Automata, Logic and Algebra for (Finite) Word Transductions

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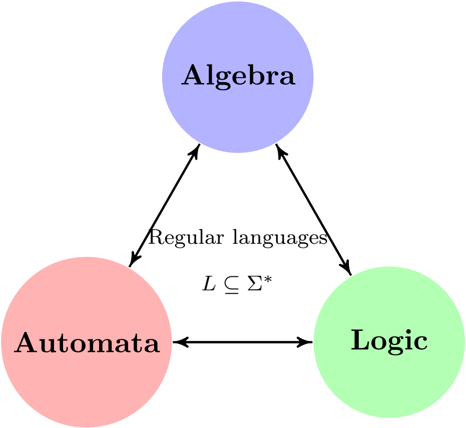
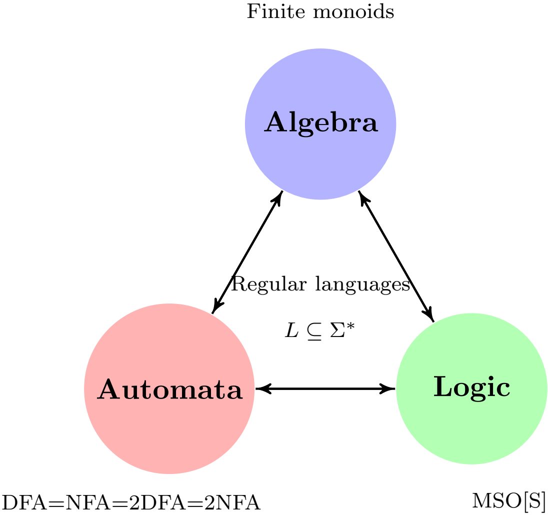
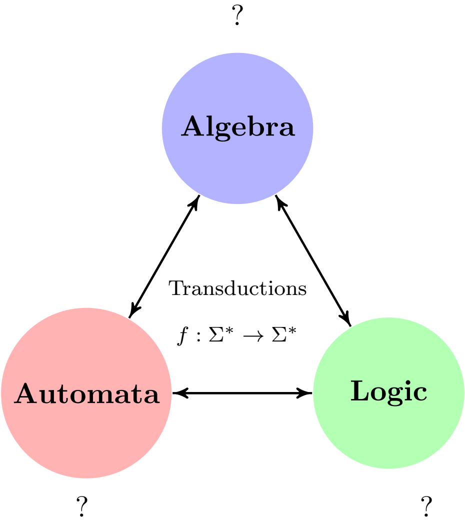
Trinity for Regular Languages



Trinity for Regular Languages



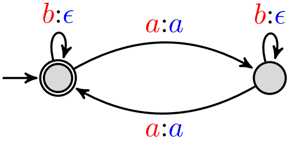
Objective of the talk



Automata models for transductions

Automata for transductions: transducers

f_{del} :



Automata for transductions: transducers

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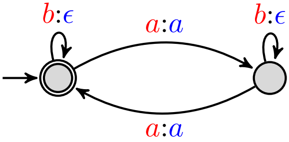

$aabaa \mapsto aaaa$

Automata for transductions: transducers

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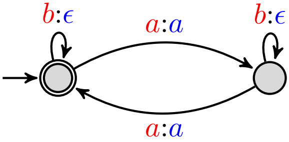

$$aabaa \mapsto aaaa$$

$$aaba \mapsto \text{undefined}$$

Automata for transductions: transducers

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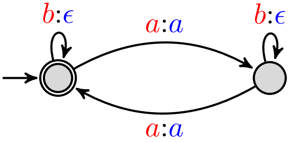
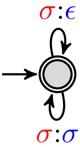


$$
\begin{aligned}
aabaa &\mapsto aaaa \\
aaba &\mapsto \text{undefined} \\
dom(f_{del}) &= \text{'even number of } a\text{'}
\end{aligned}
$$

Non-determinism

In general, transducers define binary *relations* in $\Sigma^* \times \Sigma^*$

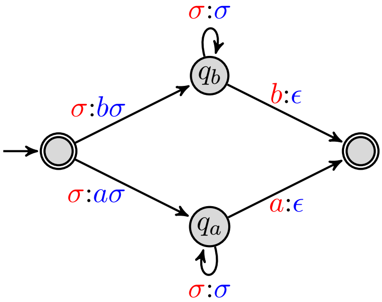


realizes $\{(u, v) \mid v \text{ is a subword of } u\}$

Sequential vs Non-deterministic functional

Non-deterministic transducers may define functions:

f_{sw} :



for all $\sigma \in \Sigma$

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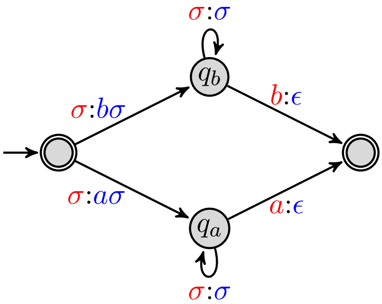

for all $\sigma \in \Sigma$

$$babaa \mapsto ababa$$

Sequential vs Non-deterministic functional

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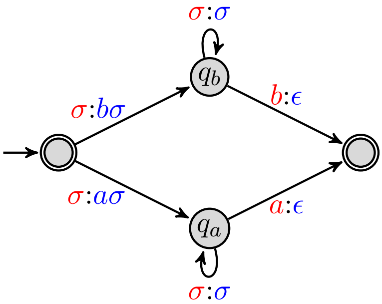


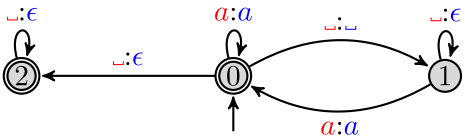
for all $\sigma \in \Sigma$

$$babaa \quad \mapsto \quad ababa$$

$$u\sigma \quad \mapsto \quad \sigma u \qquad |u| \geq 1$$

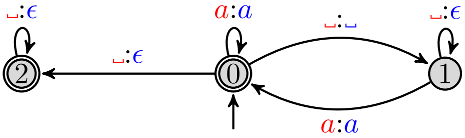
input-determinism (aka sequential) $<$ non-determinism $\cap$ functions

