Pushing the Boundaries of the Complexity of the Reachability Problem in Vector Addition Systems One Step at a Time

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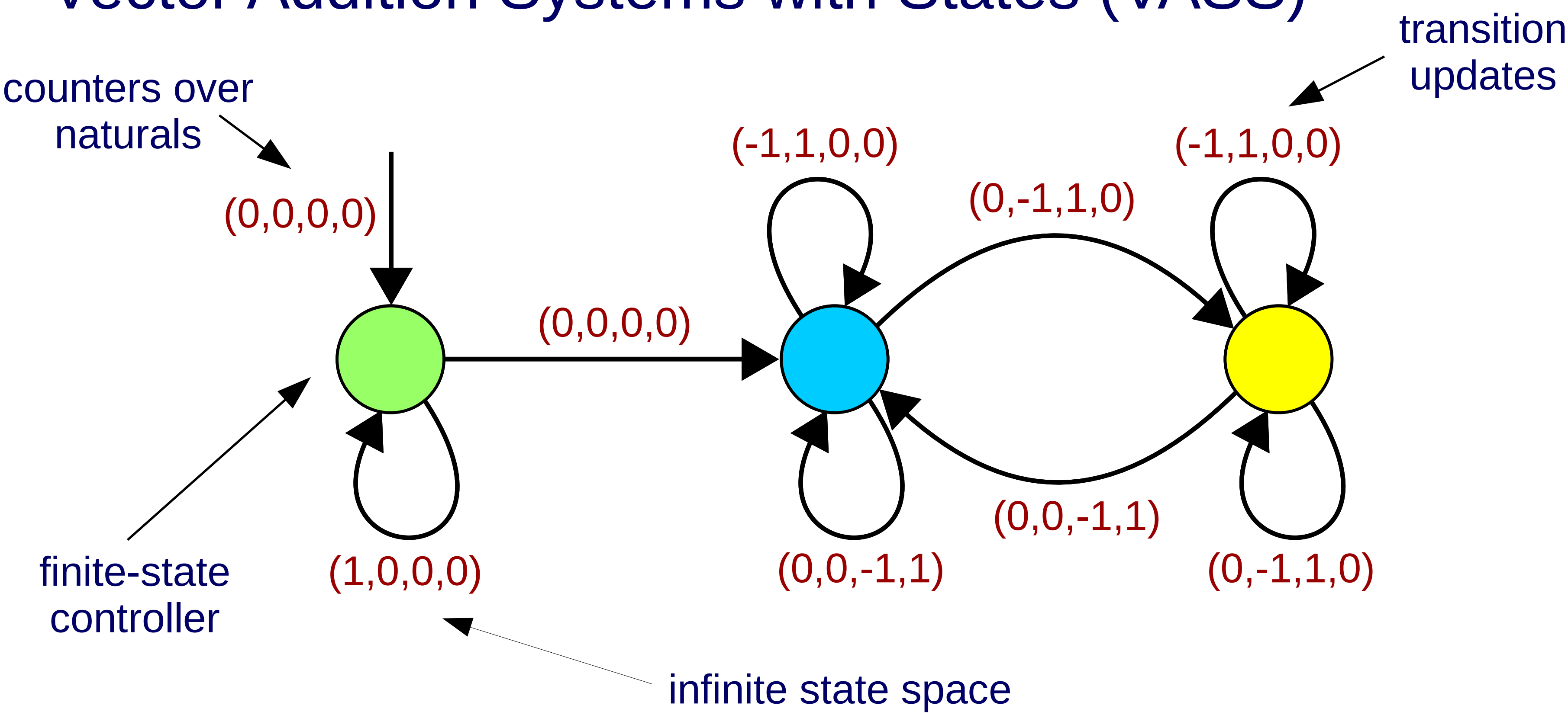
- Stephan Kreutzer (Berlin)
- Pierre McKenzie (Montreal)
- Joel Ouaknine (Oxford)
- James Worrell (Oxford)

Vector Addition Systems with States (VASS)

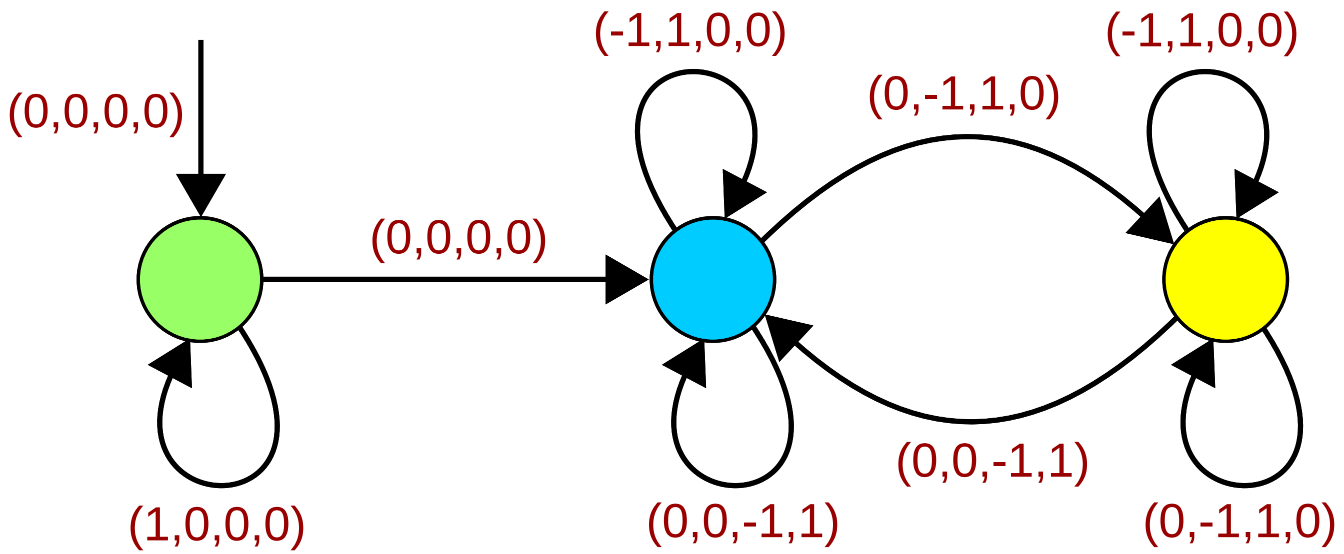


(0,0,0,0) → (1,0,0,0) → (2,0,0,0)

→ (2,0,0,0) → (1,1,0,0) → (1,0,1,0)

→ (1,0,0,1) → (0,1,0,1) → (0,0,1,1)

Vector Addition Systems with States

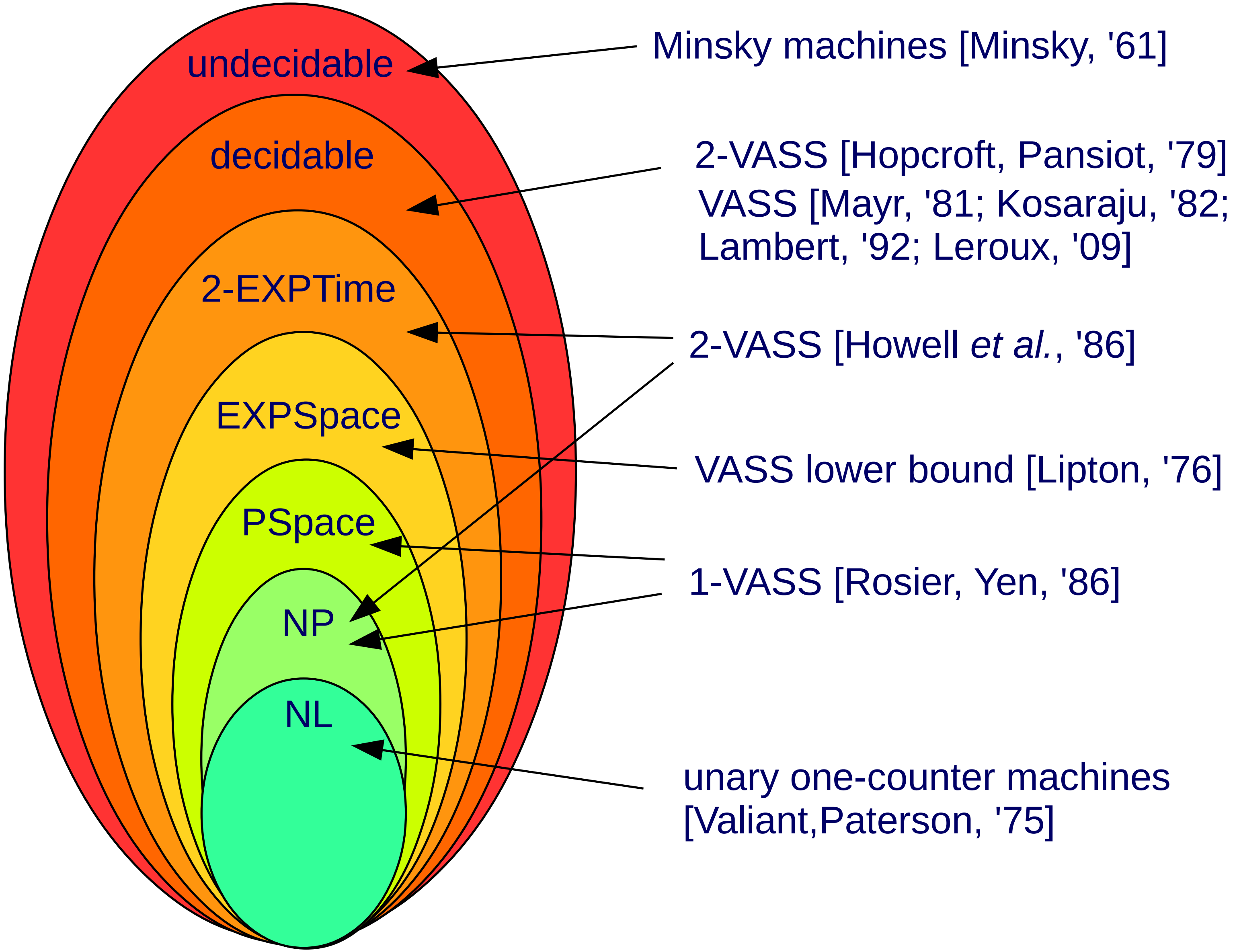


Given configurations $q(\bar{u}),\ r(\bar{v})$, does $q(\bar{u}) \Rightarrow r(\bar{v})$?

