* = R^3 \cup \{\infty\}$
but we shall NOT do so.

Consider a subset T of S^3 as follow

$$T = \{ (z_1, z_2) \in S^3 / |z_1| = |z_2| \}$$

$|z_1|^2 + |z_2|^2 = 1$, $|z_1| = |z_2|$ i.e $\frac{\phi}{2} = \frac{\pi}{4}$

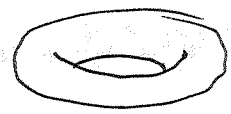
$2|z_1|^2 = 1 \Rightarrow |z_1| = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

$$= \{ (\tfrac{\sqrt{2}}{2} e^{i\theta_1}, \tfrac{\sqrt{2}}{2} e^{i\theta_2}) : \theta_1, \theta_2 \in \mathbb{R} \}$$ $\rightarrow$ 1.3 used

$\downarrow$

a torus in $\mathbb{R}^3$. $\subset S^3$

QED

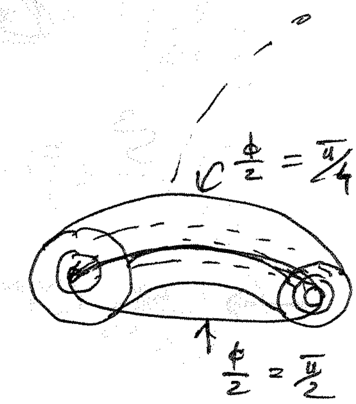


* Consider the subset

$$K_1 = \{ (z_1, z_2) \in S^3 \mid |z_1| \leq |z_2| \}$$

$\cos\frac{\phi}{2} \leq \sin\frac{\phi}{2}$

2) $\frac{\pi}{4} \leq \frac{\phi}{2} \leq \frac{\pi}{2}$



i) $\frac{\phi}{2} = \frac{\pi}{4}$ gives the torus $T \subseteq K_1 \subseteq S^3$

ii) $\frac{\phi}{2} = \frac{\pi}{2} \Rightarrow (0, e^{i\theta_2}) \in S^3$ a copy of $\{0\} \times S^1 \subset S^3$

iii) any fixed $\frac{\phi}{2} \in (\frac{\pi}{4}, \frac{\pi}{2})$ gives a torus.

So K_1 is a solid torus with boundary as T

+ and as $\frac{\phi}{2}$ increases fr. $\frac{\pi}{4}$ to $\frac{\pi}{2}$ it collapses to the central circle.

* Consider the subset

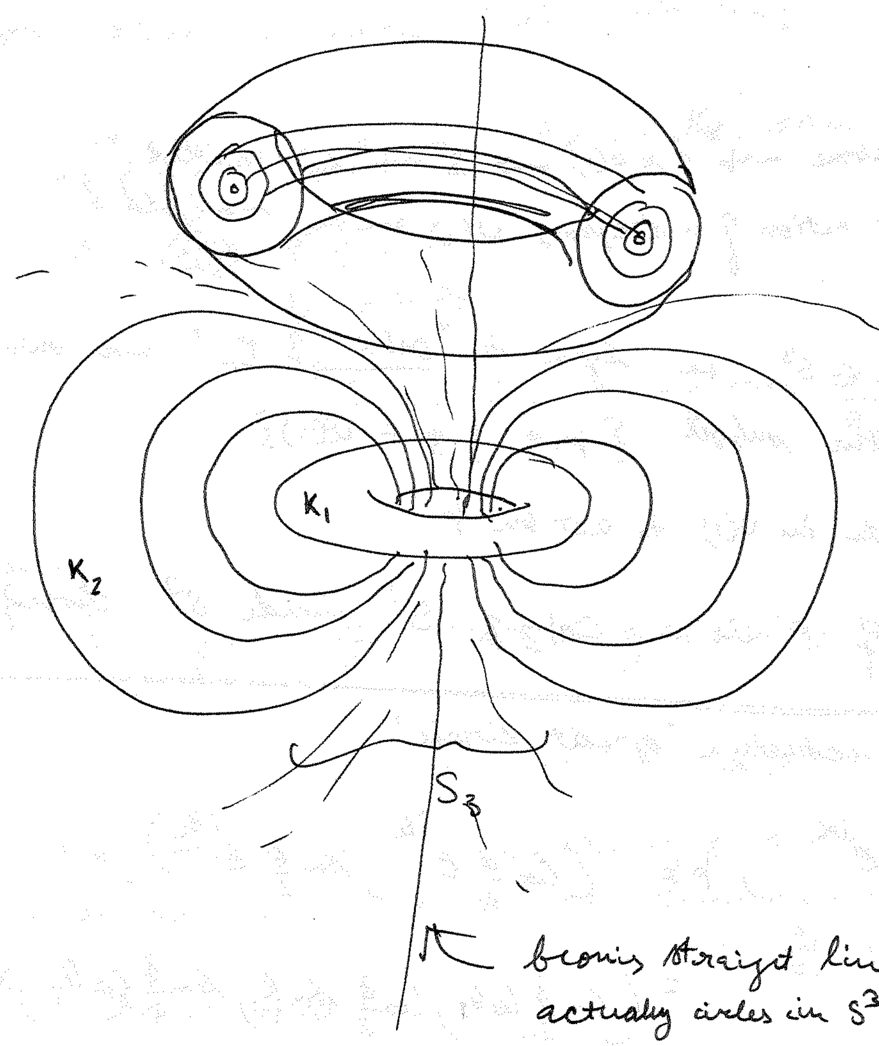
$$K_2 = \{ (z_1, z_2) \in S^3 \mid |z_1| \geq |z_2| \}$$

