Problem Set 5

Theory of Computation

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Problem 1. *Given a grammar G, give a grammar G' such that $L(G') = (L(G))^*$. Conclude that context-free languages are closed under Kleene star operation.*

Problem 2. *Linear sets are generalizations of arithmetic progression in higher dimensions. We define a subset $S \subset \mathbb{N}^m$ linear if there exists $a \in \mathbb{N}^m$ and $d_1, d_2, \ldots d_m \in \mathbb{N}^m$ such that*

$$S = \left\{ a + \sum_{i=1}^{m} k_i d_i : k_i \in \mathbb{N} \right\}$$

A finite union of linear sets is called a semi-linear set. Let $|\Sigma| = m$. For $\alpha \in \Sigma$, $\pi_\alpha(w)$ denotes the number of times the letter α occurs in w. We define

$$\pi : \Sigma^* \to \mathbb{N}^m$$

$$\pi : w \mapsto (\pi_{\alpha_1}(w), \pi_{\alpha_2}(w), \ldots, \pi_{\alpha_m}(w))$$

1. *Show that if L is regular, $\pi(L)$ can be represented as semi-linear set.*
2. *Prove or disprove that $\pi^{-1}(S)$ is regular if S is semi-linear.*
3. *If S is semi-linear, show that there exists a regular language L' such that $\pi(L') = S$.*
4. *Give an example of a language L'' which is not regular but $\pi(L'')$ is semi-linear.*
5. *Give an example of a language L''' which is not regular and $\pi(L''')$ is not semi-linear.*

Problem 3. *A context-free grammar is said to be linear if in each production rule, at most one non-terminal occurs on the right.*

1. *Show that linear context-free grammars on a singleton (unary) alphabet generate only regular languages.*
2. *Let L be a language generated by a linear grammar. Show that $\pi(L)$ is semi-linear.*
3. *Prove or disapprove that linear context-free grammars can be ambiguous.*

Problem 4. *Let* $L = \{w \cdot w : w \in \Sigma^*\}$.

1. *Show that* L *is not context-free.*
2. *Give a context-free grammar for* $\Sigma^* \backslash L$.
3. *Construct a (non-deterministic) push-down automaton that accepts* $\Sigma^* \backslash L$.
4. *Conclude that context-free languages are not closed under complementation.*

Problem 5. *Let* $\text{bin}(x)$ *denotes the binary representation of the number* x*, and* $\text{rev}(u)$ *denote reverse of a string* u*. Can the following language be accepted by a push down automaton?*

$$\mathcal{L}_1 = \{\text{bin}(n)\$(\text{bin}(n+1))\}$$

What about $\mathcal{L}_2$ *defined as follows?*

$$\mathcal{L}_2 = \{\text{bin}(n)\$\text{rev}(\text{bin}(n+1))\}$$

Problem 6. *Show that a regular language cannot be inherently ambiguous.*

Problem 7. *Give an algorithm to take as input a context-free grammar and compute if the language generated by the grammar is empty.*

Problem 8. *Let the perfect shuffle of a languages* A *and* B *be defined as follows:*

$$\text{perfect-shuffle}(A, B) = \{a_1 b_1 a_2 b_2 \dots a_k b_k \mid a_1 a_2 \dots a_k \in A, b_1 b_2 \dots b_k \in B, k \in \mathbb{N}, a_i, b_i \in \Sigma\}$$

Show that context free languages are closed under this operator of perfect shuffle.

Suppose we define shuffle to be

$$\text{shuffle}(A, B) = \{a_1 b_1 a_2 b_2 \dots a_k b_k \mid a_1 a_2 \dots a_k \in A, b_1 b_2 \dots b_k \in B, k \in \mathbb{N}, a_i, b_i \in \Sigma^*\}$$

Are context free languages are closed under shuffle?

Problem 9. *We now define the notion of a 'two-way pushdown automaton'.*

This automaton, along with the usual moves may move on its input tape in two directions, the direction is decided based on the alphabet it reads and the state that it is currently in. Suppose we assume that the begin and end of the input is marked by special symbols, describe a two-way push down automaton for the language $\mathcal{L} = \{a^n b^n c^n | n \in \mathbb{N}\}$*. Conclude that two way pushdown automaton are strictly more powerful than context-free grammar. Is the language* $\mathcal{L}' = \{w.w \mid w \in \{0,1\}^*\}$ *accepted by this model?*

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