THE PERFECT HALF-SPACE TECHNIQUE FOR MULTI-DIMENSIONAL ENERGY GAMES

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MULTI-DIMENSIONAL GAME GRAPHS

$$(V, E, d, \omega)$$

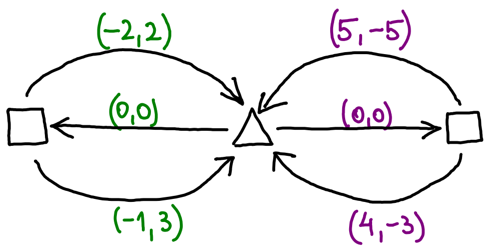
$V = V_1 \cup V_2$ finite

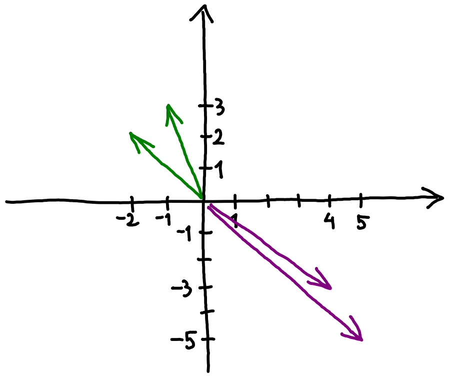
$E \subseteq V \times V$

$\omega : E \to \mathbb{Z}^d$

MULTI-DIMENSIONAL MEAN-PAYOFF GAMES

Play: $e_1, e_2, e_3, e_4, \dots$





Player 1 wins: $\boxed{\liminf_{n\to\infty} \sum_{i=1}^{n} \frac{\omega(e_i)}{n} \geq (0,0,\dots,0)}$

MULTI-DIMENSIONAL MEAN-PAYOFF GAMES

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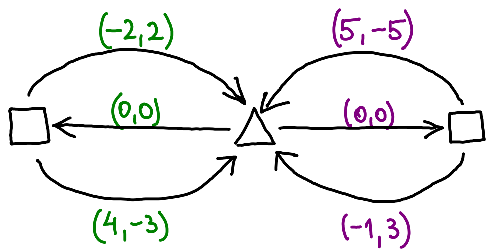
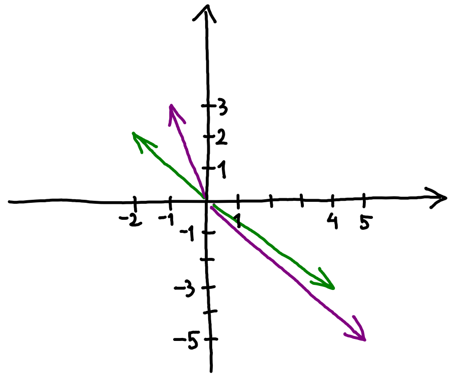
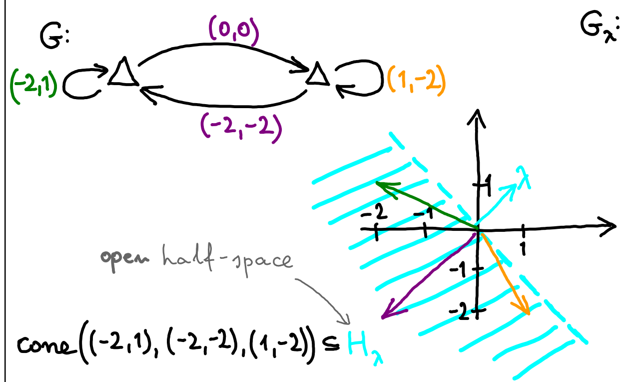
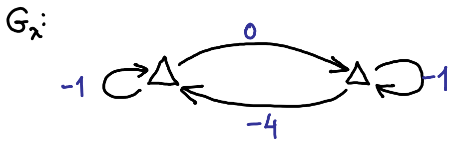




Player 1 wins: $\boxed{\liminf_{n\to\infty} \sum_{i=1}^{n} \frac{\omega(e_i)}{n} \geq (0,0,\dots,0)}$

Player 2: Chatterjee-Velner sufficient condition





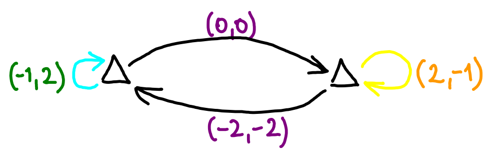
$\omega_\lambda(e) := \omega(e) \cdot \lambda$