Determinizability



␣ = white space

Determinizability

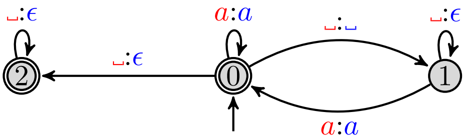


␣ = white space

␣␣aa␣␣␣a␣␣ ↦ ␣aa␣a

Determinizability

␣ = white space

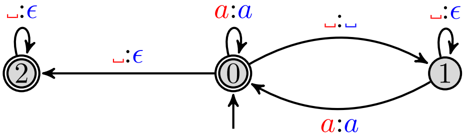


␣␣*aa*␣␣␣*a*␣␣ ↦ ␣*aa*␣*a*

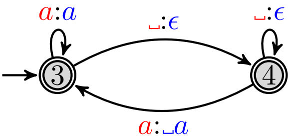
Is non-determinism needed ?

Determinizability

␣ = white space

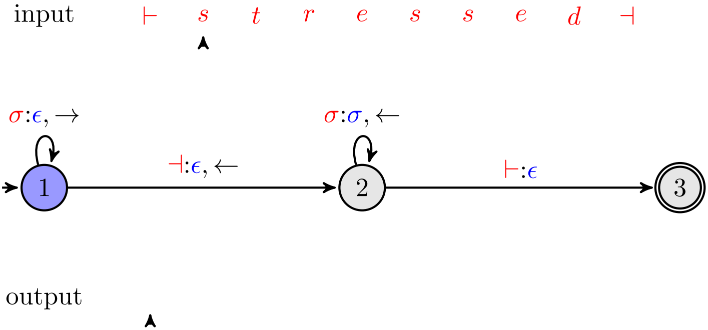


␣␣aa␣␣␣a␣␣ ↦ ␣aa␣a



Is non-determinism needed ? No.

Two-way transducers



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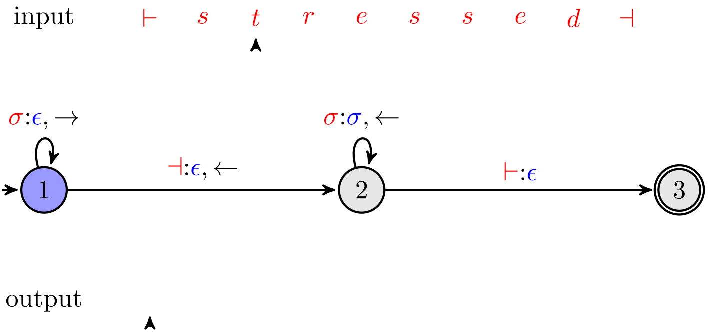

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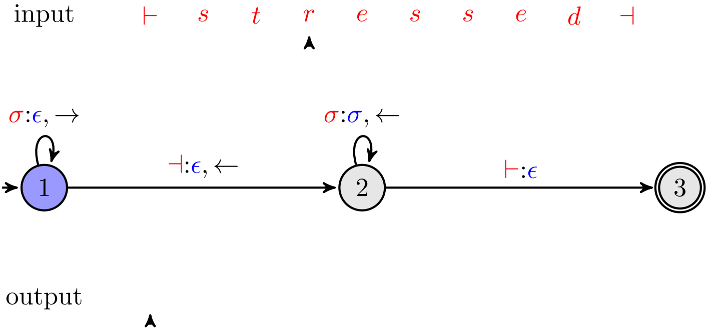
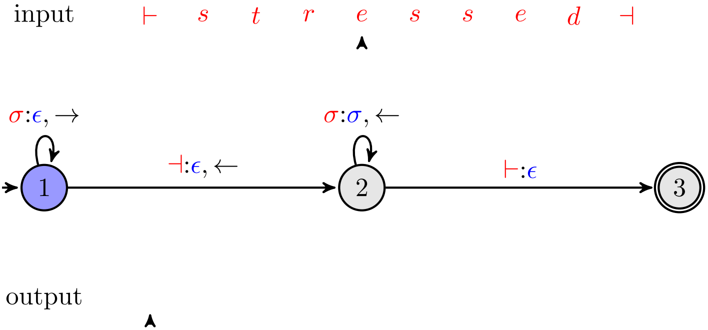


input $\vdash$ s t r e s s e d $\dashv$

output

Two-way transducers



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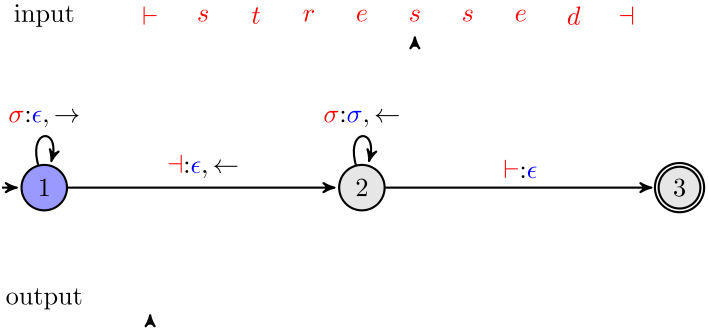

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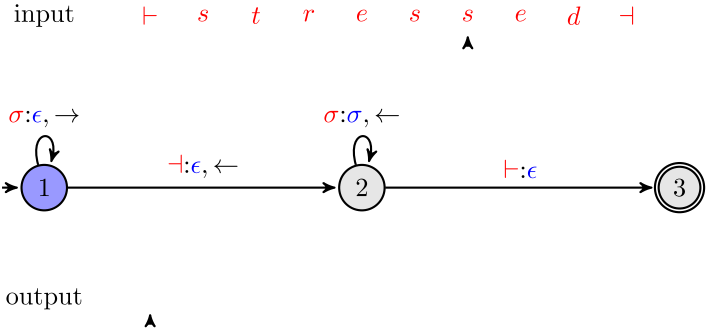