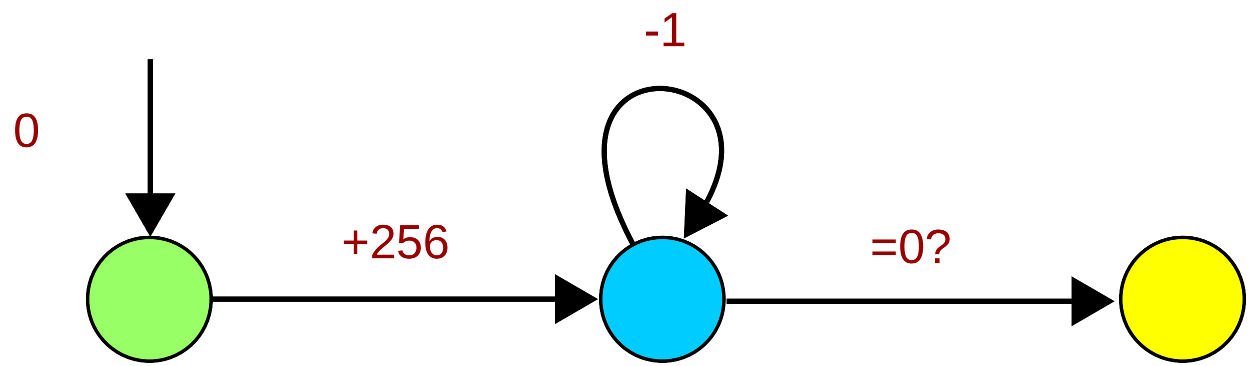
Relevant Parameters:

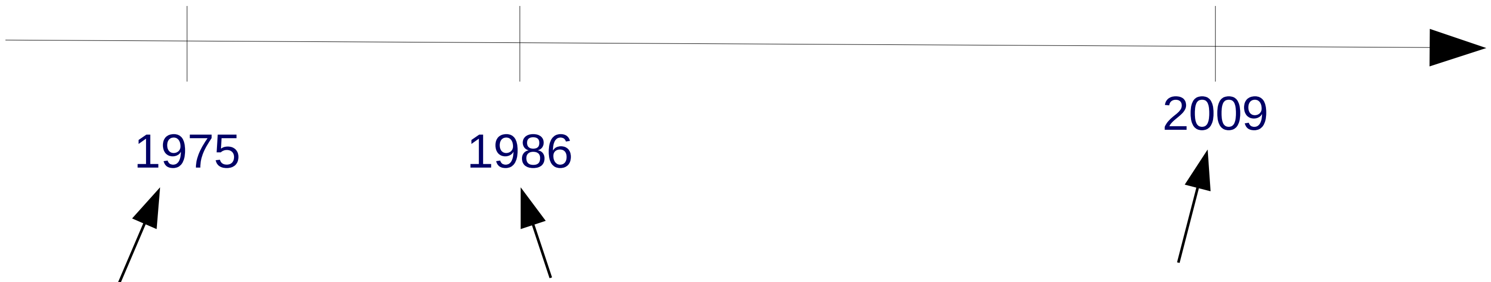
- Operations along transitions (e.g. zero tests, update types)
- Number of counters
- Encoding of numbers
- Parametric values

The Reachability Problem



One-Counter Automata





Valiant and Paterson show reachability NL-complete (unary encoding)

Rosier and Yen show reachability NP-hard and in PSpace (binary encoding)

H., Kreutzer, Ouaknine and Worrell show reachability in NP

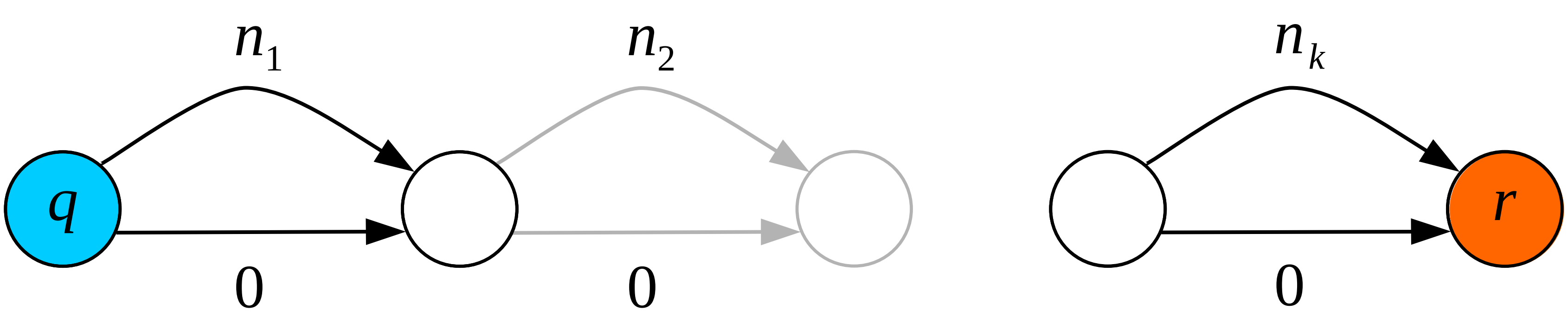
One-Counter Automata

Reachability is NP-hard via a reduction from SubsetSum

Given $S=\{n_1,\dots,n_k\}\subseteq\mathbb{N}, T\in\mathbb{N}$ there is $S'\subseteq S$ such that $\sum S'=T$

