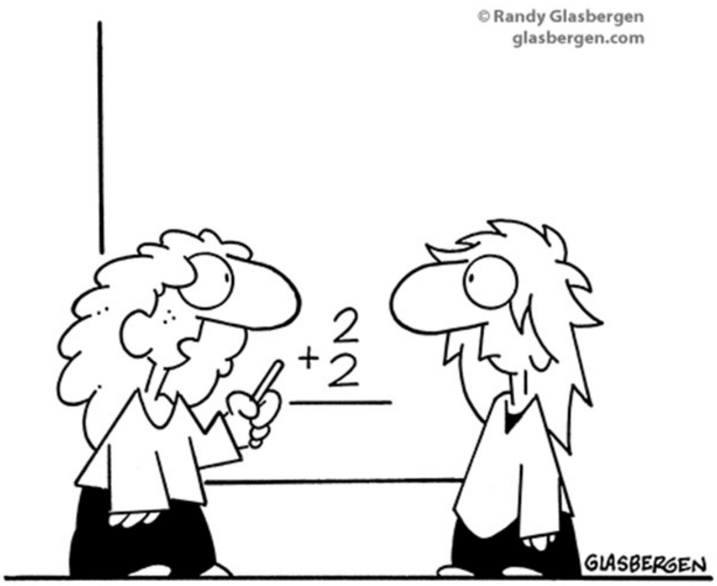

"First they build up your confidence with simple addition and subtraction, then they slam you with algebra and calculus. It's quite a clever scheme."

We shall discuss certain loose ends; discuss facts we have used. These actually belong to calculus. In any case these involve nothing more than high school material (apart from limits).

Walli's product:

$$\int_0^{\pi/2} \sin^0 x dx = \frac{\pi}{2}; \qquad \int_0^{\pi/2} \sin x dx = \cos 0 - \cos(\pi/2) = 1.$$

If $m > 1$, then integration by parts gives

$$\int_0^{\pi/2} \sin^m x dx = \int_0^{\pi/2} \sin^{m-1} x(-\cos x)' dx$$

$$= \int_0^{\pi/2} \cos x \;\; (m-1)\sin^{m-2} x \cos x dx$$

$$= (m-1)\int_0^{\pi/2} \sin^{m-2} x dx - (m-1)\int_0^{\pi/2} \sin^m x dx$$

so that

$$\int_0^{\pi/2} \sin^m x dx = \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x dx.$$

Thus

$$\int_0^{\pi/2} \sin^{2m} x dx = \frac{2m-1}{2m}\ \frac{2m-3}{2m-2}\ \frac{2m-5}{2m-4}\ \cdots \frac{3}{4}\ \frac{1}{2}\ \frac{\pi}{2}.$$

$$\int_0^{\pi/2} \sin^{2m+1} x dx = \frac{2m}{2m+1}\ \frac{2m-2}{2m-1}\ \frac{2m-4}{2m-3}\ \cdots \frac{4}{5}\ \frac{2}{3}\ 1.$$

dividing the first equation by the second

$$\frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} = \frac{(2m-1)(2m+1)}{(2m)^2}\ \frac{(2m-3)(2m-1)}{(2m-2)^2}\ \cdots$$

$$\cdots \frac{3\times 5}{4^2}\ \frac{1\times 3}{2^2}\ \frac{\pi}{2}.$$

$$\frac{\pi}{2} = \frac{2^2}{1\cdot 3}\ \frac{4^2}{3\cdot 5}\ \frac{6^2}{5\cdot 7}\cdots \frac{(2m-2)^2}{(2m-3)(2m-1)}\ \frac{(2m)^2}{(2m-1)(2m+1)}$$

$$\times \frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx}.$$

We shall now show that as $m \to \infty$;

$$\frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} \to 1. \qquad (\spadesuit)$$

It will then follow that

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2^2}{1\cdot 3}\ \frac{4^2}{3\cdot 5}\ \frac{6^2}{5\cdot 7}\cdots \frac{(2m-2)^2}{(2m-3)(2m-1)}\ \frac{(2m)^2}{(2m-1)(2m+1)}.$$

This is called wall's product.

