DOCUMENTATION FOR MY COMPUTATIONAL PHYSICS PROJECT

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Abstract. In the following document I will first give a rapid description of the background theory concerning the motion of massive test-particles on the Space-Time (without Cosmological Constant) described by the vacuum Schwarzschild solution outside the Event-Horizon. Then I will describe the program written in Octave to plot the motion of the particle in the 2-dimensional parameter space defined by the particle's radial coordinate and the azimuthal coordinate as the particle moves along geodesics in the aforesaid region of the specified Space-Time.

First 7 sections give a fast review of the background theory. The next 3 sections describe the program.

1. What is Space-Time?

For the purpose of this project one can think of **Space-Time** as the set of all "events" which has the structure of a connected 4-dimensional C^∞-manifold with a Lorentz metric (i.e a metric of signature +2).

The basic points to be noted:

- If one drops the condition of paracompactness from the definition of a manifold even then the celebrated Geroch's theorem will say that Hausdorffness condition and the existence of the Lorentz metric implies paracompactness.
- There exists versions of General Relativity where the Hausdorffness condition is dropped but here we shall not be concerned with such exotic versions of the theory.
- General Relativity is actually defined on an equivalence class of Space-Times such that 2 Space-Times are equivalent if they are isometric. For the present project where I shall deal with geodesics only of massive particles one need not get into this issue of ambiguity of representation of the Space-Time.
- To get a feel for what can be a valid Space-Time I state without proof 2 celebrated theorems regarding it:
  - A compact manifold can be a Space-Time iff its Euler Characterisic is 0.
  - Any non-compact 4-manifold can always be made into a Space-Time.

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2. Why "the" Space-Time ?

It is to be noted that Birkhoff's theorem guarantees that any spherically symmetric solution of the Einstein's equations in vacuum are stationary and asymptotically flat. Hence any such solution is isometric to the maximally extended Schwarzschild solution. In this sense, the Schawrzschild solution is unique outside the Event-Horizon.
Given the above I use "the" for the solution considered here since in this project I am focussing on the motion of test particles in the vacuum Schwarszchild Space-Time in the region exterior of the Event-Horizon of the black-hole.

3. Setting up the conventions

I will work in a system of units where the magnitude of speed of light is 1. Later I shall make the exact choice of units more precise. We assume the Cosmological Constant to be 0 for this project.

In these units the **Einstein's Field Equations** can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

where $R_{\mu\nu}$, $g_{\mu\nu}$ and $T_{\mu\nu}$ are the (μ, ν) components of the Ricci, metric and the Stress-Energy tensors. (after a choice of basis in the tangent bundle of the Space-Time). R is the Scalar Curvature and G is the Gravitational Constant. There is nothing canonical about the form of this equation and there exists variants in literature.

In this system of units one can write the Schwarszchild solution for the metric in the following form:

$$ds^2 = B(r)dt^2 - \{A(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\}$$

where the variables r, θ and ϕ are defined according to the conventional Gaussian Normal Coordinates on a 3-Riemannian manifold whose isometry group has $SO(3)$ as a subgroup and the orbits under its action are homeomorphic to S^2. The Space-Time is assumed to be foliated by such 3-manifolds (follows from assumptions of spherical symmetry) and t is the parameter whose coordinate generates a global nowhere zero time-like Killing vector field orthogonal to the foliates. (existence of such a field is equivalent to the assumption of staticity). t can be identified with our conventional notion of "time".

Conventional Schwarzschild solution has the following further specifications:

- $A(r)B(r) = constant$. I will work in the convention where $A(r)B(r) = 1$. There is nothing canonical about it and other conventions exist in literature.
- $B(r) = 1 - \frac{2GM}{r}$, where G is the Gravitational Constant and M is the mass of the black-hole measured with respect to the "infinity" of the asymptotically flat Space-Time. (Here I am skipping the very technical discussion of defining "mass" in General Relativity which is needed to justify the above choice of normalization) Here too there are other conventions in literature about the form of $B(r)$.