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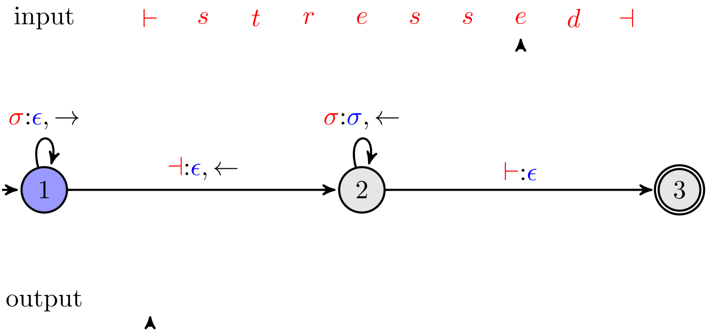

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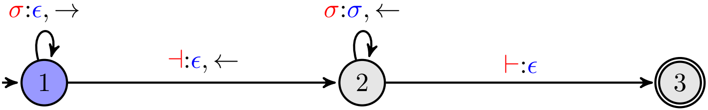
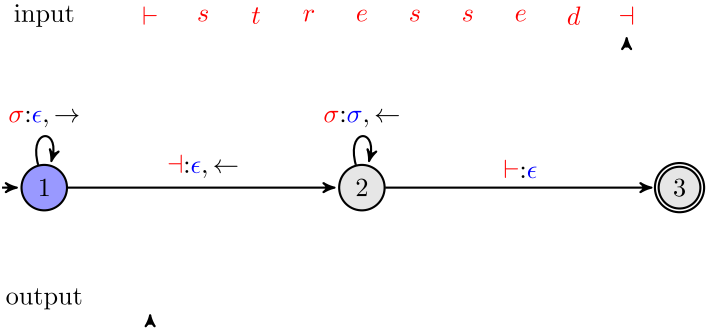


output

Two-way transducers



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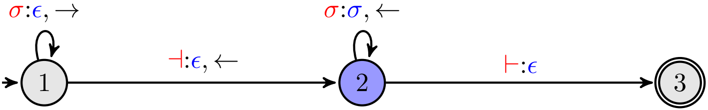
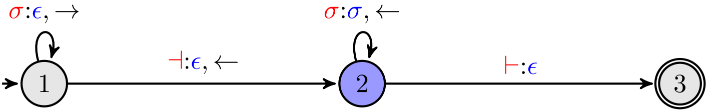


output

Two-way transducers

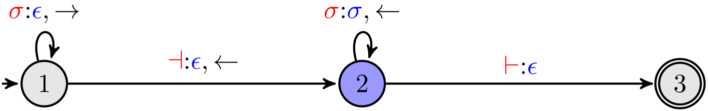
input $\vdash$ s t r e s s e d $\dashv$



output d

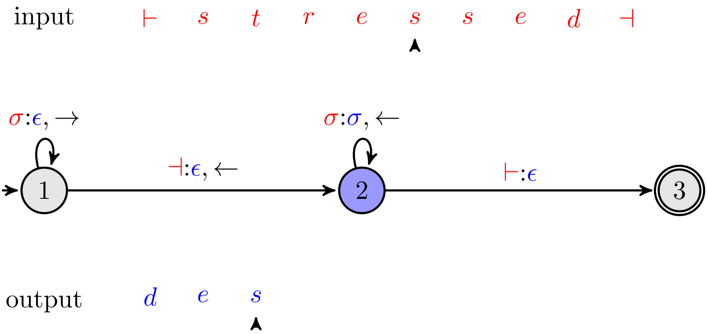
Two-way transducers

input $\vdash$ s t r e s s e d $\dashv$



output d e

Two-way transducers



Two-way transducers

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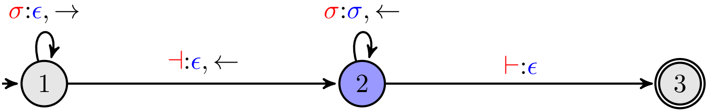
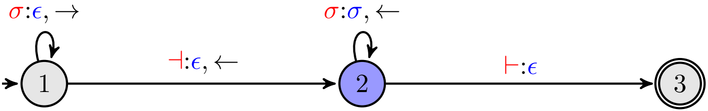


output d e s s

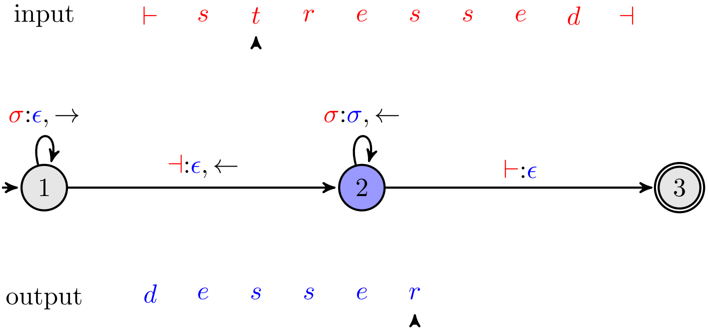
Two-way transducers

input $\vdash \; s \; t \; r \; e \; s \; s \; e \; d \; \dashv$



output $d \; e \; s \; s \; e$

Two-way transducers



Two-way transducers

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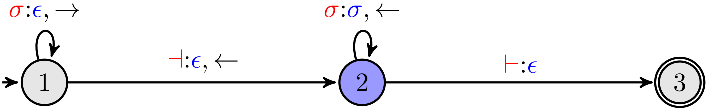

output d e s s e r t

Two-way transducers

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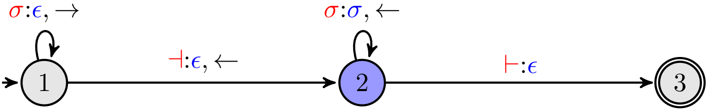