$$\frac{\pi}{2} = \lim_{m\to\infty} \ \frac{2^{2m}(m!)^2}{3^2\cdot 5^2\cdots(2m-1)^2(2m+1)} = \lim_{m\to\infty} \ \frac{2^{4m}(m!)^4}{[(2m)!]^2(2m+1)}.$$

Or

$$\sqrt{\frac{\pi}{2}} = \lim_{m\to\infty} \frac{2^{2m}(m!)^2}{(2m)!\sqrt{(2m+1)}}$$

Or

$$\sqrt{\pi} = \lim_{m\to\infty} \ \frac{2^{2m}(m!)^2}{(2m)!\sqrt{(m+1/2)}}$$

Since $\sqrt{m}/\sqrt{m+1/2} \to 1.$ we get

$$\sqrt{\pi} = \lim_{m\to\infty} \ \frac{2^{2m}(m!)^2}{(2m)!\sqrt{m}}$$

This is called Walli's formula for $\sqrt{\pi}$.

Let us now prove (♠).
Observe that for $0 \leq \pi/1/2$, using $0 \leq \sin x \leq 1$

$$\sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x$$

Hence

$$\int_0^{\pi/2} \sin^{2m+1} x dx \ \leq \ \int_0^{\pi/2} \sin^{2m} x dx \ \leq \ \int_0^{\pi/2} \sin^{2m-1} x dx$$

All quantities being positive,

$$1 \ \leq \ \frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} \ \leq \ \frac{\int_0^{\pi/2} \sin^{2m-1} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx}$$

Using the recurrence relation obtained at the beginning, the above is same as saying

$$1 \ \leq \ \frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} \ \leq \ \frac{2m}{2m-1}$$

proving (♠).

Normal integral:

You can use double integrals to show, in a painless way,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \ = \ 1.$$

Since you are not sure, we shall follow elementary methods to derive the same result. Integrand being symmetric, we only need to show

$$\int_0^{\infty} e^{-x^2/2} dx \ = \ \sqrt{\pi/2}$$

Denote

$$a_n = \int_0^{\infty} x^n e^{-x^2/2} dx; \quad n = 0, 1, 2, \cdots$$

$$a_0 = ?; \qquad a_1 = 1.$$

Integration by parts gives for $k > 1$

$$a_k = (k-1) a_{k-2}$$

leading to

$$a_{2n} = (2n-1)(2n-3)\cdots(3)(1)a_0 = \frac{(2n)!}{n!\ 2^n}a_0$$

$$a_{2n+1} = (2n)(2n-2)\cdots(2)1 = n!\ 2^n$$

Fix $k \geq 1$. Note that for any $\lambda \in R$ we have

$$\int_0^\infty x^{k-1}\ (\lambda x - 1)^2\ e^{-x^2/2}dx > 0$$

because integrand is non-negative, continuous and is not the zero function. Thus

$$\lambda^2 a_{k+1} - 2\lambda a_k + a_{k-1} > 0; \qquad \forall \lambda \in R$$

hence

$$a_k^2 \leq a_{k+1}a_{k-1}; \qquad \text{or} \qquad a_k \leq \sqrt{a_{k+1}a_{k-1}}$$

Now $a_{2n} \leq \sqrt{a_{2n+1}a_{2n-1}}$ gives you

$$\frac{(2n)!}{n!\ 2^n}a_0 \ \leq\ \sqrt{n!\ (n-1)!2^n\ 2^{n-1}} \ =\ n!\ 2^n\frac{1}{\sqrt{2n}}$$

Or

$$a_0 \ \leq\ \frac{n!\ n!\ 2^n\ 2^n}{(2n)!\ \sqrt{2n}}$$

This being true for all n and appealing to Walli we get

$$a_0 \ \leq\ \sqrt{\pi/2} \qquad\qquad (\bullet)$$

Now use $a_{2n+1} \leq \sqrt{a_{2n+2}a_n}$ to see

$$n!\ 2^n \ \leq\ \sqrt{\frac{(2n)!}{n!\ 2^n}\ a_0\ \frac{(2n+2)!}{(n+1)!\ 2^{n+1}}\ a_0} \ =\ a_0\ \frac{(2n)!}{n!\ 2^n}\sqrt{(2n+1)}$$

Or

$$a_0 \geq \frac{n!\ n!\ 2^n\ 2^n}{(2n)!}\frac{1}{\sqrt{2n+1}}$$

This being true for every n appealing to Walli once again we get

$$a_0 \ \geq\ \sqrt{\pi/2} \qquad\qquad (\bullet\bullet)$$

Clearly $(\bullet)$ and $(\bullet\bullet)$ complete evaluation of the normal integral.