Here onwards it can be understood that $B(r) = 1 - \frac{2GM}{r}$ and $A(r) = \frac{1}{1-\frac{2GM}{r}}$ though it is **NOT** necessary to assume so for the validity for the following equations. But in the Octave program I shall use the above specific forms for $B(r)$ and $A(r)$.

4. Exploiting the symmetry

It is naturally expected that for motion of particles in such a highly symmetric Space-Time there would be conserved quantities. Hence after writing the differential equations for the geodesics in their explicit form one can deduce the quantities that are conserved along the geodesic and hence rewrite the effective equations of motions in terms of these conserved equations. The point to be noted is that the conserved quantities can be deduced *without* solving the differential equations.

Thus for the purpose of purpose of finding out the geometry of the geodesics, symmetry considerations reduce the situation of solving 4 coupled first order differential equations in the 4 variables to solving 1 first order differential equations in 2 variables.

The geodesic equations yield the following results about the conserved quantities:

{In the following λ is an arbitrary affine parameter for the geodesic}

- θ is an arbitrary constant parameter. θ being constant means that the motion is happening on a fixed plane in the parameter space, as would be expected from our intuition with central force fields in Newtonian mechanics. For computational ease we choose $\theta = \frac{\pi}{2}$.
- $r^2 \frac{d\phi}{d\lambda}$ is a constant of motion and I shall denote it as J. J is the closest analogue in General Relativity of angular momentum of Newtonian mechanics.
- $A(\frac{dr}{d\lambda})^2 + \frac{J^2}{r^2} - \frac{1}{B}$ is a constant of motion and I shall denote it as $-E$. E is the closest analogue in General Relativity of energy in flat space physics.

Here too it should be noted that there is nothing canonical about the above 3 conserved quantities as stated above. In literature there exists other conventions but what is important is that there exists precisely 3 independent constants of motion.

5. The equations

So the definition of the Space-Time interval and the geodesic equations yield the following equations:

- $ds^2 = E d\lambda^2$. Hence $E > 0$ and we have the very crucial "energy positivity condition".
- $d\lambda = B dt$.
- $\frac{d\phi}{dt} = \frac{JB}{r^2}$
- $\frac{dr}{dt} = \frac{B}{\sqrt{A}}\sqrt{\frac{1}{B} - E - \frac{J^2}{r^2}}$
- If $J \neq 0$ then we can compute $\frac{d\phi}{dr}$ (the relevant equation for the geometry of the orbits) by dividing $\frac{d\phi}{dt}$ by $\frac{dr}{dt}$

Here too it should be noted that the above equations are not canonical but depend on what are chosen as the 3 independent constants of motion. In literature there exists other conventions of stating the equations of motion.

We note that for the purpose of programming it is beneficial to use the differential equation as $\frac{dr}{d\phi}$ rather than $\frac{d\phi}{dr}$ since in the later at the extremas of the motion the denominator tends to 0 and hence becomes a source of numerical instabilities.

Since the laboratory obtainable data are the time derivatives it is necessary that J and E be expressed in terms of $\frac{dr}{dt}$ and $\frac{d\phi}{dt}$ rather than in terms of derivatives w.r.t the affine parameter which is only a mathematical construction.

So we have the following equations:

- $\frac{r^2}{B}\frac{d\phi}{dt} = J$
- $\frac{A}{B^2}(\frac{dr}{dt})^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E$

6. Choice of units and values of universal constants

As already stated earlier I will be working in a system of units where the magnitude of the velocity of light is 1.

We also notice that in the functions $A(r)$ and $B(r)$ the generic expression that occurs is $\frac{2GM}{r}$. Hence for computer simulations it will be useful to choose a system of units where $2GM$ has the numerical value 1.

Further we also note that there is a natural length scale in the situation i.e the *Schwarzschild Radius*(R_s) of the black-hole which marks the Event-Horizon. (We know that $R_s = \frac{2GM}{c^2}$. For calculating this in $S.I$ units we use $G = 6.6732 \times 10^{-11} m^3/kgs^2$ and $c = 3 \times 10^8 m/s^2$) Hence it is more advantageous to communicate data in terms of this distance since intuitively one can understand the geometry of the space time depending on how far away action is happening from the Event-Horizon.