$0 \leq \frac{\phi}{2} \leq \frac{\pi}{4}$

another set of solid tori collapsing to K_1

+ bounded by T

$\therefore \quad S^3 = K_1 \cup K_2$



↖ becomes straight line in R^3 which are actually circles in S^3 through the point at ∞

Action of S^1 on S^3

Let $p = (z_1, z_2) \in S^3$ & $g \in U(1)$ which is S^1

S^1 now acts as the "gauge group."

we define a right action of S^1 on S^3 as:

$p \cdot g = (z_1, z_2) \cdot g = (z_1 g, z_2 g) =$

→ it is obvious that $p \cdot g$ is also in S^3

→ we look at the map $S^3 \times U(1) \to S^3$, $(p, g) \longmapsto pg$ is C^∞

→ e be the identity element of $U(1)$ i.e. e^{i0}

→ these qualify the above map $S^3 \times U(1) \to S^3$ to be called a "C^∞ right action of a Lie group $U(1)$ on the manifold S^3"

→ for a fixed $p \in S^3$ we define the "orbit of p" under this action to be the subset $\{p.g \mid g \in U(1)\}$ by letting everything in $U(1)$ to act on p.

→ so the orbit of p is a copy of S^1 inside S^3 through p.

Each orbit is actually a "great circle"

$$g = e^{i\alpha}, \quad p = \left(\cos\tfrac{\phi}{2} e^{i\theta_1}, \sin\tfrac{\phi}{2} e^{i\theta_2}\right)$$

$$= \left(\cos\tfrac{\phi}{2}\cos\theta_1, \cos\tfrac{\phi}{2}\sin\theta_1, \sin\tfrac{\phi}{2}\cos\theta_2, \sin\tfrac{\phi}{2}\sin\theta_2\right)$$

$p.e^{i\pi} \to -p$

so antipodal pts are passed thus

$$\left(-\cos\tfrac{\phi}{2}\cos\theta_1, -\cos\tfrac{\phi}{2}\right.$$

$$p = \left(-\cos\tfrac{\phi}{2} e^{i\theta_1}, -\sin\tfrac{\phi}{2} e^{i\theta_2}\right)$$

$$e^{i\pi}$$

$p_1 \cdot g_1 = p_3 = p_2 \cdot g_2$

$p_1 \cdot (g_1 g_2^{-1}) = p_2 \quad \therefore p_2 \in$ orbit of p_1

$\therefore$ the Orbits are disjoint or identical

$\therefore$ so all these orbits can be thought of as equivalence classes of some equivalence relation in

→ we would try to identify these orbits in S^3 to points in another space & hence the structure of S^3 is elucidated.

if (z_1, z_2) & (ω_1, ω_2) lie in the same orbit of S^3

so $z_1 \cdot g = \omega_1$

$z_2 \cdot g = \omega_2$

$\therefore \dfrac{z_1}{z_2} = \dfrac{\omega_1}{\omega_2}$ we take $\dfrac{z_1}{z_2} = \infty \in \mathbb{C}^*$ if $z_2 = 0$

Conversely:

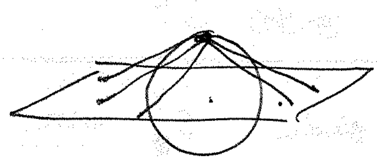
if $\dfrac{z_1}{z_2} = \dfrac{\omega_1}{\omega_2}$ then $\exists\, g \in U(1)$ s.t

$(\omega_1, \omega_2) = (z_1 g, z_2 g) = (z_1, z_2) \cdot g$

$\therefore$ the orbits in S^3 under the right action of $U(1)$ are in one-to-one correspondence with classes in $U(1)$