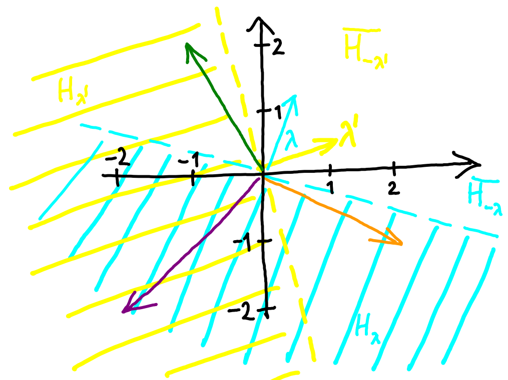
open half-space

$\mathrm{cone}((-2,1),(-2,-2),(1,-2)) \subseteq H_\lambda$

There is $\lambda \in \mathbb{Q}_+^d$, s.t. player 2 can force in G all simple cycles to be in H_λ	iff	There is $\lambda \in \mathbb{Q}_+^d$, s.t. player 2 (Min) can achieve payoff < 0 in G_λ

Consequences of MPG determinacy



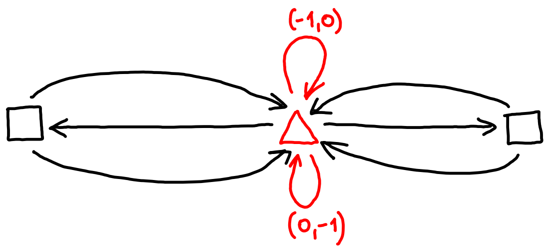


$\mathrm{cone}((-1,2),(-2,-2),(2,-1)) = \mathbb{R}^2$

For all $\lambda \in \mathbb{Q}_+^d$, player 1 can force in G all simple cycles to be in $\overline{H_{-\lambda}}$	iff	For all $\lambda \in \mathbb{Q}_+^d$, player 1 (Max) can achieve payoff ≥ 0 in G_λ

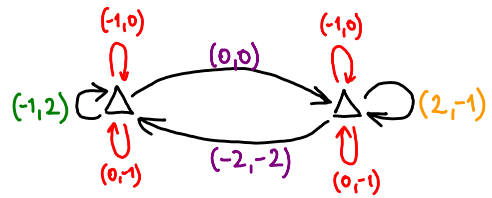
Lossy game graphs

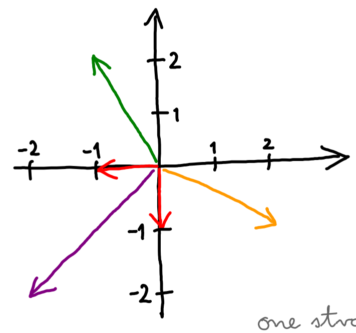
Play: $e_1, e_2, e_3, e_4, \ldots$



Player 1 wins: $\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\omega(e_i)}{n} = (0,0,\ldots,0)}$

Player 1: Chatterjee-Velner sufficient condition





many strategies

one strategy

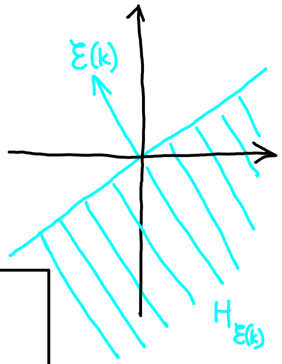
For all $\lambda \in \mathbb{Q}_+^d$, player 1 can force in G all simple cycles to be in $\overline{H_\lambda}$	$\overset{?}{\Longrightarrow}$	Player 1 has a strategy to achieve $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\omega(e_i)}{n} = (0,0,\ldots,0)$

Player 1: Chatterjee-Velner winning strategy

For all $\lambda \in \mathbb{Q}^d$, player 1 has a strategy σ_λ, that forces simple cycles to be in H_λ

Finite play: $e_1, e_2, \ldots, e_n$

Energy level: $\mathcal{E}(n) := \sum_{i=1}^{n} \omega(e_i)$



Strategy σ for player 1

- proceeds in stages $s = 1, 2, 3, \ldots$
- play at start of stage s: $e_1, e_2, \ldots, e_k$
- use $\sigma_{\mathcal{E}(k)}$ until a simple cycle is formed

Player 1: Chatterjee-Velner winning strategy

Fact If $w_1, w_2, w_3, \ldots$ bounded vectors in $\mathbb{R}^d$ s.t. $w_{n+1} \in \overline{H_{W_n}}$ where $W_n = \sum_{i=1}^{n} w_i$

then $\|W_n\| = O(\sqrt{n})$

Corollary $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{w_i}{n} = (0, 0, \ldots, 0)$

Strategy σ' for player 1

- proceeds in stages $s = 1, 2, 3, \ldots$
- play at start of stage s: $e_1, e_2, \ldots, e_k$
- use $\sigma_{\mathcal{E}(k)}$ for s^ε steps, where $\varepsilon > 0$

Fact If player 1 uses strategy σ' then $\|\mathcal{E}(n)\| = O\left(n^{\frac{1}{2} + \frac{\varepsilon}{2(1+\varepsilon)}}\right)$

Chatterjee-Velner algorithm

Theorem [Chatterjee, Velner 2013]

There is an algorithm for solving multi-dimensional mean-payoff games in time $\left(|V| \cdot \|\omega\|\right)^{O(d^2)}$

pseudo-polynomial for fixed d

Fact It suffices to consider σ_λ and σ_λ for all $\lambda \in \mathbb{Z}$ s.t. $\|\lambda\| = \left(|V| \cdot \|\omega\|\right)^{O(d)}$

Why energy/bounding games?