So from the $S.I$ system of units we move to a new system of units where length , mass and time is measured in units of m' , kg' and s' (and the $S.I$ units for them are denoted as m, kg and s respectively) as defined below:

- $1m' = R_s m$
- $1s' = \frac{R_s}{3\times 10^8} s$
- $1kg' = M kg$

{This system of units henceforth will be referred to as the **Natural Units** }

In these units the numerical value of the speed of light is 1 and the numerical value of $2GM$ is 1.

Here too there is nothing canonical about this choice of units and I made this choice after experimenting with various units and comparing the stability of the program with various choices of units which are mathematically equally valid.

The user enters all the data that he is asked to enter in $S.I$ units and once the data is taken in it is converted in units of kg', m' and s' before it is processed.

7. Bounds on the initial conditions of the test particle

In the conventions and units defined above we have:

{ A dot on the top of the variable name denotes its derivative w.r.t "t" }

- $B(r) = 1 - \frac{1}{r}$ and $A(r) = \frac{1}{1-\frac{1}{r}}$
- $J = \frac{r^2}{1-\frac{1}{r}}\dot{\phi}$
- $E = \frac{1}{1-\frac{1}{r}} - \frac{J^2}{r^2} - \frac{\dot{r}^2}{1-\frac{1}{r}}$ 3

Hence during the motion of a test-particle along a geodesic in this Space-Time, for a change in the parameter values the distance covered on the spatial manifold (say dl) is given as:

$$dl^2 = \frac{1}{1-\frac{1}{r}} dr^2 + r^2 d\phi^2$$

So the velocity of the particle $v = \frac{dl}{dt}$ is given as:

$$v^2 = \frac{1}{1-\frac{1}{r}} \dot{r}^2 + r^2 \dot{\phi}^2$$

Since the motion to be tracked is outside the Event-Horizon we have $1 - \frac{1}{r} > 0$. This coupled with the condition $E > 0$ gives us the following inequality:

$$r^2 \dot{\phi}^2 + \frac{\dot{r}^2}{1-\frac{1}{r}} < 1 - \frac{1}{r}$$

The above is equivalent to:

$$v^2 < 1 - \frac{1}{r}$$

Hence we see that given the radial coordinate of the particle, the bound on the velocity of the particle is *stronger* than just the speed of light.

The above implies that once the radial position r and the $\dot{r}$ is given there is a bound on the values of $\dot{\phi}$ as:

$$\dot{\phi}^2 < \frac{1}{r^2}\{(1 - \frac{1}{r}) - \frac{\dot{r}^2}{(1 - \frac{1}{r})}\}$$

Hence the checks have to done on the variable values supplied by the variables to test whether they lie in the intervals deduced above.

8. Basic structure of the program

The basic organisation blocks in the program are:

- The user enters the values of the following parameters in $S.I$ units:
  - M
  - Values of the variables r,ϕ,$\dot{r}$ and $\dot{\phi}$ at some fixed t which will be considered as the initial conditions.
- From the above values the constants of motion J and E are calculated.
- Then differential equation solver is started which finds values of r at changing values of ϕ. To do this at every stage of iteration it first evaluates the $A(r)$, $B(r)$, $\frac{d\phi}{dt} = \frac{JB}{r^2}$, and $\frac{dr}{dt} = \frac{B}{\sqrt{A}}\sqrt{\frac{1}{B} - E - \frac{J^2}{r^2}}$. If $J \neq 0$ then the differential equation solver evaluates $\frac{dr}{d\phi}$ (by dividing $\frac{dr}{dt}$ by $\frac{d\phi}{dt}$) and updates the value of r for each value of ϕ . If $J = 0$ then only $\frac{dr}{dt}$ is evaluated.

Apart from the input data and the universal constants there is 1 arbitrary constants in the program called h (variable has been explained below). The value of this was arrived at by trial and error trying to balance between various factors explained below.