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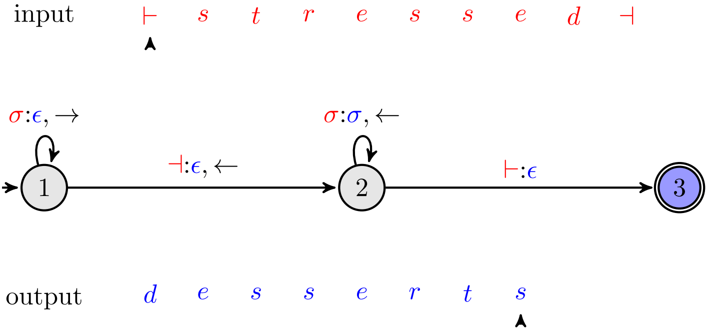

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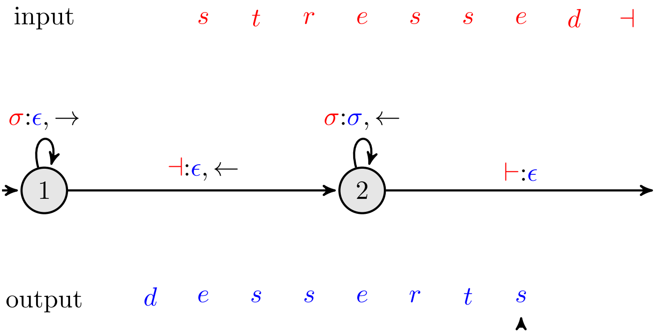


one-way < two-way

☺ decidable equivalence problem (Culik, Karhumaki, 87).

☺ closed under composition ∘ (Chytil, Jakl, 77)

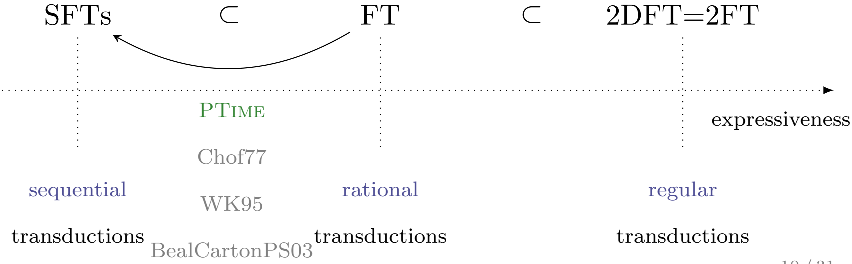
Landscape of Transducer Classes

SFTs ⊂ FT ⊂ 2DFT=2FT

expressiveness

sequential transductions	rational transductions	regular transductions

Landscape of Transducer Classes



SFTs ⊂ FT ⊂ 2DFT=2FT

expressiveness

PTime

Chof77

WK95

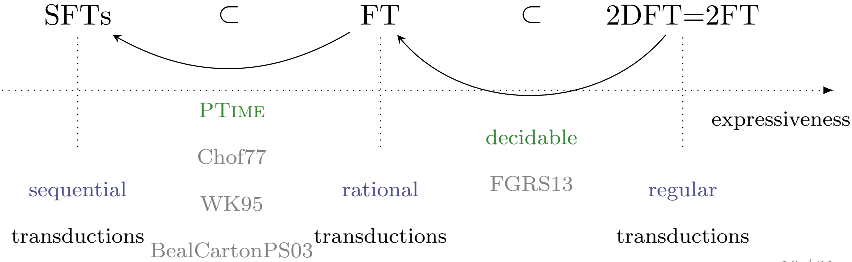
BealCartonPS03

sequential transductions

rational transductions

regular transductions

Landscape of Transducer Classes



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decidable

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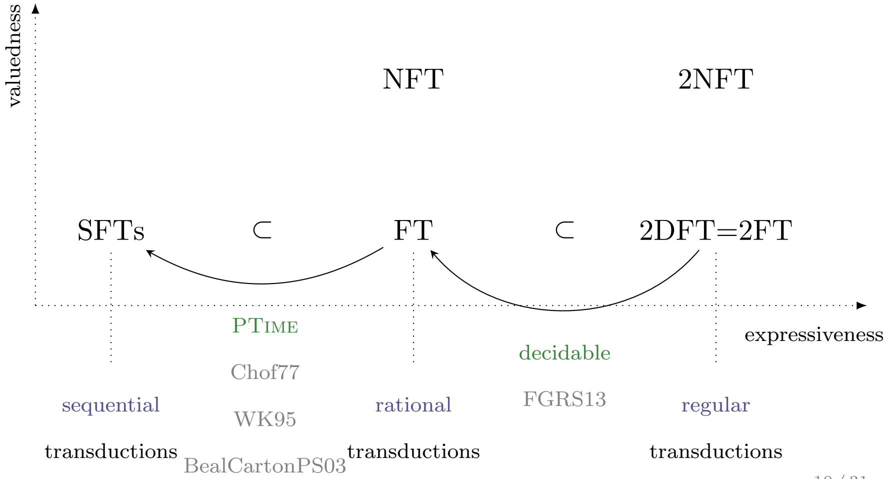

valuedness

NFT 2NFT

SFTs ⊂ FT ⊂ 2DFT=2FT

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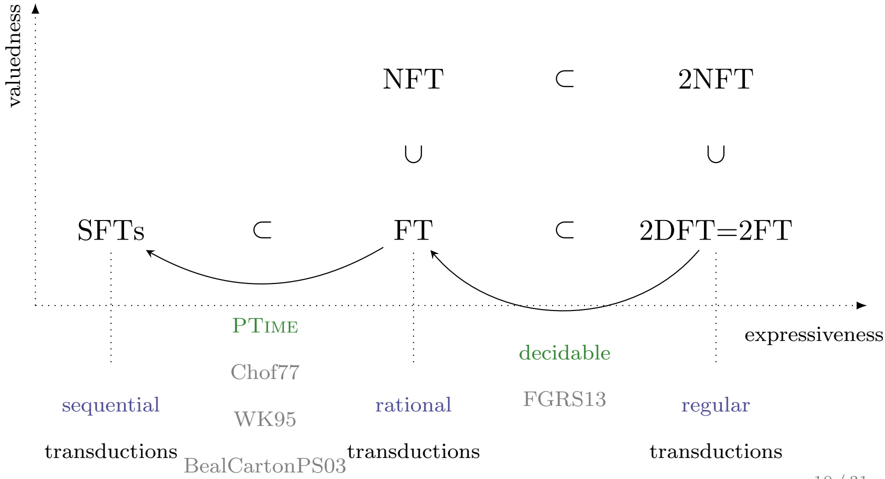

