"*Machines take me by surprise with great frequency.*" – Alan Turing

1. Given below is a list of operations on two languages L_1, L_2. For each of them, comment if regular languages are closed under the operation.

   (a)
   $$L_1 \setminus |L_2| = \{w \mid \exists x, wx \in L_1, \exists u \in L_2, |u| = |x|\}$$

   (b)
   $$alt(L_1, L_2) = \{alt(w, x) \mid |w| = |x|, w \in L_1, x \in L_2\}$$
   where, for two words of equal length $w = a_1a_2 \dots a_n$ and $x = b_1b_2 \dots b_n$
   $$alt(w, x) = a_1b_1a_2b_2 \dots a_nb_n$$

   (c) We saw the *shuffle* operator in Problem Set 2. We shall now add an additional constraint to it. Recall
   $$\text{shuffle}(L_1, L_2) = \{w \in \text{shuffle}(u, v) \mid u \in L_1, v \in L_2\}$$
   where, for two words u_n, v_n
   $$\text{shuffle}(u, v) = \{u_1v_1 \dots u_nv_n \mid u = u_1 \dots u_n, v = v_1 \dots v_n, u_i, v_i \in \Sigma^*\}$$
   Adding our additional constraint, we get
   $$\text{shuffle\_equals}(L_1, L_2) = \{w \in \text{shuffle}(u, v) \mid u \in L_1, v \in L_2, |u| = |v|\}$$

2. A homomorphism $h : \Sigma^* \to \Gamma^*$ is *injective* when
   $$\forall u, v \in \Sigma^*, h(u) = h(v) \Rightarrow u = v$$
   i.e. no two words in Σ^* have the same image in Γ^* under h.
   Construct an algorithm to check if a given homomorphism h is injective.

3. Consider the following operations that can be performed on a finite automaton $A = (Q, \Sigma, \Delta, Q_0, F)$:

   - The *trim* of A is defined as:
     $$trim(A) = (\hat{Q}, \Sigma, \hat{\Delta}, \hat{Q}_0, \hat{F})$$
     where
     $$\hat{Q} = \{q \in Q \mid \exists \text{ a path } q_0 \rightsquigarrow q \rightsquigarrow q_f, q_0 \in Q_0, q_f \in F\}$$
     $$\hat{\Delta} = \Delta \cap (\hat{Q} \times \Sigma \times \hat{Q})$$
     $$\hat{Q}_0 = Q_0 \cap \hat{Q}$$
     $$\hat{F} = F \cap \hat{Q}$$
     i.e. we remove states that cannot be part of an accepting path.

- The *reverse* of A is defined as:

$$rev(A) = (Q, \Sigma, \Delta, F, \{q_0\})$$

$$(q, a, q') \in \Delta \iff \delta(q', a) = q$$

- $det(A)$ is defined to be the output of the subset construction algorithm on A. Of course if A is a DFA, then $A = det(A)$.

Given a regular language L and a finite automaton $A = (Q, \Sigma, \delta, q_0, F)$ that accepts L, show that the following procedure:

$$A \xrightarrow{trim} A_1 \xrightarrow{rev} A_2 \xrightarrow{det} A_3 \xrightarrow{trim} A_4 \xrightarrow{rev} A_5 \xrightarrow{det} A_6 \xrightarrow{trim} A_7 = B$$

results in B being the minimal DFA that accepts L.

4. L be the set of binary representations of prime numbers (leading zeroes allowed). Is L regular? Prove your assertion.