iff

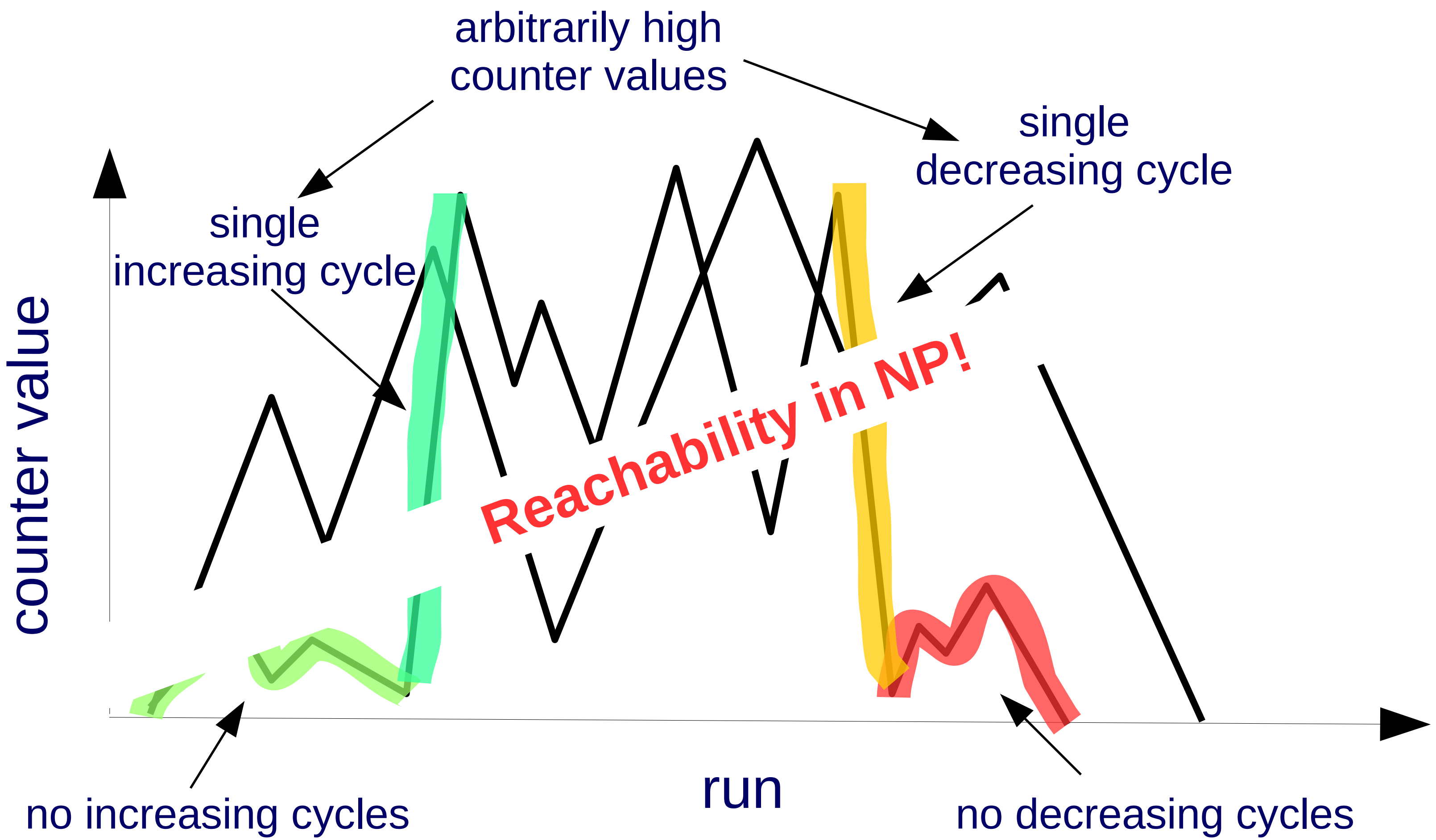
$$q(0)\Rightarrow r(T)$$



One-Counter Automata

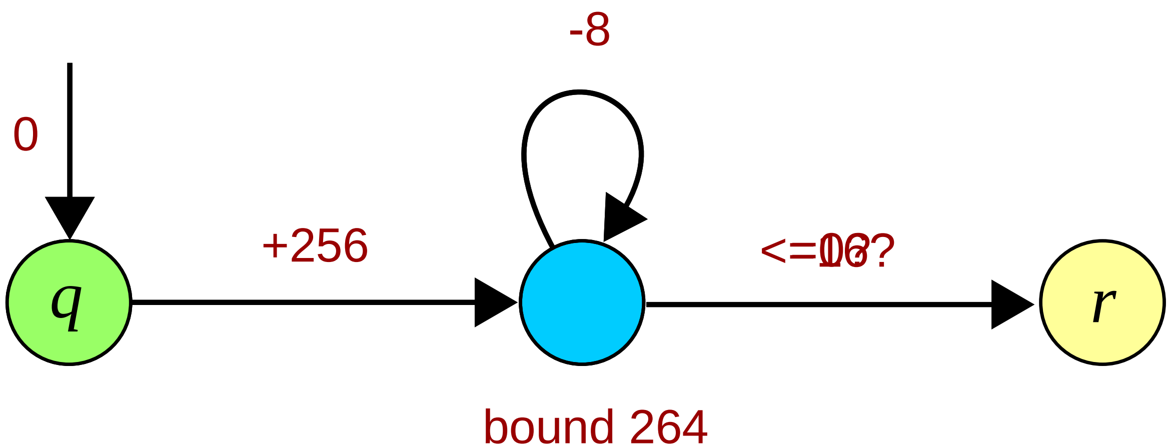
“If there is a run then there is one whose maximum counter value is polynomially bounded.” [Lafourcade et al., '04]

One-Counter Automata



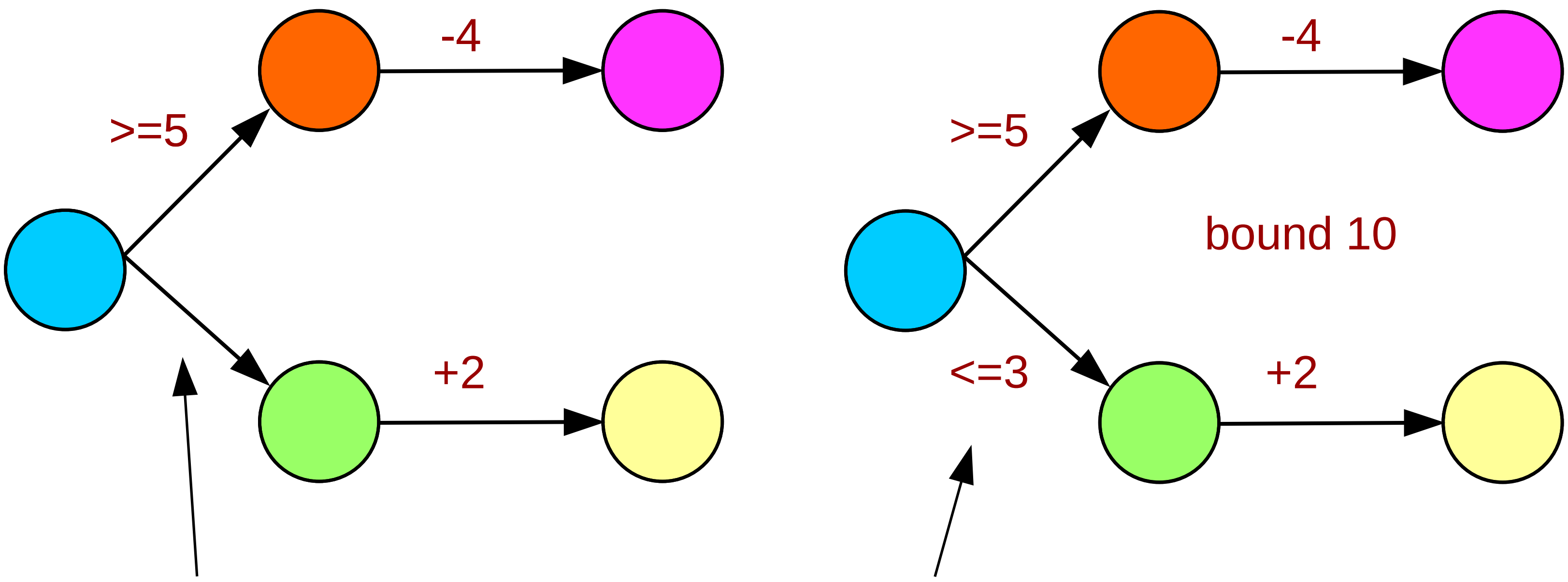
“Runs can be structured.” [H. *et al.*, '09]

Bounded One-Counter Automata



- State-space bounded by arbitrary but fixed constant
- Reachability PSpace-complete [Fearnley, Jurdzinski, '13]

Bounded One-Counter Automata



no way to enforce
counter below a
value

if-then-else
constructions possible

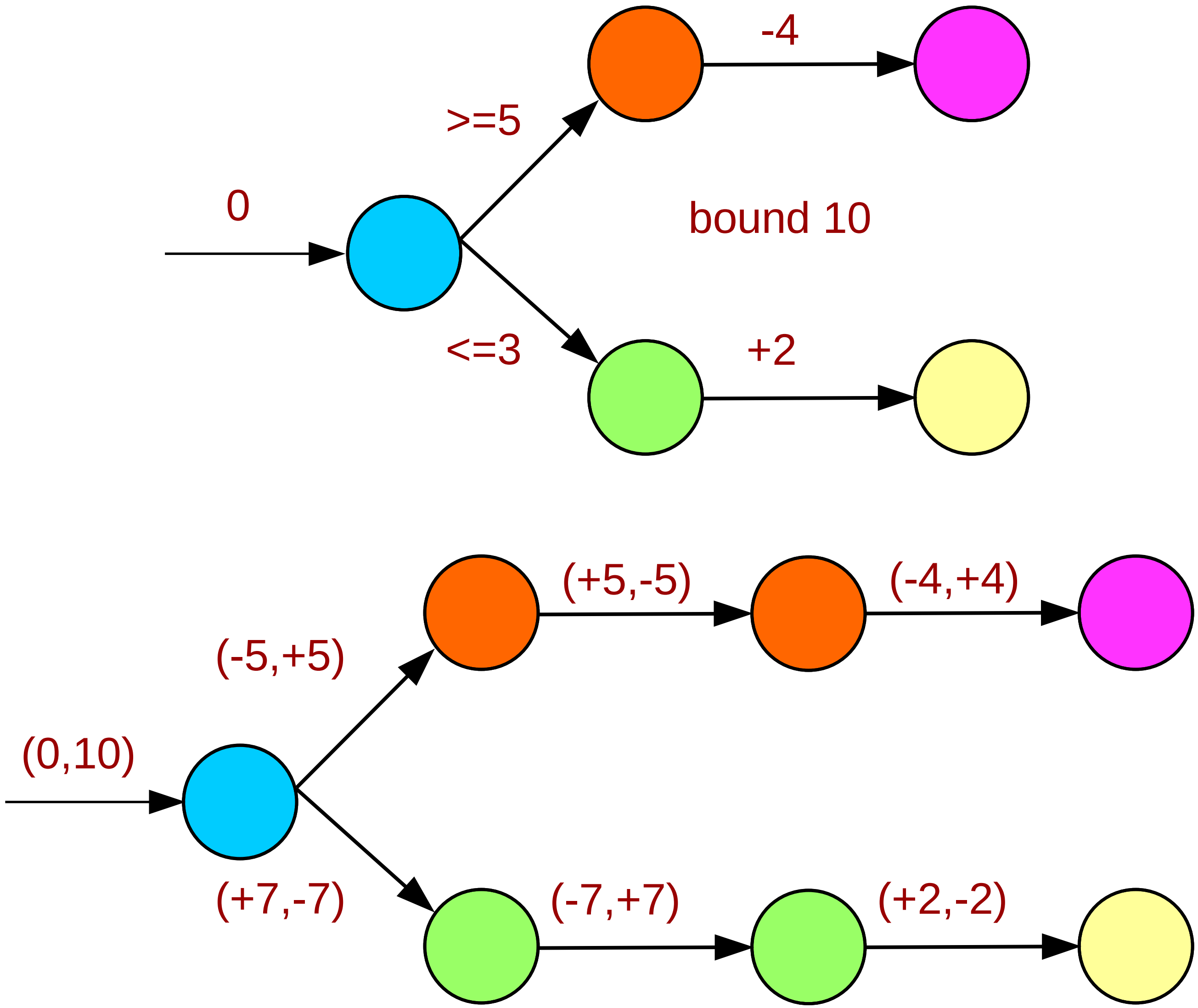
Bounded One-Counter Automata

Fearnley and Jurdzinski show that reachability is PSpace-hard via a reduction from Quantified-SubsetSum