valuedness

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∪ ∪

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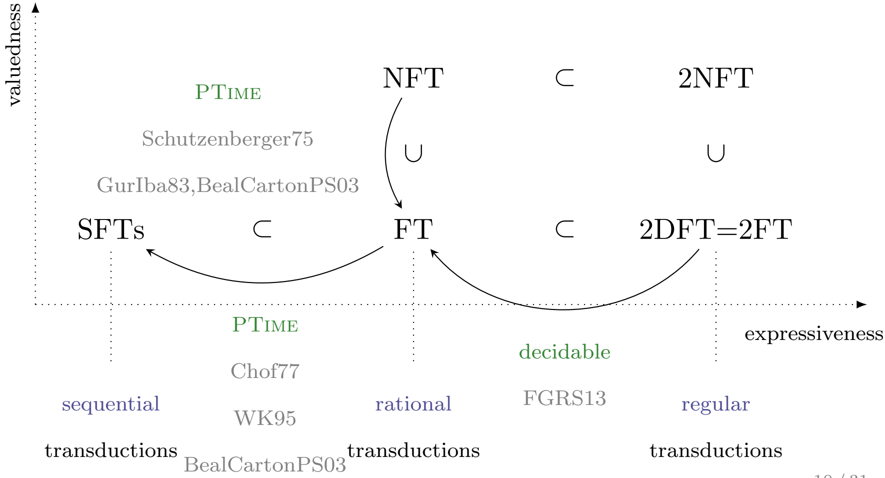

valuedness

PTIME

Schutzenberger75

GurIba83,BealCartonPS03

NFT ⊂ 2NFT

∪ ∪

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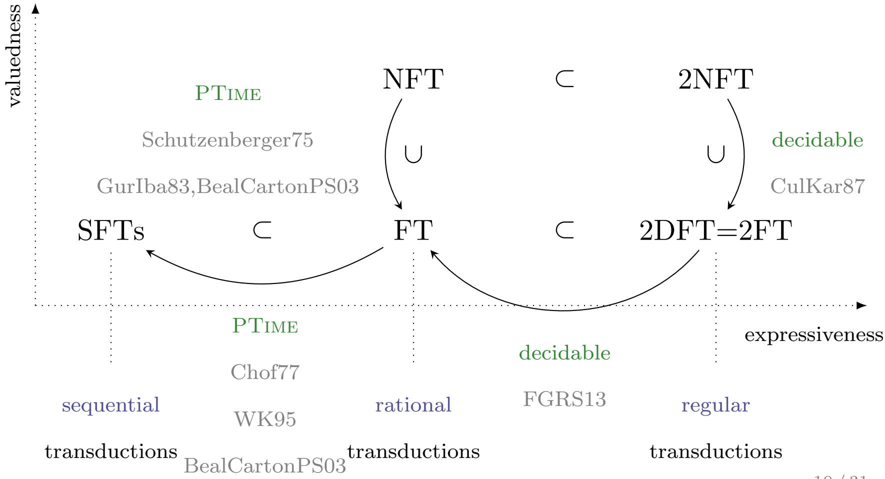

valuedness

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NFT ⊂ 2NFT

∪ ∪

decidable

CulKar87

SFTs ⊂ FT ⊂ 2DFT=2FT

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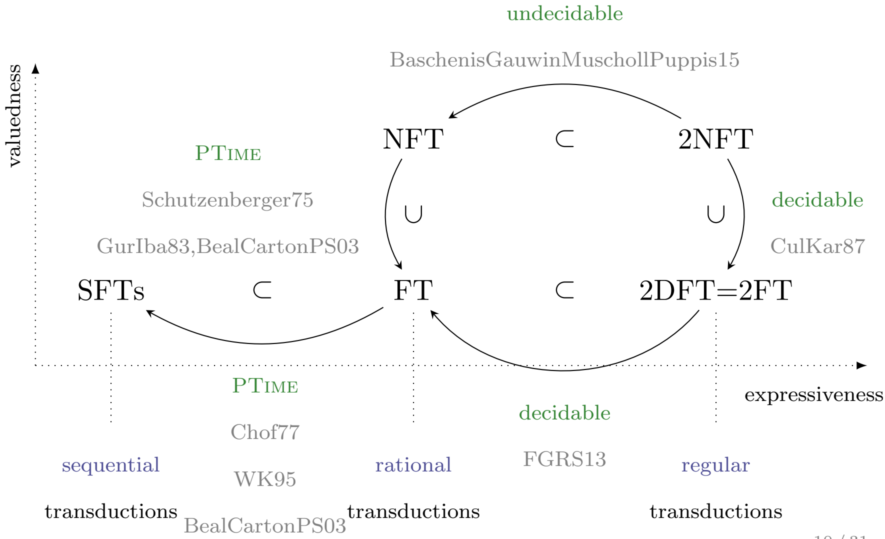

undecidable

BaschenisGauwinMuschollPuppis15

NFT ⊂ 2NFT

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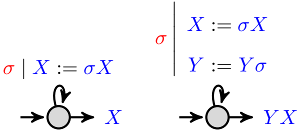
Other recent results

Transducers with registers

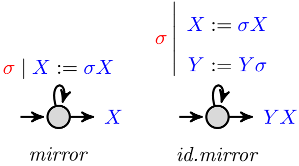


mirror *id.mirror*

- deterministic one-way
- equivalent to 2DFT if copyless updates (Alur, Cerny, 10)
- decidable equivalence problem (F., Reynier) ∼ HDT0L