Stirling formula for $n!$:

We used this in discussing Random walks.
it says that $n!$ is like $\sqrt{2\pi}e^{-n}n^{n+1/2}$.

This is to be interpreted in the folloing sense. Their ratio converges to one. When we say that a sequence (a_n) is like another sequence (b_n) (both are sequences of strictly positive numbers) there are two ways of understanding.

Either $(a_n - b_n) \to 0$

or $(a_n/b_n) \to 1$.

Of course when the first happens, then the second also happens. However the other way is not in general true.

For example (n) is like $(n + 1/n)$ in the first sense and hence also in the second sense. the sequence (n^2) is like $(n^2 + n)$ in the second sense, but not so in the first sense. In fact their difference is n which becomes larger and larger. But then in what sense are they like each other? well, Both numbers are becoming large, when you replace one by the other, the relative error is going to zero.

If you are measuring length of this room, if you are off by a mile then the error is indeed very huge. On the other hand if you are measuring distance (of earth) to sun, if you are off by a mile or even hundred miles, the error is very very small. So the absolute error is many times unimportant and it is the relative error that matters.

Thus we need to show

$$\frac{n!}{\sqrt{2\pi}e^{-n}n^{n+(1/2)}} \to 1$$

or

$$\frac{n!}{e^{-n}n^{n+(1/2)}} \to \sqrt{2\pi}$$

This is achieved in two steps: first show limit exists and non-zero. Next show (using a test case) that it must the right side.

You see we have expression like $e^{-n}n^n$. To understand it, take logarithm, then this becomes: $n\log n - n$. This should remind you $x\log x - x$ which is integral of $\log x$. The proof of Stirling in just a clever approximation of the area under the curve $f(x) = \log x$ from $x = 1$ to $x = n$.

Let us first make a few observations which depend on the fact that

$$f'' = -1/x^2 \leq 0.$$

Let g be a twice differentiable function on an interval (a, b) with $g'' \leq 0$.

Consider any two points $u < v$ in the interval (a, b). We claim that the chord (or secant) joining the two points $(u, f(u))$ and $(v, f(v))$ lies below the graph of f. There are several ways of seeing this. Here is a way.

First observe that the equation of the chord is

$$y \;=\; f(u) + \frac{f(v) - f(u)}{v - u}(x - u).$$

Consider any point $w \in [u, v]$. We need to show

$$f(u) + \frac{f(v) - f(u)}{v - u}(w - u) \leq f(w).$$

That is,

$$f(u) - f(w) + \frac{f(v) - f(u)}{v - u}(w - u) \leq 0.$$

Or

$$[f(u) - f(w)](v - u) + [f(v) - f(u)](w - u) \leq 0.$$

$$[f(u) - f(w)](v - u) + [f(v) - f(w) + f(w) - f(u)](w - u) \leq 0.$$

$$[f(v) - f(w)](w - u) - [f(w) - f(u)](v - w) \leq 0.$$

By MVT, there are points $\theta \in (u, w)$ and $\eta \in (w, v)$ such that $f(w) - f(u) = f'(\theta)(w - u)$ and $f(v) - f(w) = f'(\eta)(v - w)$. So we need to show

$$f'(\eta)(v - w)(w - u) - f'(\theta)(v - w)(w - u) \leq 0.$$

That is,

$$[f'(\eta) - f'(\theta)](v - w)(w - u) \leq 0.$$

First factor above is $f''(\zeta)(\eta - \theta)$ for some ζ. Now $f'' \leq 0$ and $\theta < \eta$ tell you that the first factor above is negative. Since $u, w < v$, the other two factors are positive and hence the inequality is true.