$\therefore \dfrac{z_1}{z_2} \in \mathbb{R}^{2*}$

we recall that



$\phi_s^* : S^2 \longrightarrow \mathbb{R}^{2*}$

$(k_1, k_2, k_3) \mapsto \left(\dfrac{k_1}{1-k_3}, \dfrac{k_2}{1-k_3}\right)$

$(0, 0, 1) \mapsto \{\infty\}$

$\therefore \phi_s^{*-1} : \mathbb{R}^{2*} \longrightarrow S^2$

$(z) \mapsto \left(\dfrac{z+\bar{z}}{z\bar{z}+1}, \dfrac{z-\bar{z}}{i(z\bar{z}+1)}, \dfrac{z\bar{z}-1}{z\bar{z}+1}\right)$

$\{\infty\} \mapsto (0,0,1)$

if $z = \dfrac{z_1}{z_2}$ then $\dfrac{z+\bar{z}}{z\bar{z}+1} = \dfrac{z_1\bar{z}_2 + \bar{z}_1 z_2}{|z_1|^2 + |z_2|^2} = z_1\bar{z}_2 + \bar{z}_1 z_2$

& similarly

$\therefore$ we define.

$\mathcal{P} : S^3 \longrightarrow S^2$

$\mathcal{P}(z_1, z_2) = (\phi_s^{*-1})\left(\dfrac{z_1}{z_2}\right)$

for record

$\mathcal{P}(x_1, y_1, x_2, y_2) = \left(2x_1x_2 + 2y_1y_2,\ 2x_2y_1 - 2x_1y_2,\ (x_1^2+y_1^2) - (x_2^2+y_2^2)\right)$

$\mathcal{P}(\phi, \theta_1, \theta_2) = (\sin\phi \cos(\theta_1 - \theta_2), \sin\phi \sin(\theta_1 - \theta_2), \ldots)$

$\therefore$ if $\theta_1 - \theta_2 = \theta$

then $P(\phi, \theta_1, \theta_2) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$$\| \\ S^2$$

$P(x_1, y_1, x_2, y_2)$ is a C^∞ map from $\mathbb{R}^4$ to $\mathbb{R}^3$

& its restriction to S^3 is also C^∞

So let $x \in S^2$

& (z_1, z_2) be s.t. $P(z_1, z_2) = x$

$\therefore P^{-1}(x) = \{ (z_1, z_2)\cdot g \in S^3 \mid g \in U(1) \}$ ie $P((z_1,z_2)\cdot g) = P(z_1, z_2)\ \forall g$

$\therefore$ P identifies each orbit on S^3 under the right action of $U(1)$ to a pt on S^2 // P is C^∞ and maps S^3 onto S^2

$\therefore$

$$\begin{tikzcd} & S^3 \arrow[d, "P"'] & U(1) \\ & S^2 \arrow[dl, shift left, "\phi_S^*"] & S^3 \times U(1) \\ \mathbb{R}^{2*} \arrow[ur, shift left, "\phi_S^{*-1}"] & & \end{tikzcd}$$

Riemann sphere.

$\therefore$ P provides S^3 with a "principle $U(1)$-bundle structure over S^2"

P is the Hopf. Map.

So we have the

Hopf map $\wp : S^3 \longrightarrow S^2$ (C^∞ + onto)

& the projection map $P: S^2 \times U(1) \to S^2$ (C^∞ + onto)

above each pt on S^2 there is a copy of S^1.

but these are tied up differently in S^3 + $S^2 \times U(1)$

so altho' S^3 + $S^2 \times U(1)$ are not globally the same but are locally the same i.e.

We consider the openset on S^2 as:

$$U_S = S^2 - (0,0,1) \quad , \quad U_N = S^2 - (0,0,-1)$$

$$\therefore \wp^{-1}(U_S) = \{(z_1, z_2) \in S^3 \mid z_2 \neq 0\}$$

$$\wp^{-1}(U_N) = \{(z_1, z_2) \in S^3 \mid z_1 \neq 0\}$$

each in the complement of either of the degenerate tori i.e. the lines circles

$$\Psi_S : \wp^{-1}(U_S) \longrightarrow U_S \times U(1) \quad , \quad \Psi_N : \wp^{-1}(U_N) \longrightarrow U_N \times U(1)$$

$$\Psi_S (z_1, z_2) \mapsto \left(\wp(z_1, z_2), \frac{z_2}{|z_2|}\right) \quad , \quad \Psi_N (z_1, z_2) = \left(\wp(z_1, z_2), \frac{z_1}{|z_1|}\right)$$

$$\Psi_S^{-1} (x, g) \mapsto (z_1, z_2) \cdot \left(g \frac{|z_2|}{z_2}\right) \quad \Psi_N^{-1} (x, g) = (z_1, z_2) \cdot \left(g \frac{|z_1|}{z_1}\right)$$

ψ_S