1. Finite memory strategies?
2. Deciding simulation between a finite-state system and a VASS/Petri net.
3. Large complexity gap:
   - (a) 2-ExpTime lower bound [Courtois, Schmitz 2014]
   - (b) (d-1)-ExpTime algorithm [Brázdil, Jančar, Kučera 2010]

Energy games

- Given-initial-charge energy games
  - Initial charge: $C \in \mathbb{N}$
  - Energy level: $\mathcal{E}(n) = \sum_{i=1}^{n} \omega(e_i)$
  - Player 1 wins: $\mathcal{E}(n) \geq -C$ for all $n \in \mathbb{N}$
- Arbitrary-initial-charge energy games
  - Player 1 wins: there exists initial charge $C \in \mathbb{N}$, s.t. $\mathcal{E}(n) \geq -C$ for all $n \in \mathbb{N}$

Mean-payoff vs energy games

Fact

Player 1 (Max) can guarantee mean payoff ≥ 0

iff

Player 1 wins in the arbitrary-initial-charge energy game

Fact (lossy games)

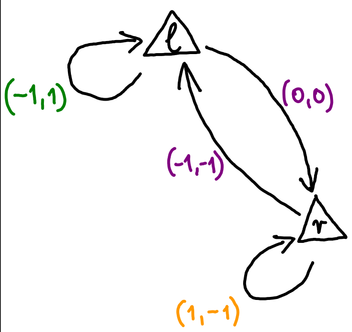
Player 1 (Max) can guarantee mean payoff $\underbrace{= 0}_{\text{in the lossy game}}$

iff

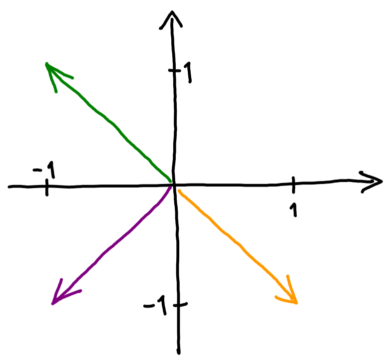
Player 1 wins in the lossy bounding game

Multi-dimensional mean-payoff vs energy games

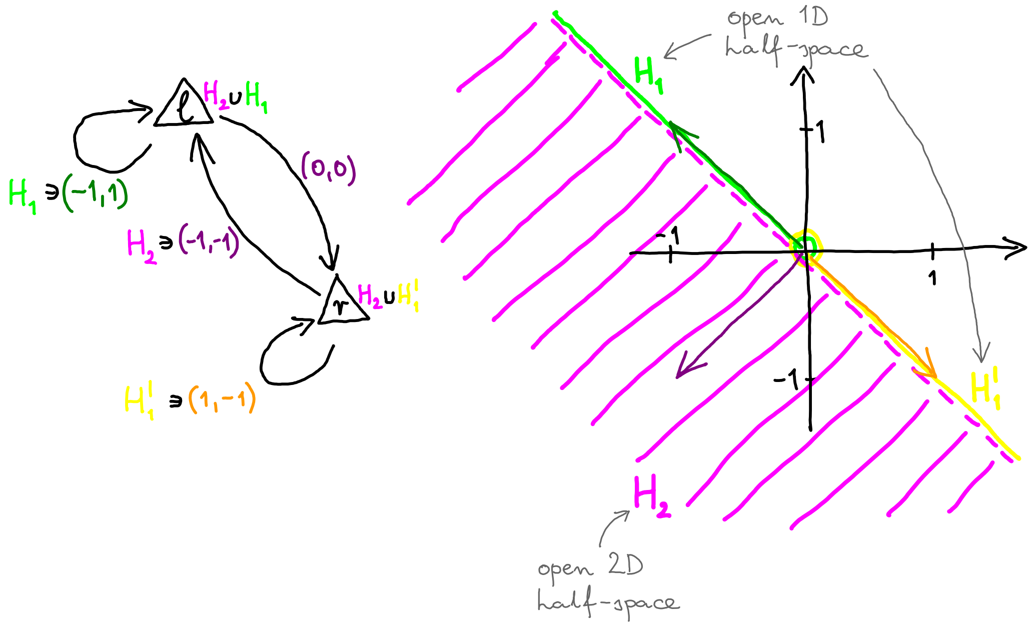
Winning for player 1



Winning for player 2!