Other recent results

Transducers with registers



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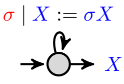
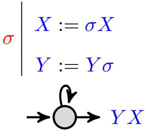


mirror



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Two-way to one-way transducers

- decidable, but non-elementary complexity in (FGRS13)
- elementary complexity first obtained for subclasses (sweeping) by (BGMP15)
- recently for the full class (BGMP17)

Logic for transductions

(Courcelle) MSO Transformations

"interpreting the output structure in the input structure"

- output predicates defined by MSO[S] formulas interpreted over the input structure

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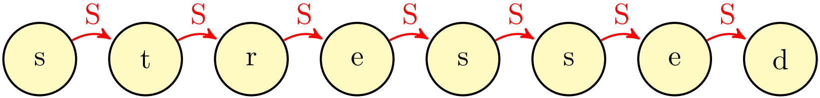

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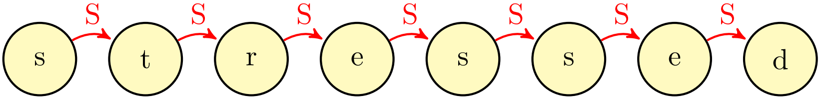

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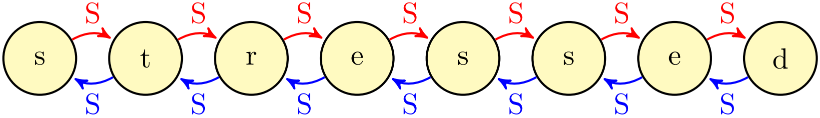

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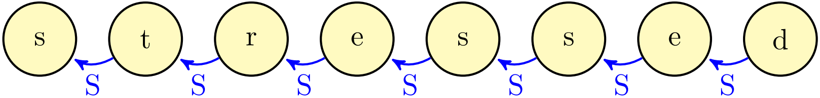

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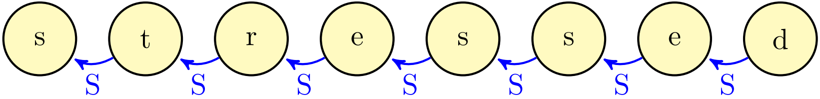
"interpreting the output structure in the input structure"

- output predicates defined by MSO[S] formulas interpreted over the input structure



$$\phi_S(x,y) \quad \equiv \quad S(y,x)$$

$$\phi_\sigma(x) \quad \equiv \quad \sigma(x)$$

- input structure can be copied a fixed number of times: $u \mapsto uu$, or $u \mapsto u.\mathrm{mirror}(u)$.

Büchi Theorems for Word Transductions

Let $f : \Sigma^* \to \Sigma^*$.

Theorem (Engelfriet, Hoogeboom, 01)

f is 2FT-definable iff f is MSO-definable.

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Theorem (Bojanczyk 14, F. 15)

f is (1)FT-definable iff f is order-preserving MSO-definable.

Order-preserving MSO: $\phi_S^{i,j}(x,y) \models x \preceq y$.

First-order transductions

Replace MSO by FO formulas.

Results

- equivalent to aperiodic transducers with registers (F., Trivedi, Krishna S., 14)
- and to aperiodic 2DFT (Carton, Dartois, 15) (Dartois, Jecker, Reynier, 16)

Algebraic characterizations of transductions

Myhill-Nerode congruence for

- $u \sim_L v$ if: for all $w \in \Sigma^*$, $uw \in L$ iff $vw \in L$
- u and v have the same “effect” on continuations w
- **Myhill-Nerode's Thm**: L is regular iff $\Sigma^*/_{\sim_L}$ is finite
- canonical (and minimal) deterministic automaton for L, $\Sigma^*/_{\sim_L}$ as set of states

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Goal

Extend Myhill-Nerode's theorem to classes of transductions

Sequential transductions (Choffrut)

Refinement of the MN congruence.

Two ideas

1. produce asap: $F(u) = LCP\{f(uw) \mid uw \in dom(f)\}$

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   2.1 $u \sim_{dom(f)} v$
   2.2 $F(u)^{-1}f(uw) = F(v)^{-1}f(vw)$ $\forall w \in u^{-1}dom(f)$

"u and v have the same effect on continuations w w.r.t. domain membership and produced outputs"

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Theorem (Choffrut)

f is sequential iff $\sim_f$ has finite index

$\sim_f$ is a right congruence ⇝ canonical and minimal transducer !

Transitions: $[u] \xrightarrow{\sigma|F(u)^{-1}F(u\sigma)} [u\sigma]$

Rational transductions are almost sequential

- $f_{sw} : u\sigma \mapsto \sigma u$ is not sequential
- but sequential modulo *look-ahead information* $\mathcal{I} = \{a, b, \epsilon\}$.

$a\,b\,b\,a\,a\,a\,a\,b\,b\,b\,b\,a\,b$

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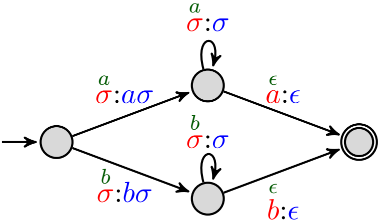
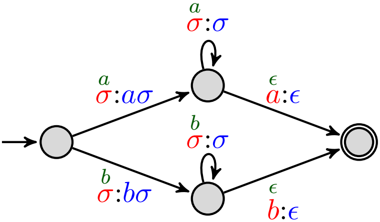


Rational transductions are almost sequential

- $f_{sw} : {\color{red}u\sigma} \mapsto {\color{blue}\sigma u}$ is not sequential
- but sequential modulo *look-ahead information* $\mathcal{I} = \{a, b, \epsilon\}$.

$$\overset{b}{a}\,\overset{b}{b}\,\overset{b}{b}\,\overset{b}{a}\,\overset{b}{a}\,\overset{b}{a}\,\overset{b}{a}\,\overset{b}{b}\,\overset{b}{b}\,\overset{b}{b}\,\overset{b}{b}\,\overset{b}{a}\,\overset{\epsilon}{b}$$



- look-ahead information: $\mathcal{L} : \Sigma^* \to \mathcal{I}$
- $f[\mathcal{L}]$: f with input words extended with look-ahead information

Results

Theorem (Elgot, Mezei, 65)

f is rational iff $f[\mathcal{L}]$ is sequential, for some finite look-ahead information $\mathcal{L}$ computable by a right sequential transducer.

