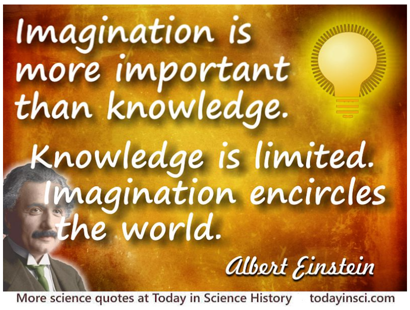

CMI/BVR Probability notes 12

We shall now prove the theorem about extinction probability. We assume below that $p_1 \neq 1$.

(a) Consider the events

$$A_n = (X_n = 0).$$

They are increasing and their union is precisely the event that some X_n is zero. In other words

$$P(A_n) \uparrow \alpha$$

Clearly, $P(A_1) = \varphi(0)$

$P(A_2) = \varphi(\varphi(0))$

Inn other words, if $a_n = P(A_n)$ then

$$a_{n+1} = \varphi(a_n)$$

Since $a_n \uparrow \alpha$ and φ is continuous on $[0,1]$, taking limits in the above equality leads to

$$\varphi(\alpha) = \alpha$$

Let now $c \in [0,1]$ satisfy, $\varphi(c) = c$. since φ is increasing and $c \geq 0$ we see

$$\varphi(0) \leq \varphi(c) = c; \qquad a_1 \leq c$$

Proceeding by induction, if we proved $a_n \leq c$ then the same reasoning as above gives

$$a_{n+1} = \varphi(a_n) \leq \varphi(c) = c$$

Thus for all n, we have $a_n \leq c$, showing $\alpha \leq c$.

This proves (1) of the theorem.

(b). Since $\varphi(1) = 1$, we only need to show that there can be at most one solution of $\varphi(t) - t = 0$ in $[0, 1)$.

If there are two solutions a and b then

$$\varphi(a) - a = \varphi(b) - b = 0.$$

Mean value theorem applied to $\varphi(t) - t$ on $[a, b]$ and $[b, 1]$ leads to points $c, d < 1$ such that

$$\varphi'(c) - 1 = \varphi'(d) - 1 = 0$$

One more MVT leads to t_0 with $0 < t_0 < 1$ such that

$$\varphi''(t_0) = 0$$

In case $p_0 + p_1 < 1$ then this is a contradiction because

$$\varphi''(t) = (2!)p_2 + (3!)t + \cdots, \quad 0 < t < 1.$$

Since at least one term on right side non-zero, we can not have $\varphi''(t_0) = 0$.

On the other hand if $p_0 + p_1 = 1$, then $y = \varphi(t)$ is a straight line and meets the straight line $y = t$ at at most one $t \in R$ and $t = 1$ is visible. So meets at no other point.

Thus in any case there is at most one solution in $[0, 1)$. proving (2) of the theorem.

(c) If $\varphi(t) = t$ has a solution $a \in [0, 1)$ then MVT applied on the interval $[a, 1]$ to $\varphi(t) - t$ will give a point $c \in (0, 1)$ with $\varphi'(c) = 1$. In case $p_0 + p_1 < 1$ then as noted above $\varphi'' > 0$ in $(0, 1)$ so that φ is strictly increasing there. This shows

$$m = \lim_{t<1; t\uparrow 1} \varphi'(t) > 1.$$

Recall that m is the mean offspring of a fellow.

Of course The case $p_0 + p_1 = 1$ can not occur, because in this case there is no solution of our equation in $[0, 1)$. Thus

if $\varphi(t) = t$ *has a solution in* $[0, 1)$ *then* $m > 1$. $\quad (\bullet)$

Assume now that 1 is the only solution of $\varphi(t) = t$. Clearly $p_0 > 0$, otherwise zero is a solution of the above equation. Thus the graph $y = \varphi(t)$ is above graph $y = t$ at the point $t = 0$ and since they do not meet any where till 1, we conclude

$$\varphi(t) > t \quad (i.e.) \quad 1 - \varphi(t) < 1 - t \quad \forall t < 1$$

or

$$\frac{1-\varphi(t)}{1-t} \leq 1$$

Taking limit as $t \uparrow 1$ you get

$$m = \varphi'(1) \leq 1$$

[even though each difference quotient is smaller than one, unless I show that the limit exists, I can not conclude that $m \leq 1$. Though we can do this using our assumption that mean is finite, it is not necessary to justify. We can argue as follows. No matter how close to 1 you take $t < 1$ the above inequality with MVT tells you that $\varphi'(u) \leq 1$ for some $u \in (t, 1)$. Since φ' is increasing this is enough to conclude $\lim_{t<1;\, t\uparrow 1} \varphi'(t) \leq 1$. But this is m as proved last time. It is not necessary to talk about $\varphi'(1)$.]