Player 2: a more generous sufficient condition



Perfect half-spaces

$$H_d \cup H_{d-1} \cup \ldots \cup H_1$$

open half-space of $\mathbb{R}^d$

open half-space of the boundary of H_d

open half-space of the boundary of H_2

Player 2: a more generous sufficient condition
First-cycle bounding game

- If $v \in V_1$ then
  - Player 2 picks a $|V| \cdot \|\omega\|$-generated perfect half-space $\mathcal{H}(v)$
  - Player 1 follows an edge to another vertex
- If $v \in V_2$
  - Player 2 follows an edge to another vertex
- When the first cycle C is formed
  - Player 2 wins if $\omega(C)$ is in common part of $\{\mathcal{H}(v) : v \in C\}$
  - Player 1 wins otherwise

Player 2: a more generous sufficient condition
First-cycle bounding game

Cycle decomposition: $C_1, C_2, C_3, \ldots$

Semi-perfect half-spaces: $\mathcal{H}(C_1), \mathcal{H}(C_2), \mathcal{H}(C_3), \ldots$

Theorem $\lim_{n \to \infty} \left\| \sum_{i=1}^{n} \omega(C_i) \right\| = +\infty$

Proof idea

There is a semi-perfect half-space $H_d \cup \ldots \cup H_k$ s.t.

- $\omega(C_n) \in \overline{H_k}$ for all sufficiently large n
- $\omega(C_n) \in H_k$ for infinitely many n

Consequences of determinacy of FCBG

If player 2 does not win then

for every perfect half-space $\mathcal{H} = H_d \cup \ldots \cup H_1$
player 1 can force a cycle $C \notin \mathcal{H}$

Player 1: a winning strategy

Memory

- a simple path
- a $|V|\cdot\|\omega\|$-generated perfect half-space $\mathcal{H} = H_d \cup \dots \cup H_1$
- counters $\mathcal{C}(k,W)$ for $k=0,1,2,\dots,d$
  W: simple cycle weights

Memory update

- extend path if still simple
- if a cycle C formed
  - remove C from simple path
  - increment counters $\mathcal{C}(k,\omega(C))$ for $k=0,1,\dots,d$
  - maintain the Invariants

Player 1: a winning strategy

$$\boxed{\text{Let } \mathcal{U}(k) := \left(4|V|\cdot\|\omega\|\right)^{2k(d+2)^2}}$$

Invariants

1. The current energy level is $\boxed{\sum_W \mathcal{C}(d,W)\cdot W}$
2. If $W\in\langle H_k\rangle$ then $\mathcal{C}(k,W) < 2\cdot\mathcal{U}(k)$
3. If $W\in\widehat{H}_k$ then $\mathcal{C}(k,W) < \mathcal{U}(k)$
4. If 3 is violated a k-shift (a change of $\mathcal{H}$) is done
5. If 4 is not possible a k-reset (of counters) is done (respecting 1)

Technical tools

1. **An Alternatives Lemma**

Take $A\subseteq\{-M,\dots,M\}$ contained in M-generated subspace $S\subseteq\mathbb{Q}^d$.

Either • A contained in an M-generated closed half-space of S

or • $(0,0,\dots,0)\in \mathrm{cone}_{\mathbb{Q}}(A)$

2. **A Small Lemma**

Take matrix $A\in\{-M,\dots,M\}^{d\times n}$.

If $Ax=\mathbb{0}$ has a solution in $\mathbb{Q}^n_+$
then it has one in $\left\{1,\dots,\left(2(M+1)\right)^{(\mathrm{rank}(A)+2)^2}\right\}$.

Complexity of bounding games

Theorem [Chaloupka 2013]
2-dim bounding games with $\|\omega\|=O(1)$
can be solved in polynomial time

Theorem
If player 1 wins then she can keep $\|\mathcal{E}(\pi)\| \le \left(|V|\cdot\|\omega\|\right)^{O(d^3)}$

Corollary
d-dim bounding games can be solved in time $\left(|V|\cdot\|\omega\|\right)^{O(d^4)}$

pseudo-polynomial for fixed d

Theorem [Chatterjee, Doyen, Henzinger, Raskin 2010]
Deciding if player 1 wins is co-NP-complete.

Complexity of Energy Games

Theorem [Brázdil, Jančar, Kučera 2010]

Given-initial-charge energy games can be solved in $(d-1)$-ExpTime

Theorem [Courtois, Schmitz 2014]

Given-initial-charge energy games are 2-ExpTime-hard

Theorem

Given-initial-charge energy games are 2-ExpTime-complete