9. The steps of the program

The program proceeds along the following major segments (the **subsections**) and steps (the **items** in each subsection):

9.1. Acquiring and Validating the input data.

- The user is asked to enter the value of M in kg (data is stored in the variable $M0$) and if the number entered is negative then the program exits with an error message.
- Then the *Schwarzschild Radius* of the Black-Hole is calculated in $S.I$ units (and the data is stored in the variable R_s) and is displayed.
- The user is asked to enter a valid initial value of r (i.e $r > Rs$) and the data is stored in the variable $r00$. If the value of $r00$ is invalid then the program exists with an error message(s) depending on whether $r = R_s$ or $r < R_s$. If $r00$ is a valid data then it is converted to the Natural Units by dividing it by R_s and the quotient is stored in $r0$.
- Two more variables are defined to store two more constants that shall be frequently used i.e $x = 3 \times 10^8$ and $y = 1 - \frac{1}{r0}$.
- The user is asked to enter the initial value of $\dot{r}$ in $S.I$ and it is stored in $tr00$. $tr00$ is converted to Natural Units by dividing it by x and the quotient is saved in $tr0$.
- Then the maximum possible initial value of $\dot{\phi}$ is calculated as $\frac{x}{Rsr0}\sqrt{y - \frac{tr0^2}{y}}$ and it is stored in the variable $validangvel$.
- The user is asked to enter a valid initial value for $\dot{\phi}$ (i.e $-validangvel < \dot{\phi} < validangvel$). The data entered is stored in the variable $tp00$. If $tp00$ does not lie in the required interval then the program exits with an error message.
- If $tp00$ is valid then it is converted to Natural Units by multiplying it with $\frac{R_s}{x}$ and saved in $tp0$.

- Then a variable called f is defined which saves the fraction of speed of light at which the test-particle is initially moving and this is calculated as $f = \sqrt{\frac{tr0^2}{y} + (r0tp0)^2}$. This value is displayed to the user for information.
- If $tp00 \neq 0$ then the user is asked to enter the number of revolutions of the black-hole until which he/she wants to track the particle and this is stored in the variable n.
- Then user is asked to enter a multiple of the Schwarzschild Radius till which he/she wants to track the particle and the data is stored in l. If this is negative then the program exits with an error message.
- A message is displayed to the user that the particle's motion will be tracked till just before it reaches the Event Horizon.

9.2. **Defining the functions and constants.** Henceforth all calculations are done in the Natural Units.

- A function called **B** is defined which takes a value of r and evaluates $B(r) = 1 - \frac{1}{r}$.
- Global variables j and er are defined which store the values of the J and E after calculating them as $j = \frac{r0^2}{B0}tp0$ and $er = \frac{1}{B0} - \frac{1}{B0^3}tr0^2 - \frac{j^2}{r0^2}$ where $B0$ is the initial value of B.
- A function called **rhsr** is defined which for a given value of r evaluates the value of $\frac{1}{B} - er - \frac{j^2}{r^2}$.
- A function called **rhsp** is defined which for a given value of r evaluates the value of $\frac{jB}{r^2}$.
- A function called **rhs** is defined which for a given value of r evaluates the value of $\frac{dr}{d\phi}$. To do this it calls the functions rhsr and rhsp. Theoretically the output of the function rhsr should be positive definite and if $tp0 \neq 0$ then the output of rhsp should be non-zero. But numerical instabilities might affect this scenario. Hence it is checked whether numerical instabilities have affected the values (by pushing them out of the valid ranges) and if not then $\frac{dr}{d\phi}$ is calculated and returned. If the output of the functions rhsr and rhsp are in the invalid ranges then the string "Numerical Instabilities" is returned by the function rhs.

9.3. **The Runge-Kutta algorithm for solving differential equations.** For this program I chose to use the standard 4^{th} order version for the Runge-Kutta algorithm for a differential equation of the form $\frac{dy}{dx} = f(y)$. Runge-Kutta algorithm is an iterative process such that at every stage of the iteration a value y_n is found for a corresponding x_n. At the end of the n^{th} stage x_n is changed by an amount h and hence for the next cycle the definition of the independent variable updates as $x_{n+1} = x_n + h$. Now the corresponding y_{n+1} is found through the following sequence of calculations:

(1) $k_1 = hf(y_n)$
(2) $k_2 = hf(y_n + \frac{k1}{2})$
(3) $k_3 = hf(y_n + \frac{k_2}{2})$
(4) $k_4 = hf(y_n + k_3)$
(5) $y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$

Detailed analysis shows that in equating $f(x_n) = y_n$ the error is of $O(h^5)$.