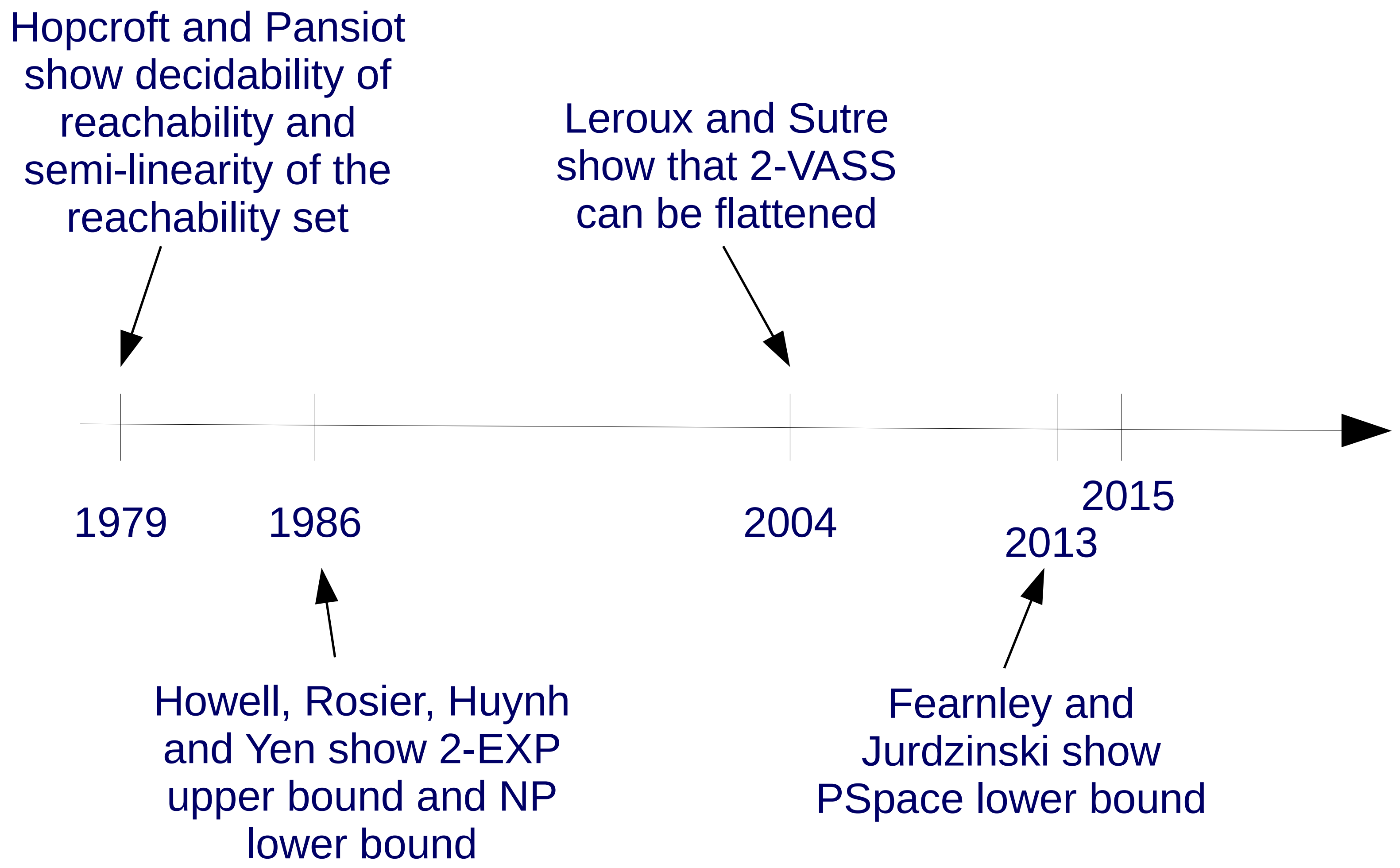
Given $S=\{n_{1,}\ldots,n_k\}\subseteq\mathbb{N}, T\in\mathbb{N}$, does the following hold:

$$\exists x_1\in\{0,1\}\ \ \forall x_2\in\{0,1\}\ \ \cdots\forall x_n\in\{0,1\}:\sum_{i=1}^{n} x_i\cdot n_i=T$$

Vector Addition Systems with States in Dimension Two

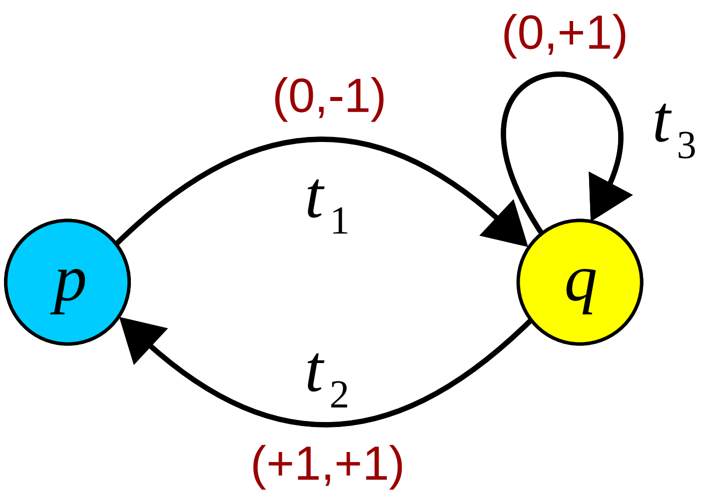


Vector Addition Systems with States in Dimension Two



Vector Addition Systems with States in Dimension Two

$V = (Q, T)$



Leroux and Sutre, 2004:

$p(\bar{u}) \Rightarrow q(\bar{v})$ iff

$$p(\bar{u}) \Rightarrow t_1 (t_3)^* \Rightarrow q(\bar{v})$$

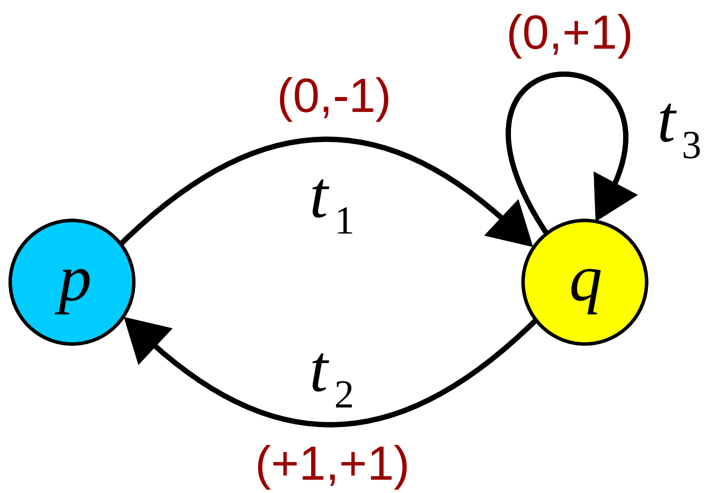
or

$$p(\bar{u}) \Rightarrow t_1 (t_3)^* t_2 (t_1 t_2)^* t_1 \Rightarrow q(\bar{v})$$