Thus

if $\varphi(t) = t$ *has no solution in* $[0, 1)$ *then* $m \leq 1$. (●●)

It is easy to conclude from (●)(●●) part (3) of the theorem.

Back to Markov Chains:

Actually Branching process is also a markov chain.

Transition matrix is easy to see. If $X_n = i$, then there are i fellows now and the total offspring of all these have pgf $[\varphi(t)]^i$. Hence, p_{ij} is the coefficient in t^j in $[\varphi(t)]^i$.

In the Branching process, it is possible to show that X_n has only two alternatives: either it becomes zero sooner or later OR it diverges to infinity. Thus this kind of process has different flavour than usual markov chains that we have seen earlier: in those you essentially keep on moving among states.

So we have a countable set S and transition matrix $(p_{ij} : i, j \in S)$. if today you are in i, tomorrow you will go to j with probability p_{ij}. The matrix (p_{ij}) is also called one-step transition probability matrix. Also recall $P_i(A)$ of an event A is the probability of the event under $X_0 = i$, the starting initial state being i.

Though our interest in this course is finite state chains, we shall, for a short while, consider generalities. This is for three reasons. Firstly, to show you the general landscape of Markov chains. Secondly to answer som interesting questions about random walks. For example, if you start at zero you return to zero surely. Would you surely hit 89? Thirdly, to impress upon you that the technique you learnt in understanding Random walk is so powerful and useful that you can carry it over for markov chains.

Recall
$$p_{ij}^{(n)} = P_i(X_n = j)$$
probability of being in state j on day n; having started at i on day zero. This is also the (ij)th entry of P^n, the n-th power of the matrix P. We take $p_{ij}^{(0)} = 1$ or zero according as $i = j$ or not. In other words P^0 is the identity matrix.

Similarly
$$f_{ij}^{(n)} = P_i\{X_m \neq j \quad \forall m < n; \quad X_n = j\}$$
Probability that first return to j occurs on day n; having started at i on day zero. We take
$$f_{ij}^{(0)} \equiv 0$$
This stands to reason because even if you started at i, *return* has not yet occurred.

We have the renewal equation, you will see that proof is also exactly as in Random Walk case.

■ **Renewal equation:**
For $n \geq 1$, we have
$$p_{ij}^{(n)} = \sum_{m=0}^{n} f_{ij}^{(m)}\, p_{jj}^{n-m)} \qquad ■$$
of course, we could have omitted the term $(m = 0)$ on right side since $f_{ij}^{(0)} = 0$. The above equation simply says the following: To visit j on day n; you must make a first visit to j on some day $m \leq n$; and then starting at j on that day you should return to j on day n.

Is the renewal equation true for $n = 0$?
$$p_{ij}^{(0)} = f_{ij}^{(0)} p_{jj}^{(0)} = 0 \quad \forall i \neq j$$

and is hence true when $i \neq j$. However when $i = j$, we see

$$p_{ii}^{(0)} = 1 \neq 0 = f_{ii}^{(0)} p_{ii}^{(0)}.$$

Before proving the renewal equation, let us see the importance of it.

■ Theorem:

For $i \neq j$, the convolution of the two sequences $\{f_{ij}^{(n)}; n \geq 0\}$ and $\{p_{jj}^{(n)}; n \geq 0\}$ is the sequence $\{p_{ij}^{(n)}; n \geq 0\}$.

($i = j$): Convolution of the two sequences $\{f_{ii}^{(n)}; n \geq 0\}$ and $\{p_{ii}^{(n)}; n \geq 0\}$ is the sequence $\{p_{ii}^{(n)}; n \geq 1; \text{ and } 0 \quad \text{for} \quad n = 0\}$ ■