9.4. **Plotting the geometry of the path in parameter space for the geodesic in Space-Time.** The above version of the Runge-Kutta algorithm, is used to plot the geometry of the path in the parameter space corresponding to the geodesic in Space-Time by solving the differential equation for $\frac{dr}{d\phi}$ (if $J \neq 0$) or $\frac{dr}{dt}$ (if $J = 0$) .

Important points about the implementation are as follows:

- At the start of the process values for the variables h and s are fixed. h is the same as in the Runge-Kutta algorithm and s is the factor by which h is reduced whenever the function rhs returns "Instability". I have fixed h to be 0.01 and s to be 0.1. These were fixed through trial and error to strike a balance between speed of execution of the program and the precision.
- Before the Runge-Kutta algorithms begins the decision is made regarding which differential equation to solve depending on whether or not $J = 0$.
- The running variable for ϕ is p and for radial coordinate it is r. At the start of every cycle of Runge-Kutta the check is made whether $p - p0 \leq 2n\pi$ and $1 < r < lr0$. If the current values of p and r do not satisfy the above then the process is halted since the program is complete. For $J = 0$ case one need not check anything for p since it is a constant.
- As per the Runge-Kutta algorithm the function rhs or rhsr (depending on whether or not $J = 0$) will be called 4 times. After every call a check is made as to whether it has returned "Instability" or a negative number (later check is needed if $J = 0$). If it has not then the result of the call is used for the next step. If it does return a error value then the *the program execution is paused.* If $J = 0$ then ϕ is not updated at the end of each cycle.
- At the end of every cycle of Runge-Kutta the current position of the particle in the 2-dimensional parameter space of r and ϕ is plotted.

A practical point

Theory shows that if the particle has J less than a critical value then the massive particle will definitely reach the Event-Horizon after an infinite amount of time as seen by the external observer (for whom "t" is time) but in finite proper time of the particle. Hence for such cases the program runs extremely slow and it seems as if nothing is happening since the program sees the external observer's time. This phenomenon will be typically seen if $tp00$ is set to 0.