Original statement: $RAT = SEQ \circ RightSEQ$.

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Original statement: $RAT = SEQ \circ RightSEQ$.

Reutenauer, Schützenberger, 91

- *canonical look-ahead* given by a congruence $\equiv_f$
- identify suffixes with a 'bounded' effect on the transduction of prefixes
- characterization of rational transductions
  - f is rational
  - $\equiv_f$ has finite index and $f[\equiv_f]$ is sequential
  - $\equiv_f$ and $\sim_{f[\equiv_f]}$ have finite index.

Definability problems

Rational Transductions

Given f defined by T, is it definable by some $\mathcal{C}$-transducer ?

- sufficient conditions on $\mathcal{C}$ to get decidability (F., Gauwin, Lhote, LICS'16)
- includes aperiodic congruences: decidable FO-definability
- even PSpace-c (F., Gauwin, Lhote, FSTTCS'16)

Regular Transductions

- existence of a canonical transducer if *origin* is taken into account (Bojanczyk, ICALP'14)

- decidable FO-definability *with* origin, open without

A new logic for transductions

joint with Luc Dartois and Nathan Lhote

MOTIVATIONS

- specify properties of transductions in a high-level formalism: a logic
- decidable model-checking

$$T \models \varphi$$

a transducer (NFT, 2DFT, ...)

a formula

$$[\![T]\!] \subseteq [\![\varphi]\!]$$

a transduction $\Sigma^* \to \Sigma^*$ (function)

a relation $\subseteq \Sigma^* \times \Sigma^*$

EXAMPLES

- "there exists at least one 'a' in the output"

  $$\{ (u, v_1 a v_2) \mid u, v_1, v_2 \in \Sigma^* \}$$

- "every request is processed exactly one"

  $$\{ (r_{i_1} \cdots r_{i_k}, g_{\pi(i_1)} \cdots g_{\pi(i_k)}) \mid \pi \text{ permutation} \}$$

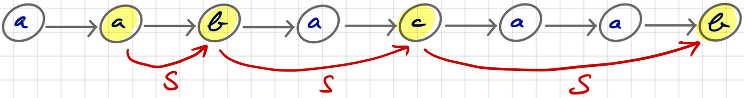
- more generally : shuffle

  $$\{ (\sigma_1 \cdots \sigma_n, \sigma_{\pi(1)} \cdots \sigma_{\pi(n)}) \mid \pi \text{ permutation} \}$$

NON-DETERMINISTIC MSOT (NMSOT)   COURCELLE

use second-order parameters $X_1, \dots, X_n$

$\{ (u, v) \mid v \text{ is a subword of } u, |v| \text{ even} \}$



$$\varphi_{dom}(X) \equiv even(X)$$

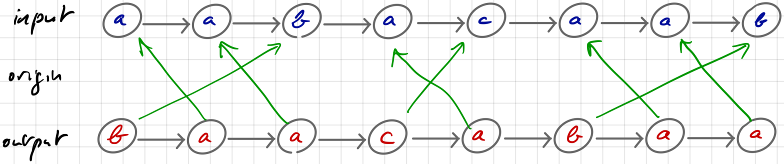
$$\varphi_S(x,y,X) \equiv x,y \in X \wedge x < y \wedge \neg(\exists z \in X \, . \, x < z < y)$$

FROM NMSOT to NEW LOGIC

NMSOT is not satisfactory as a specification language

- * too "operational"
- * the previous examples are not NMSOT

IDEA: SEE TRANSDUCTIONS AS SINGLE STRUCTURES WITH ORIGIN



Predicates: $\leq_{in}$, $\leq_{out}$, o

EXAMPLES

- "there exists at least one 'a' in the output"

$$\exists^{out} x \,.\, a(x)$$

- shuffle

$$BIJ(\sigma) \wedge \underbrace{\bigwedge_{\sigma \in \Sigma} \forall^{in} x \, \forall^{out} y \,.\, \begin{array}{c} \sigma(x,y) \wedge \\ \sigma(x) \end{array} \rightarrow \sigma(y)}_{LABPRES(\sigma)}$$

- identity

$$BIJ(\sigma) \wedge LABPRES(\sigma) \wedge ORDERPRES(\sigma)$$

where $ORDERPRES(\sigma) \equiv \forall^{in}_{x,x'} \forall^{out}_{y,y'} \begin{array}{c} \sigma(x,y) \\ \sigma(x',y') \\ x \leq_{in} x' \end{array} \rightarrow y \leq_{out} y'$

RESULTS

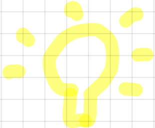
$FO[\leq_{in}, \leq_{out}, \theta]$ is undecidable

$FO_2[\leq_{out}, \theta, MSO_{bin}[\leq_{in}]]$ is decidable

→ capture MSOT (for functions)

→ all previous examples are definable

→ $T \models \varphi$ decidable for T: 2DFT



Reduction to a data word logic

$$FO_2[\leq, MSO_{bin}[\lesssim]]$$

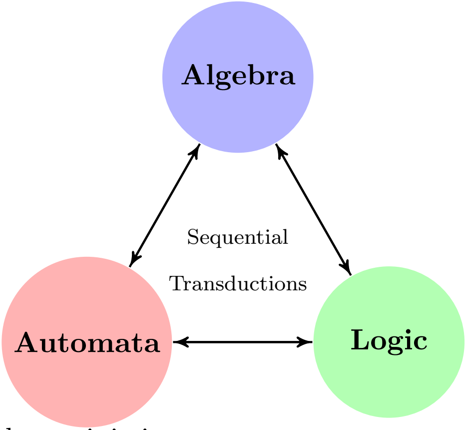
linear-order

total pre-order (data comparison)