Thus if we define the generating functions

$$P_{ij}(t) = \sum_{n \geq 0} p_{ij}^{(n)} t^n$$

$$F_{ij}(t) = \sum_{n \geq 0} f_{ij}^{(n)} t^n$$

then we have

■ Theorem:

$$P_{ij}(t) = F_{ij}(t) P_{jj}(t) \qquad i \neq j$$

$$P_{ii}(t) = F_{ii}(t) P_{ii}(t) + 1. \qquad ■$$

The second equation implies

$$P_{ii}(t) = \frac{1}{1 - F_{ii}(t)} \qquad 0 < t < 1$$

Note that $\{f_{ij}^{(n)} : n \geq 0\}$ are probabilities of disjoint events and hence their gf is always strictly smaller than one for $t < 1$. Thus, in particular, by taking limits as $t \uparrow 1$ in the above equality we are led to the following result. Remember

$$\lim_{t<1;\, t\uparrow 1.} P_{ii}(t) = \sum_n p_{ii}^{(n)}; \qquad \lim_{t<1;\, t\uparrow 1.} F_{ii}(t) = \sum_n f_{ii}^{(n)}$$

■ Theorem:

$$\sum_n p_{ii}^{(n)} = \infty \quad \longleftrightarrow \quad \sum_n f_{ii}^{(n)} = 1. \qquad ■$$

Definition: We say that a state i is recurrent if starting from i we surely return to i. Otherwise the state is called transient

In other words i is recurrent if

$$P_i(X_n = i \quad \text{for some} \quad n \geq 1) = 1$$

and i is transient if

$$P_i(X_n = i \quad \text{for some} \quad n \geq 1) < 1$$

Equivalently, i is recurrent if $\sum\limits_n f_{ii}^{(n)} = 1$ and transient if $\sum\limits_n f_{ii}^{(n)} < 1$.

In view of the previous theorem we have,

■ Theorem:
A state i is recurrent if $\sum\limits_n p_{ii}^{(n)} = \infty$ and transient if $\sum\limits_n p_{ii}^{(n)} < \infty$. ■

We need to now justify the terminology. Is recurrent state 'really' recurrent. After all, recurrent means something that happens again and again. So if i is recurrent will you keep on visiting again and again? Is transient state really 'transient' that is, visited only finitely many times , that is, after some day you stop visiting it. Yes these are all true.

Consider one dimensional random walk. Start at zero. Since you are sure to return to zero, you feel that after returning to zero the walk starts again 'afresh like random walk' and so will visit zero agin for a second time. having visited second time, it starts 'afresh again like a random walk' and visits again for a third time and so on. Thus it visits infinitely many times. Though the intuition is correct, there is a subtle point that needs to be justified and understood.

The markov property says that on any specific day if you are at zero, you will go to ± 1 with equal probability. thus on day 30 you are at zero then on be assured that on day 31 you will be at ± 1 etc. similarly, if on day 89 you are at zero on day 90 you will be at ± 1 with probability 1/2 and 1/2.

But return to zero occurs 'not on a specific day' but on some random day. is the above statement still correct for a random day? To understand more clearly, if the outcomes are

$$0, 1, 0, -1, \cdots$$

then first return happens on day 2. If the outcomes are

$$0, -1, -2, -1, 0, -1 \cdots$$

then first return to zero happens on day 4. If the outcomes are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1, 0, -1 \cdots$$

then first return happens on day 16.

Thus the day of first return is random, you can not say that it is surely 24 or 46 or 2 etc. So the question is: can we assume that even if the day is random; if we find ourselves at zero then the walk starts 'afresh' and next day it is at ± 1 with equal probabilities? *unfortunately this is false!*

To bring home the point, start at zero. I assure you that you will surely visit -1. Let T^* be the day of first visit to -1. In the above three 'runs' of the game the day happens to be 3 in the first scenario; 1 in the second scenario and 17 in the third scenario. thus depending on the scenario, there is a day of first visit to -1. It is well defined and $T^* \geq 1$. (Chances of never visiting -1 is zero).

Consider the previous day, $T = T^* - 1$. This is a random time. In the above scenarios it is 2,0,16 respectively. Now ask yourself: Ok on this day T I am at zero. Next day, that is, day $T + 1$ am I at ± 1 with equal probability? Think till you get the answer.

Thus the simple 'markov property'; or the innocent statement 'on any day if you are at i, next day you move to j with probability p_{ij}' is not as easy a concept as the English makes it out to be. The fact that 'any day' means 'any specific day' and 'not random day' is very very important and subtle. This distinction is profound. Matters that I have explained now are NOT for a first course and we shall not dwell on these. These matters lead to interesting concepts.

Unfortunately, unless you understand these matters clearly, unless you stop making loose statements without justification, you will be committing mistakes. This does not mean you should kill intuition, intuition is very essential and important. You must encourage and develop it, but ask: is my intuition right and how do I justify it if some one questions (actually you should justify to yourself, even if no one questions!)