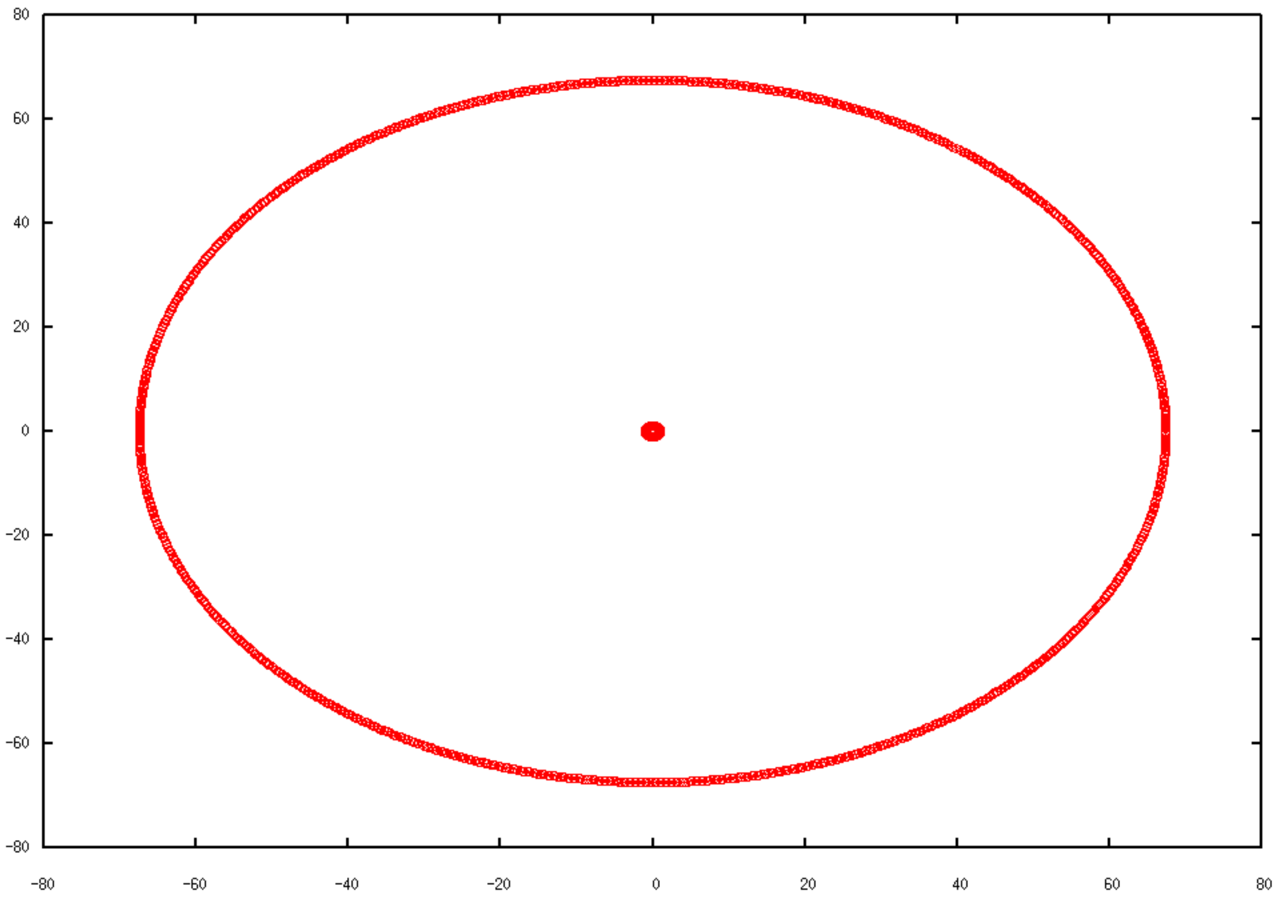
10. Some Sample Output

10.1. **An elliptic orbit in the parameter space!** Theoretical analysis shows that the conditions to get an elliptic orbit in the parameter space are much more stringent than for other geometries. Hence it is highly unlikely that a randomly chosen valid initial condition is going to produce an elliptic orbit. Hence the below given data set which does produces one is a very rare case:

The initial data is:

- $M = 10^{33} Kg$
- $r00 = 10^8 m$
- $p0 = 0$
- $tr00 = 2m/s$
- $tp00 = 1s^{-1}$
- $n = 2$
- $l = 10$

The trajectory looks as below:

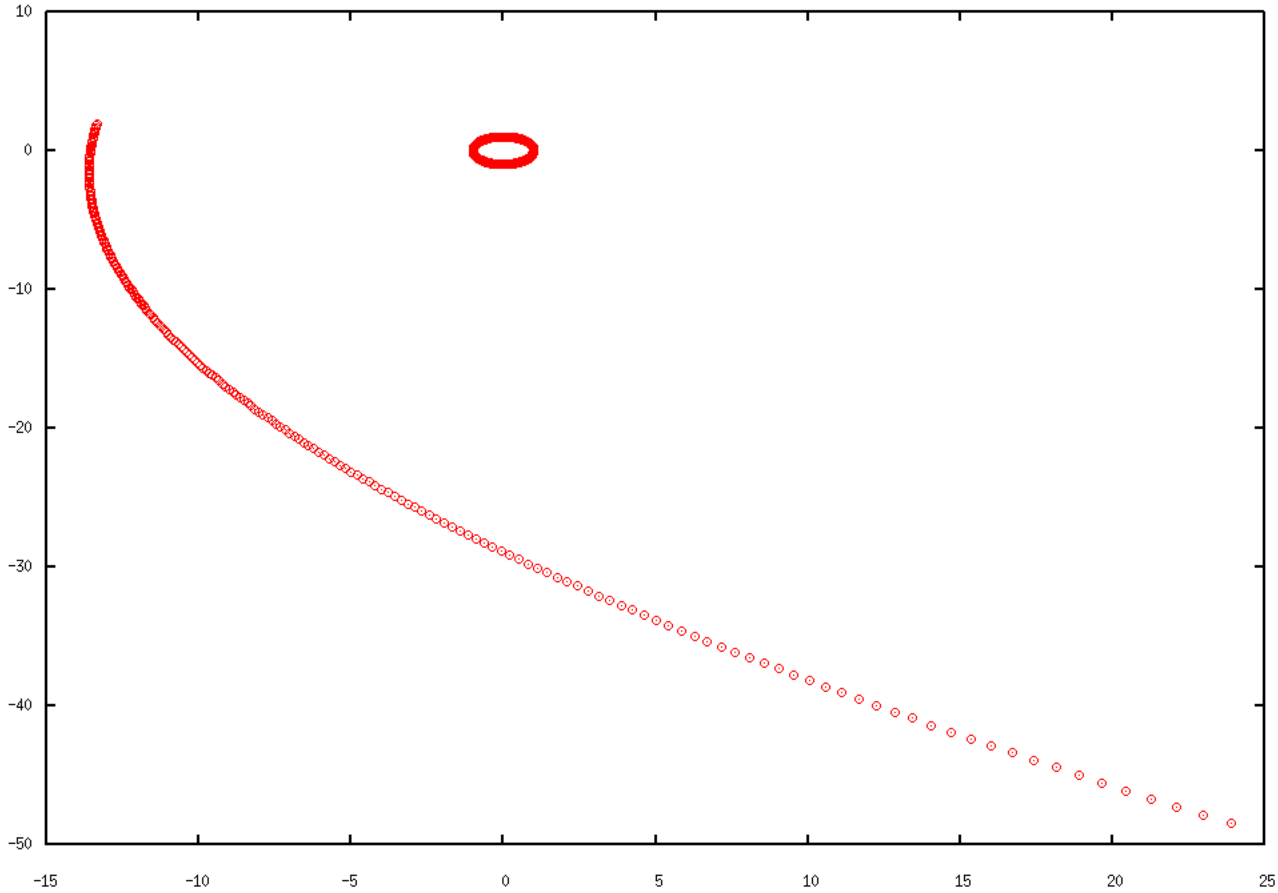


{ The small circle in the centre is the Event-Horizon. The larger graph is the orbit. }

10.2. **The typical case.** An initial data which would generate the typical orbits would be:

- $M = 10^{33} Kg$
- $r00 = 2 \times 10^{7} m$
- $p0 = 3$
- $tr00 = 3m/s$
- $tp00 = 4s^{-1}$
- $n = 2$
- $l = 4$

The trajectory looks as below:

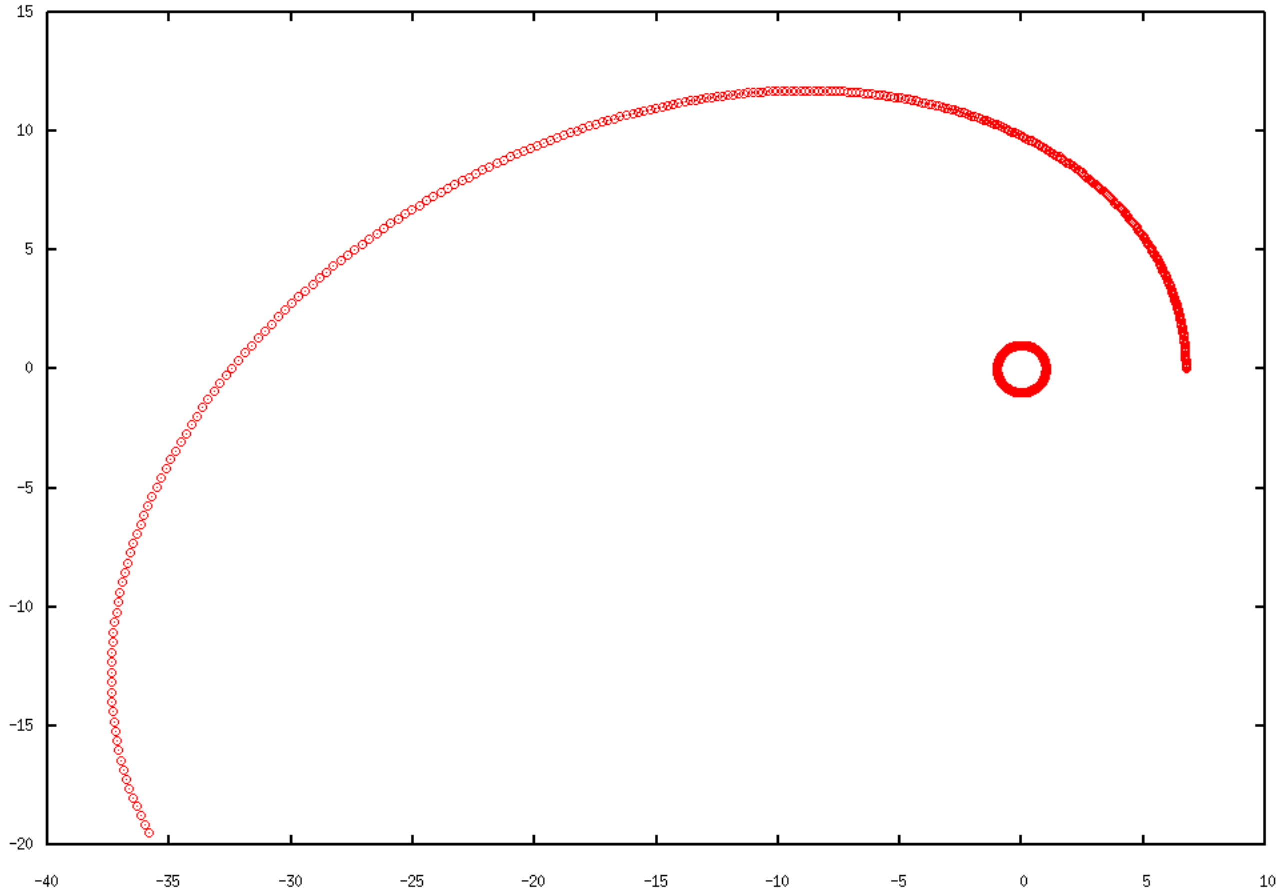


{ The small circle in the centre is the Event-Horizon. The larger graph is the orbit. }

10.3. **A slightly atypical case.** An initial data to get these would be:

- $M = 10^{34} Kg$
- $r00 = 10^8 m$
- $p0 = 0$
- $tr00 = 20m/s$
- $tp00 = 1s^{-1}$
- $n = 2$
- $l = 10$

The trajectory looks as below:



{ The small circle in the centre is the Event-Horizon. The larger graph is the orbit. }

The above data set reaches an instability at the point shown and the program gets paused.

11. Some Remarks

In such systems like the above the very crucial physical aspect is the existence of invariants of motion arising out of symmetries of the Space-Time. But in numerical simulations numerical approximations and instabilities make this scenario very different. The quantities that should be constant do waver. Even in simple Space-Times like the one considered here I found that there are initial conditions for which the program runs into instabilities. I am aware that there exists more sophisticated algorithms for numerical integration which are designed to keep these conserved quantities numerically constant. I would like to study these techniques further and that would enable me to do simulations in more involved Space-Time geometries where intense curvature of it would otherwise make simple simulations unstable.

Lastly doing this project was a great experience in many ways. Primarily it gave me a chance to brush up the basic theory once again. Further it gave me a numerical feel for Einstein's Gravity. The process of analysing and testing the conditions for valid initial conditions was very illuminating and developed a lot of intuition.

12. Thanks

I would like to thank Prof.G.Date for his exciting graduate course in General Relativity during Aug-Nov 2007 which helped me a lot in understanding this subject. Even during this project I was fortunate that he agreed to have discussion sessions to clarify various subtle points. His insights helped me find a way out of the various corners where I got stuck.

I would also like to express thanks to Prof.R.Adhikari for not only this course in Computational Physics but also for providing this creative opportunity of doing a project as opposed to the rigmarole of examinations.