linear path scheme

Vector Addition Systems with States in Dimension Two

$V = (Q, T)$

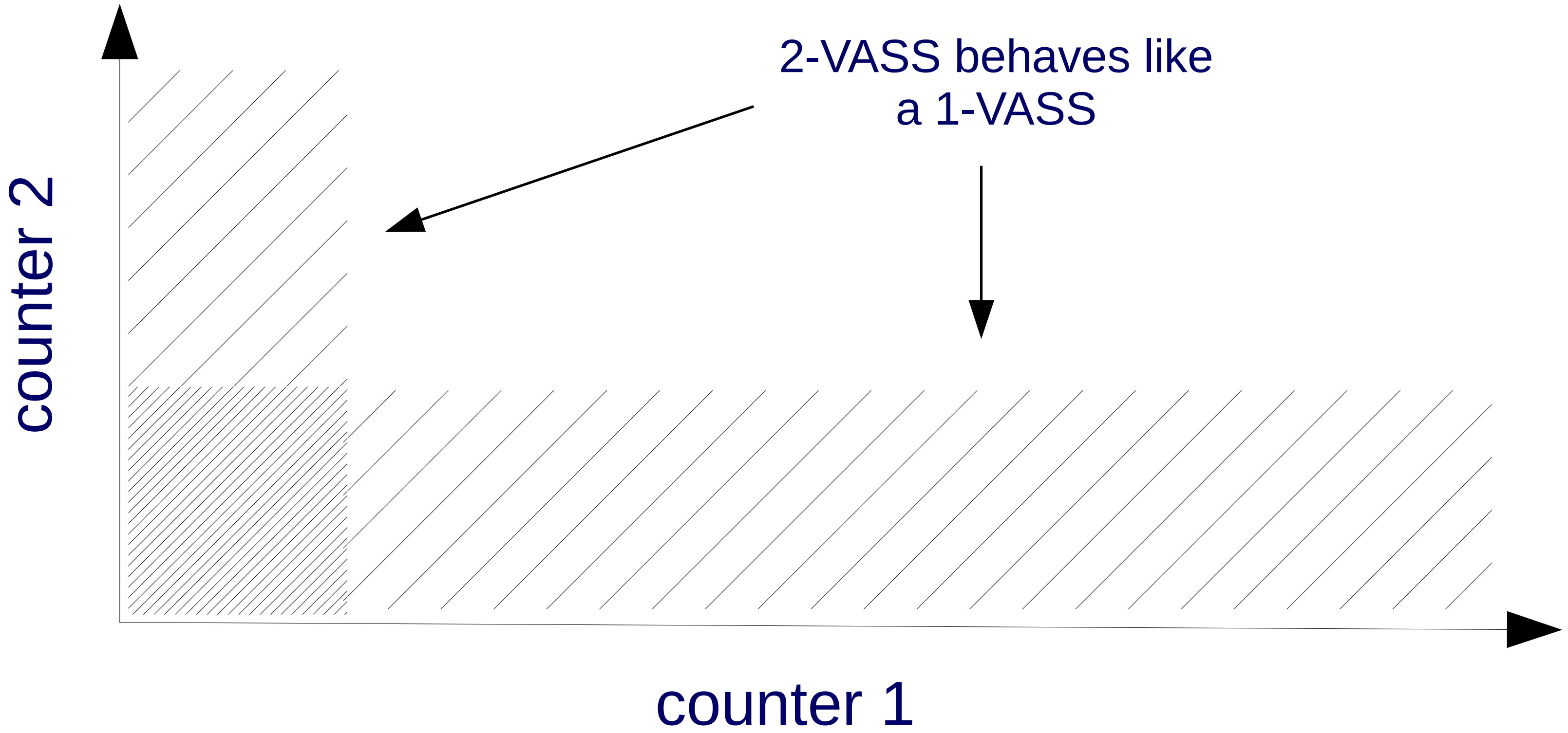


Blondin, Finkel, Göller, H., McKenzie, 2015:

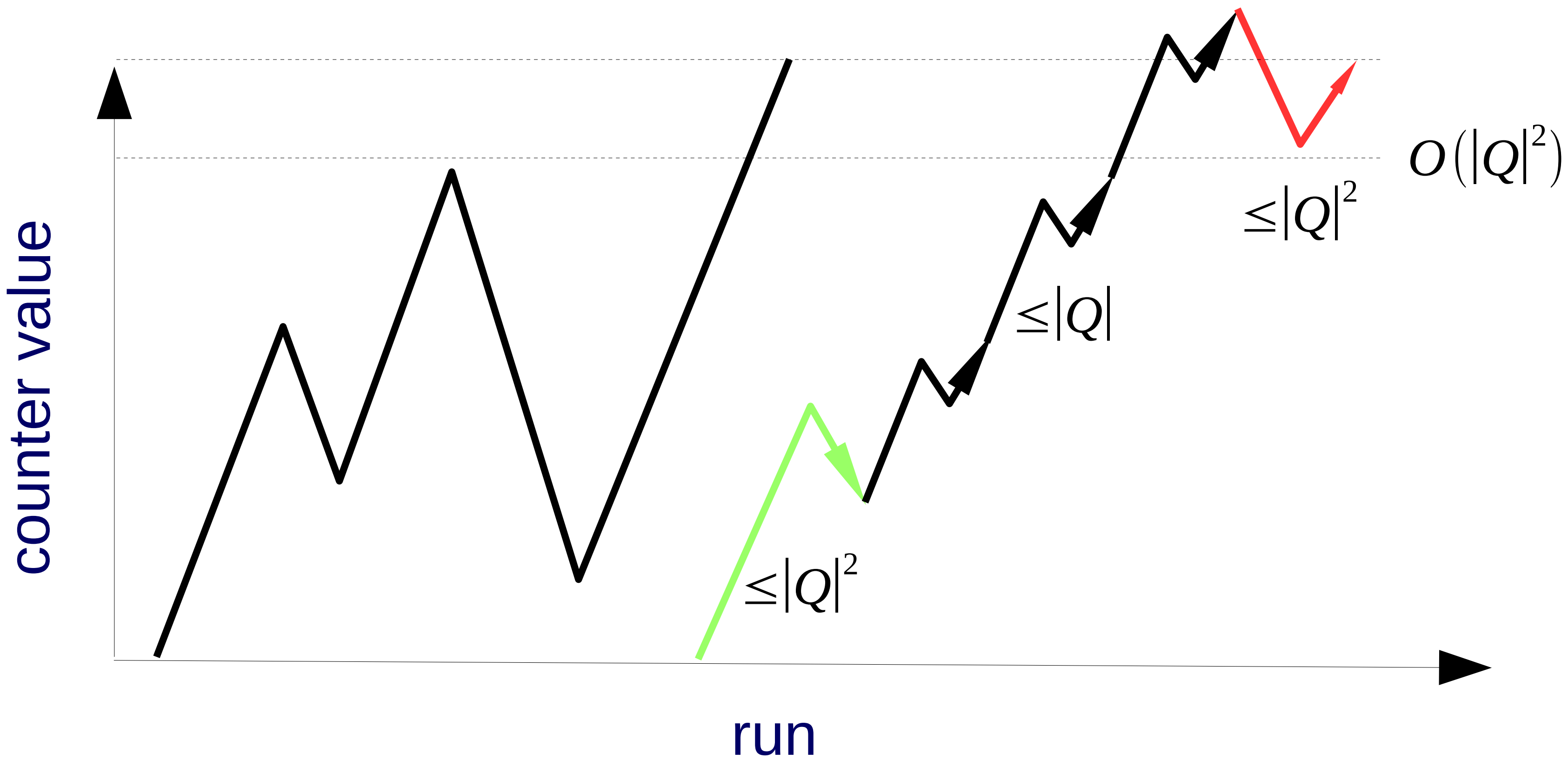
$p(\bar{u}) \Rightarrow q(\bar{v})$ iff $p(\bar{u}) \Rightarrow \alpha_0 \beta_1^* \cdots \alpha_{k-1} \beta_k^* \alpha_{k+1} \Rightarrow q(\bar{v})$