"origin = data"

Summary: sequential transductions

finiteness of $\sim_f$ (Choffrut)

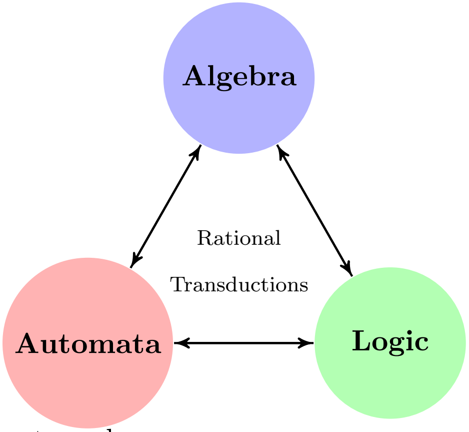


input-deterministic
one-way transducers

prefix-independent MSO, ?

Summary: rational transductions

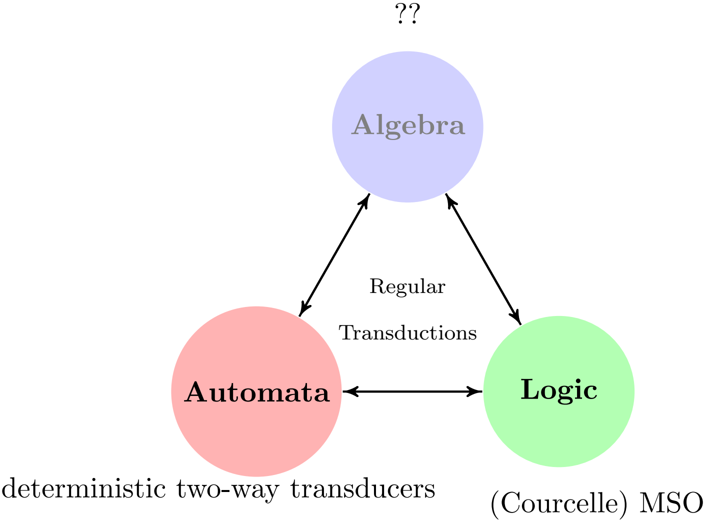
finiteness of $\equiv_f$ and $\sim_{f[\equiv_f]}$ (Reutenauer, Schützenberger)



one-way transducers

order-preserving MSO

Summary: regular transductions



Other works

- AC^0 transductions (Cadilhac,Krebs,Ludwig,Paperman,15)
- variants of two-way transducers (Guillon, Choffrut, 14,15,16), (Carton, 12) (McKenzie, Schwentick, Thérien, Vollmer, 06)
- model-checking and synthesis problems for rational transductions with "similar origins" (F., Jecker, Löding, Winter, 16)
- non-determinism
- infinite words, nested words, trees

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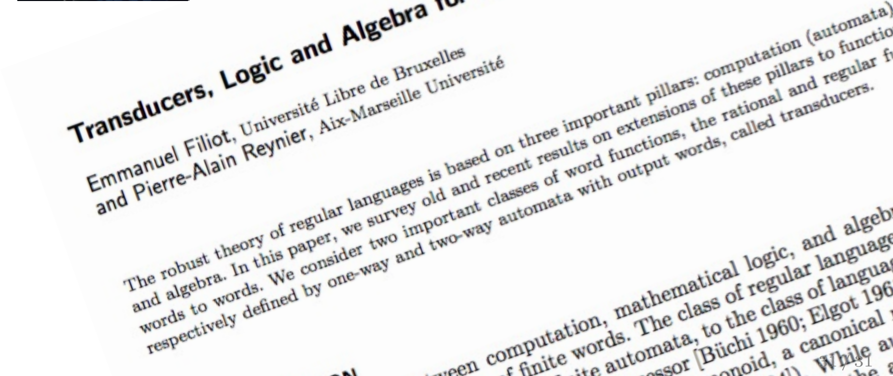

Transducers, Logic and Algebra for Functions of Finite Words

Emmanuel Filiot, Université Libre de Bruxelles
and Pierre-Alain Reynier, Aix-Marseille Université

The robust theory of regular languages is based on three important pillars: computation (automata)
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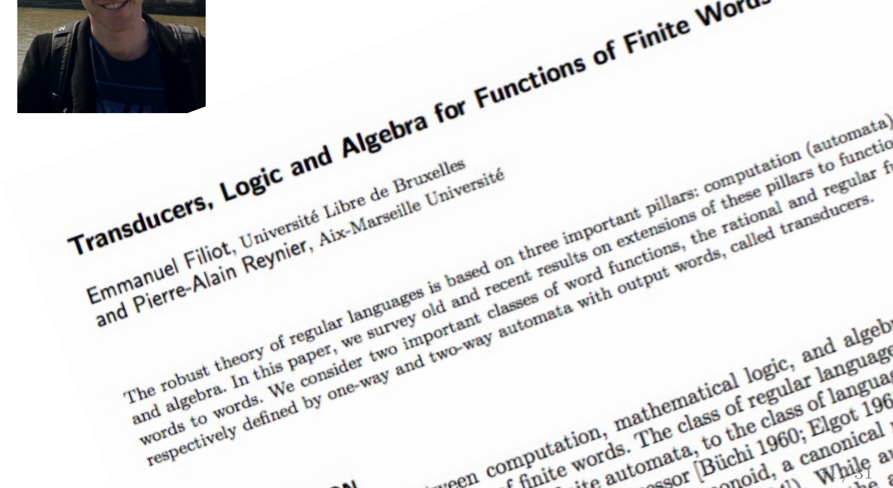
Thank You.



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