Consider any point u in the interval (a, b). We claim that the tangent (to the graph of f) at u lies above the graph.

The equation of the tangent is

$$y \;=\; f(u) + f'(u)(x-u).$$

Let us take any other point $w \in (a, b)$. We need to show

$$f(w) \leq f(u) + f'(u)(w-u).$$

That is,

$$\frac{f(w)-f(u)}{w-u} \leq f'(u).$$

But the left side is $f'(\theta)$ for some $u < \theta < w$ and since f' is decreasing (remember $f'' \leq 0$, the inequality is verified.

Thus, for $k \geq 1$,

the area under the curve $y = \log x$ from k to $k+1$ is in between

the area under the chord joining $(k, \log k)$, $(k+1, \log(k+1))$ and

area under the tangent at $x + (1/2)$ between k to $k+1$.

Draw graphs and see. Thus

$$\frac{1}{2}\log(k+1) + \frac{1}{2}\log k \leq \int_k^{k+1} \log x \; dx \leq \log(k+1/2).$$

Adding these for $k = 1, 2, \cdots, n-1$ and remembering that $x \log x - x$ is a primitive for $\log x$ we get

$$\log(n!) - \frac{1}{2}\log n \leq n \log n - n + 1 \leq \sum_1^{n-1} \log(k+1/2).$$

Let

$$a_n = n \log n - n + 1 - [\log(n!) - \frac{1}{2}\log n] = \log\left\{\frac{e^{-n} n^{n+1/2}}{n!}\right\} + 1.$$

Then a_n is the area between the curve $y = \log x$ and the 'chords' explained above, from $x = 1$ to $x = n$ Thus we see

$$a_n \geq 0; \qquad a_n \uparrow . \qquad\qquad (\spadesuit)$$

Also

$$a_n \leq \sum_1^{n-1} \{\log(k + 1/2) - \frac{1}{2}\log(k+1) - \frac{1}{2}\log k\}.$$

$$= \frac{1}{2}\sum_1^{n-1}\left\{\log\frac{(k+1/2)}{k} - \log\frac{(k+1)}{k+1/2}\right\}$$

$$\leq \frac{1}{2}\sum_1^{n-1}\left\{\log(1+\frac{1}{2k}) - \log(1+\frac{1}{2(k+1/2)})\right\}$$

$$\leq \frac{1}{2}\sum_1^{n-1}\left\{\log(1+\frac{1}{2k}) - \log(1+\frac{1}{2(k+1)})\right\}$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\log(1+\frac{1}{2n}).$$

As a consequence a_n is bounded above . So ($\spadesuit$) implies that a_n converges to a finite limit. Say $a_n \uparrow c$

$$\log\left\{\frac{e^{-n}n^{n+1/2}}{n!}\right\} = a_n - 1 \uparrow c - 1.$$

Or

$$\frac{e^{-n}n^{n+1/2}}{n!} \to e^{c-1}.$$

Or

$$\frac{n!}{e^{-n}n^{n+1/2}} \to e^{1-c} = k \text{ say} \quad k \neq 0.$$

Or

$$\frac{n!}{ke^{-n}n^{n+1/2}} \to 1.$$

We shall now evaluate the constant k by using the above limit in a known case, namely, Walli's product. We know

$$\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} \to \sqrt{\pi}.$$

Suppose that we have strictly positive numbers a_n and b_n and $a_n/b_n \to 1$. If $\alpha_n \times a_n \to c$ then $\alpha_n \times b_n \to c$. This is because

$$\alpha_n \times b_n = \alpha_n \times a_n \times \frac{b_n}{a_n} \to c \times 1.$$

Similarly if $\alpha_n/a_n \to c$ then $\alpha_n/b_n \to c$. In other words we can replace a_n by b_n. As a consequence the above result of Walli implies

$$\frac{2^{2n}k^2e^{-2n}n^{2n+1}}{ke^{-2n}(2n)^{2n+1/2}\sqrt{n}} \to \sqrt{\pi}.$$

That is,

$$k = \sqrt{2\pi}.$$

Thus

$$\frac{n!}{\sqrt{2\pi}e^{-n}n^{n+1/2}} \to 1.$$

This completes proof of Stirling.