$|\alpha_i|, |\beta_i| \leq (|Q| + \|T\|)^{O(1)}, \; k \leq O(|Q|^2)$

Vector Addition Systems with States in Dimension Two

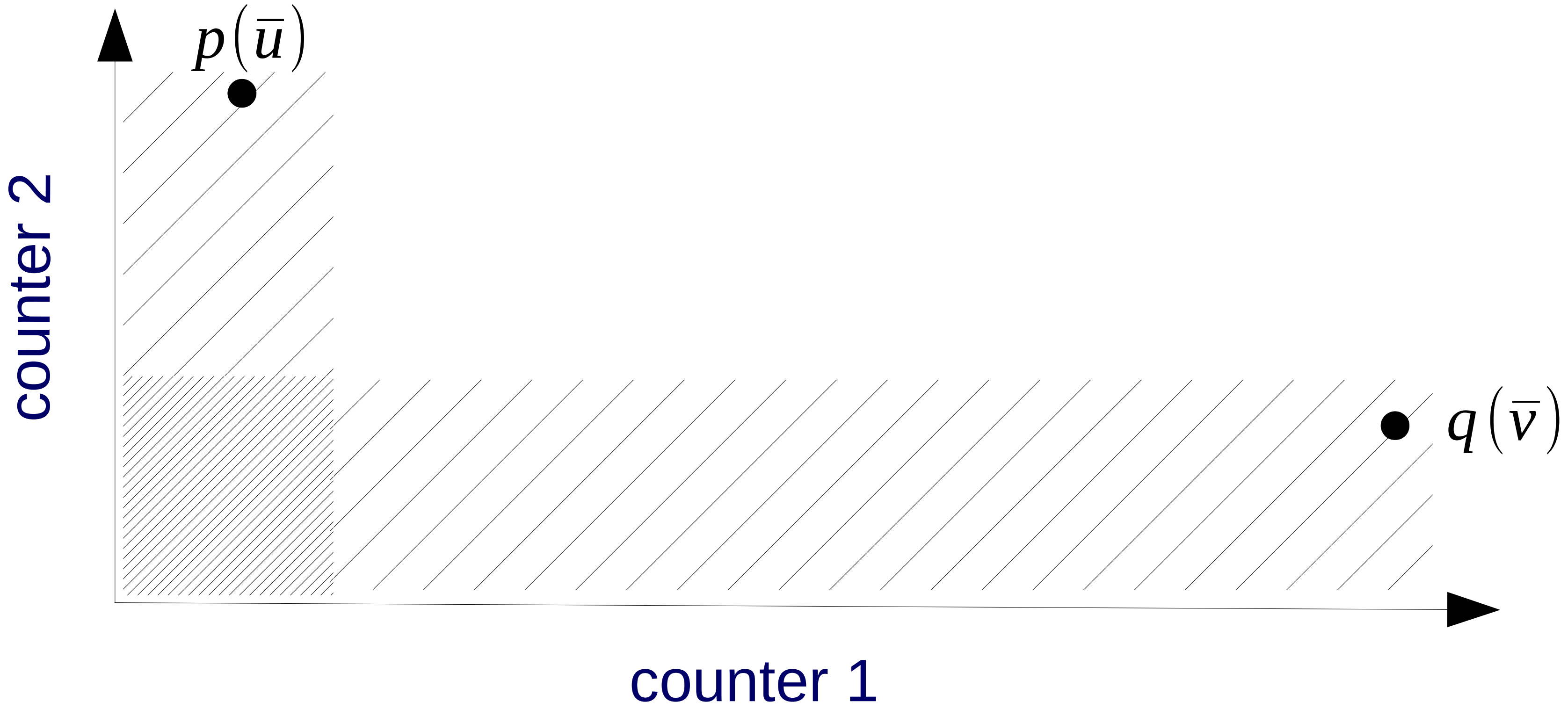


Returning Back to One-Counter Automata



"Paths whose counter values grow sufficiently high have an easy description" [Valiant, Paterson, '75].

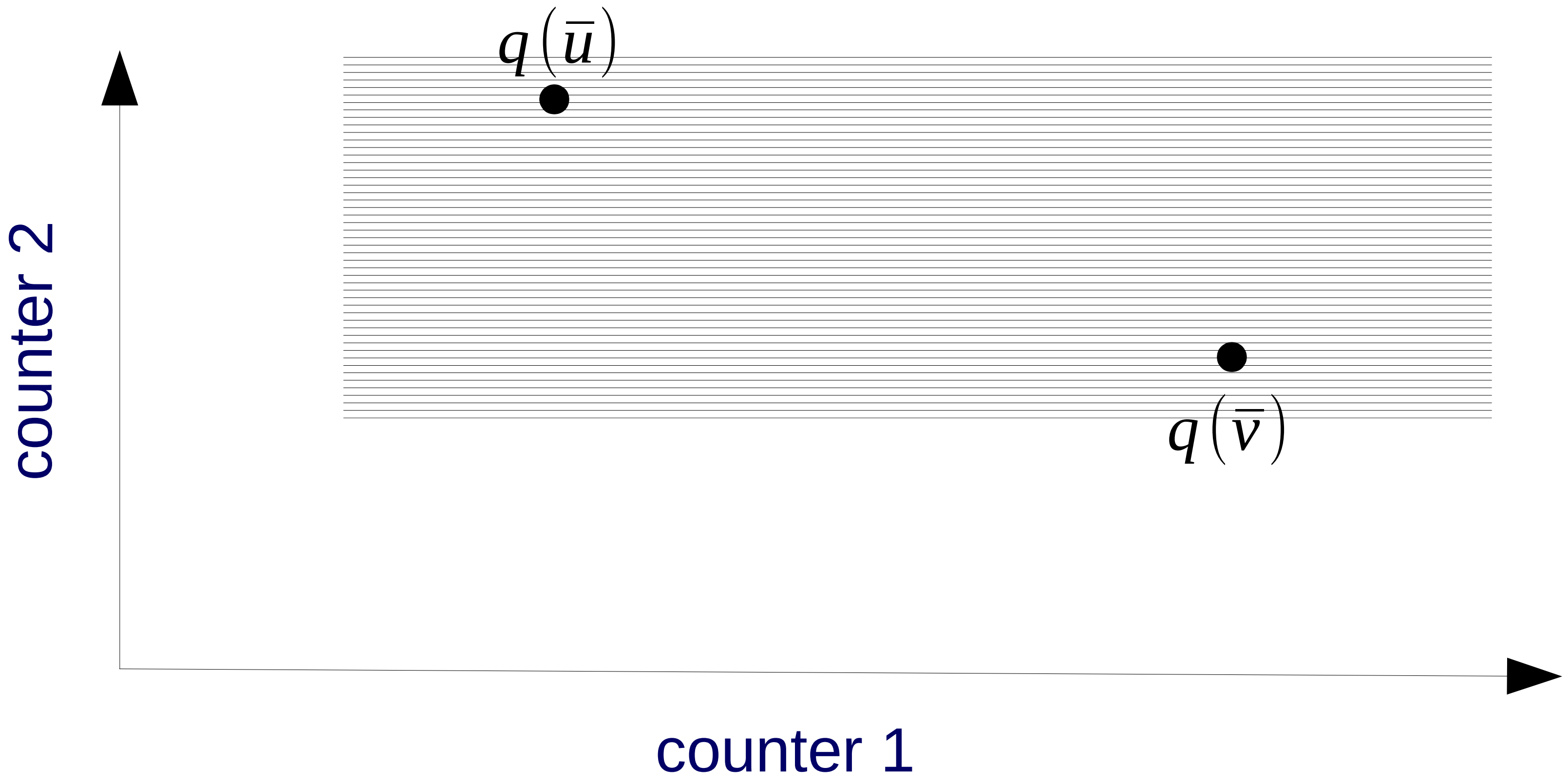
Vector Addition Systems with States in Dimension Two



$p(\bar{u}) \Rightarrow q(\bar{v})$ iff $p(\bar{u}) \Rightarrow \alpha_0 \beta_1^* \alpha_1 \beta_2^* \alpha_2 \Rightarrow q(\bar{v})$

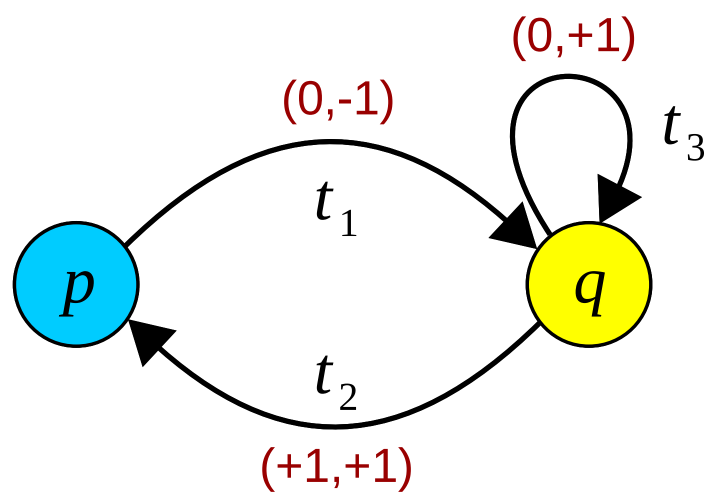
$|\alpha_i|, |\beta_i| \le (|Q| + \|T\|)^{O(1)}$

Vector Addition Systems with States in Dimension Two



Vector Addition Systems with States in Dimension Two

Represent net effect of cyclic paths as semi-linear sets:

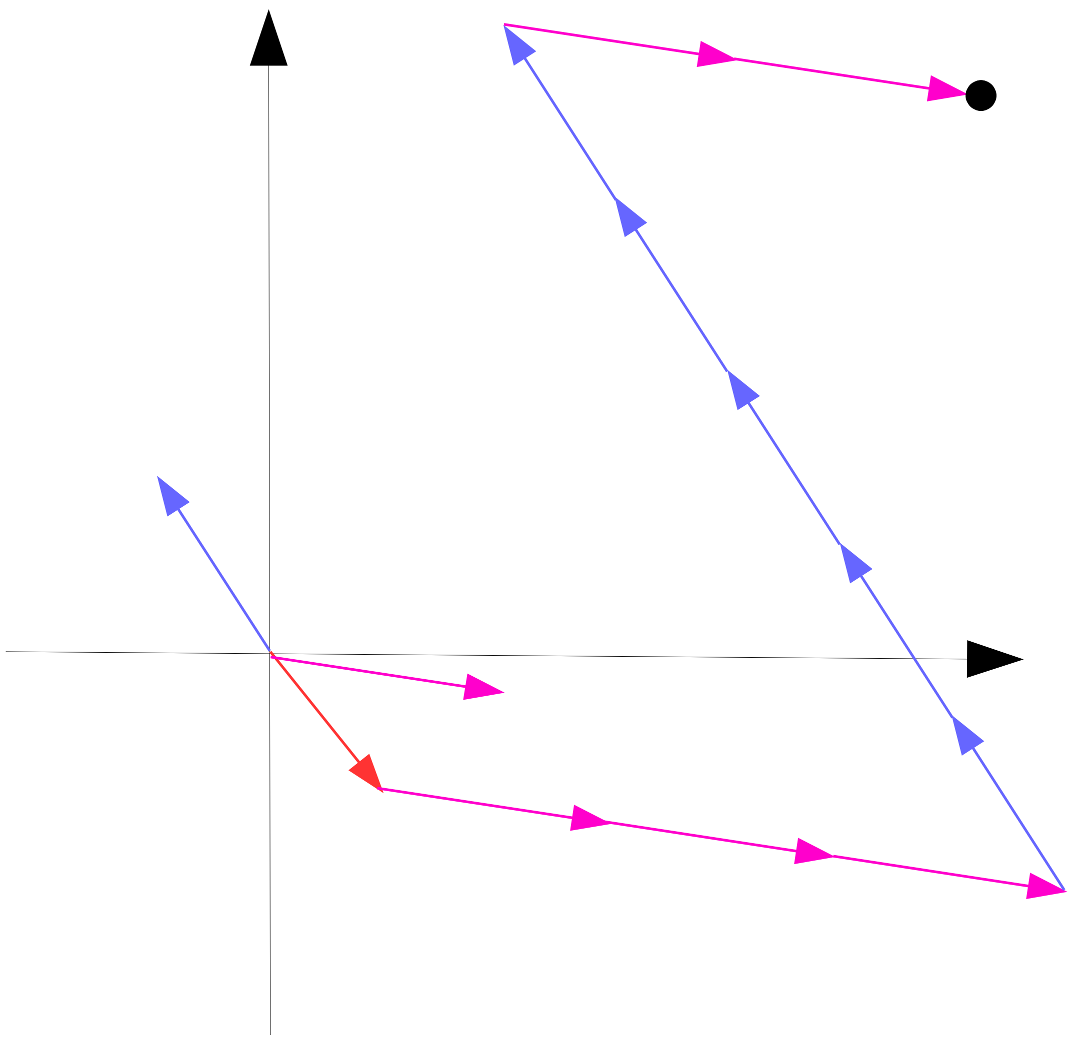


$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} : p(0,0) \Rightarrow p(u,v) \right\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \lambda \in \mathbb{N} \right\}$$

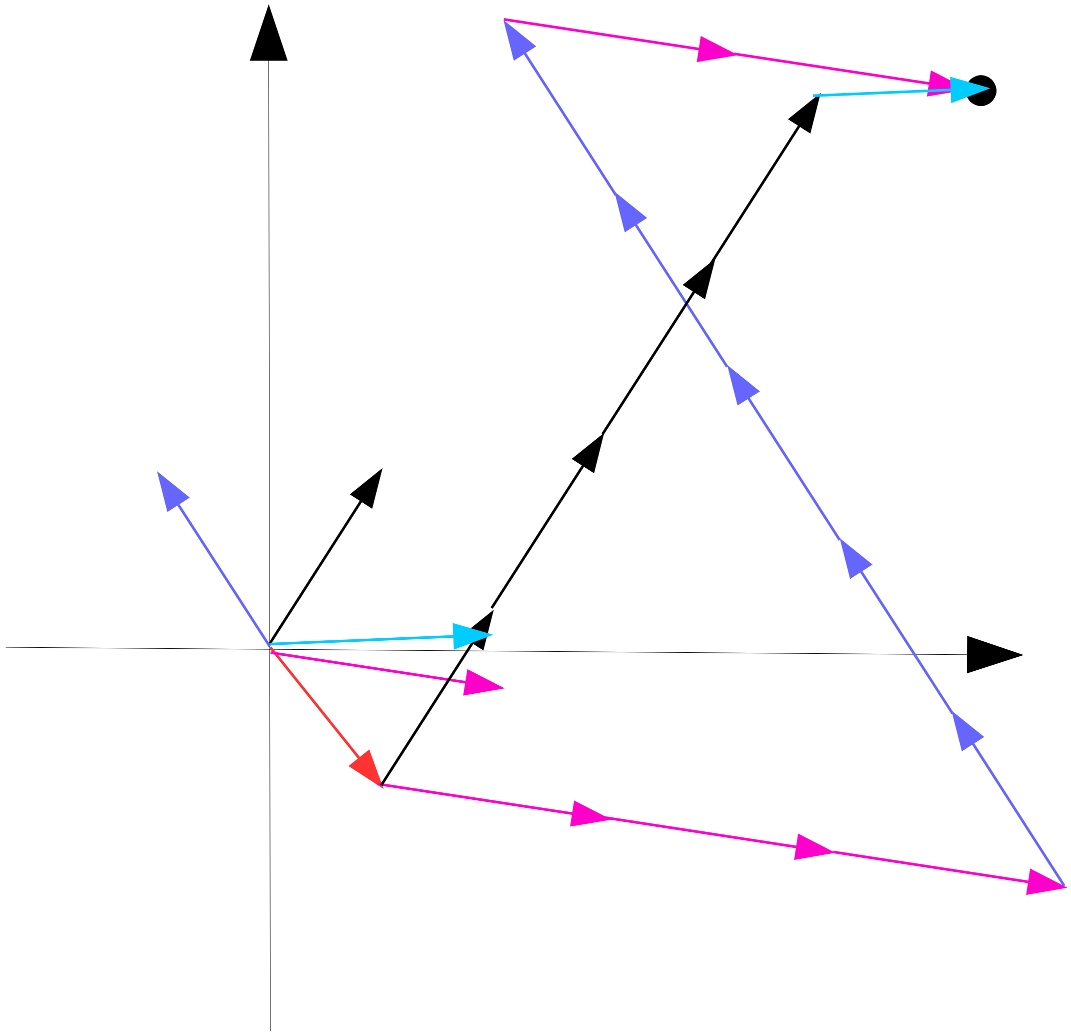
period vector

base vector

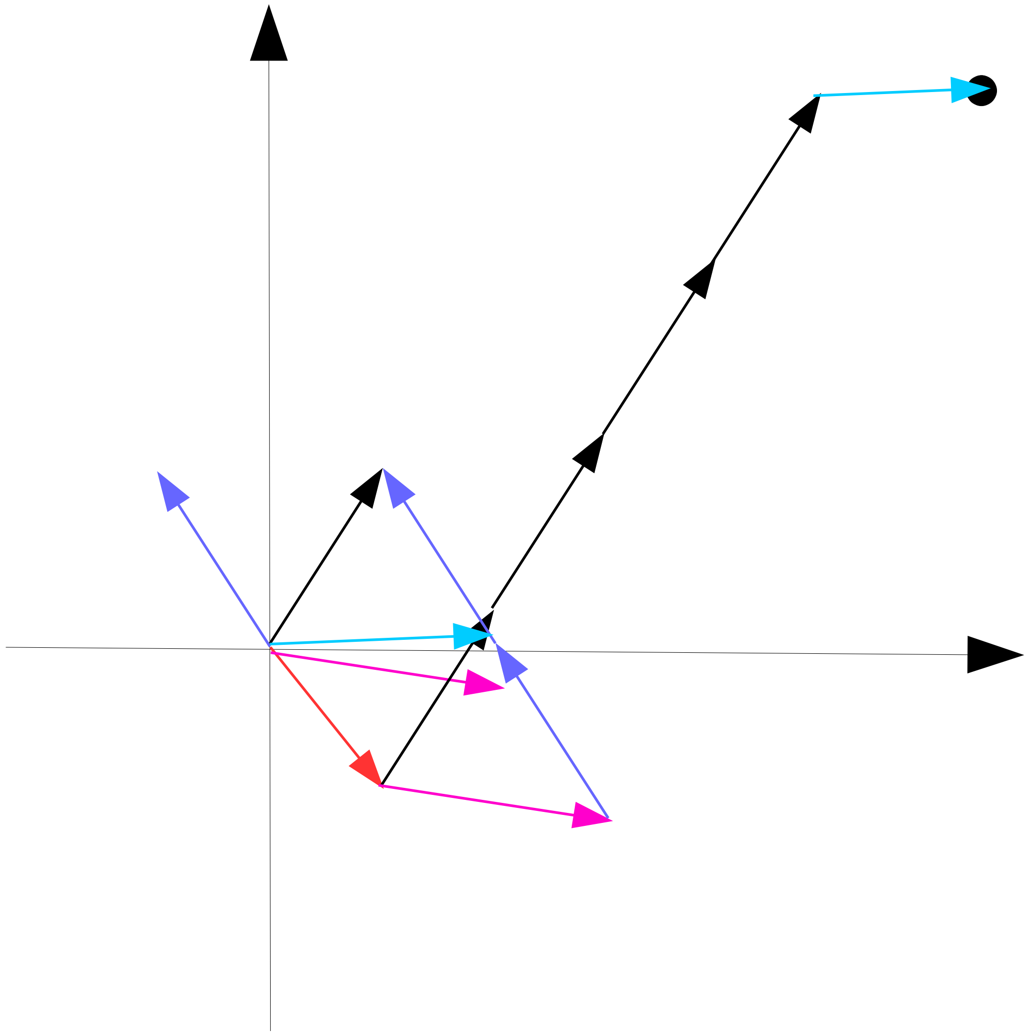
Decomposing Linear Sets in Dimension Two



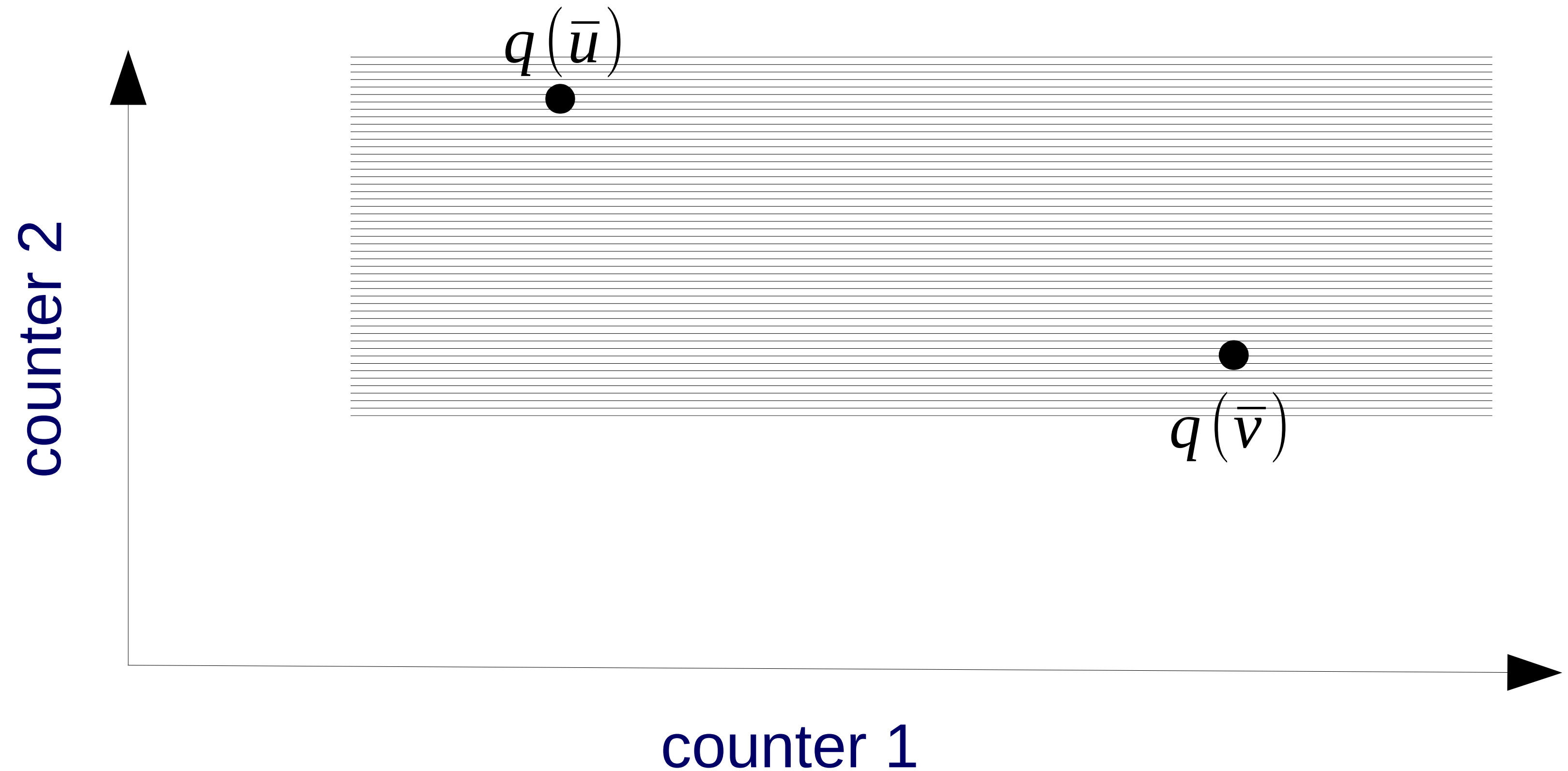
Decomposing Linear Sets in Dimension Two



Decomposing Linear Sets in Dimension Two



Vector Addition Systems with States in Dimension Two

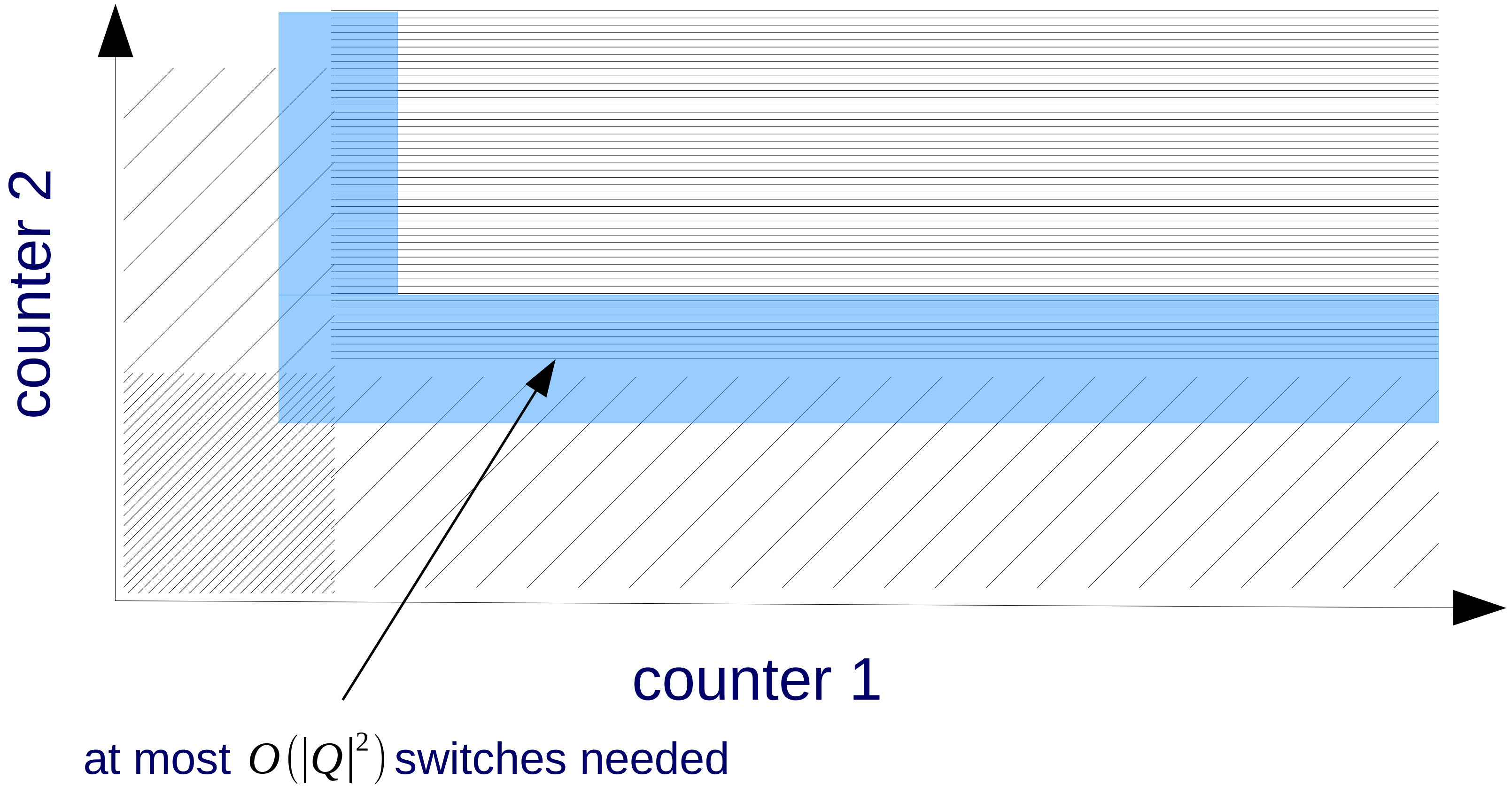


$$q(\bar{u}) \Rightarrow q(\bar{v}) \quad \text{iff} \quad q(\bar{u}) \Rightarrow \alpha_0 \beta_1^* \alpha_1 \beta_2^* \alpha_2 \Rightarrow q(\bar{v})$$

$$|\alpha_i|, |\beta_i| \leq (|Q| + \|T\|)^{O(1)}$$

net effects of $\beta_{1,} \beta_2$ point towards $\bar{v} - \bar{u}$

Vector Addition Systems with States in Dimension Two



$$p(\bar{u}) \Rightarrow q(\bar{v}) \qquad \text{iff} \qquad p(\bar{u}) \Rightarrow \alpha_0 \beta_1^* \cdots \alpha_{k-1} \beta_k^* \alpha_{k+1} \Rightarrow q(\bar{v})$$

$$|\alpha_i|, |\beta_i| \leq (|Q| + \|T\|)^{O(1)}, \quad k \leq O(|Q|^2)$$

Vector Addition Systems with States in Dimension Two

Given $p(\bar{u})$, $q(\bar{v})$ and $\alpha_0\beta_1^*\cdots\alpha_{k-1}\beta_k^*\alpha_{k+1}$.

If $p(\bar{u}) \Rightarrow \alpha_0\beta_1^*\cdots\alpha_{k-1}\beta_k^*\alpha_{k+1} \Rightarrow q(\bar{v})$

then $p(\bar{u}) \Rightarrow \alpha_0\beta_1^{e_1}\cdots\alpha_{k-1}\beta_k^{e_k}\alpha_{k+1} \Rightarrow q(\bar{v})$ with $e_i \leq 2^{(|V|+\log\|\bar{u}\|+\log\|\bar{v}\|)^{O(1)}}$.

Reachability in PSpace!

Vector Addition Systems with States in Dimension Two

Approach yields NP upper bound under unary encoding.
Can we do better?

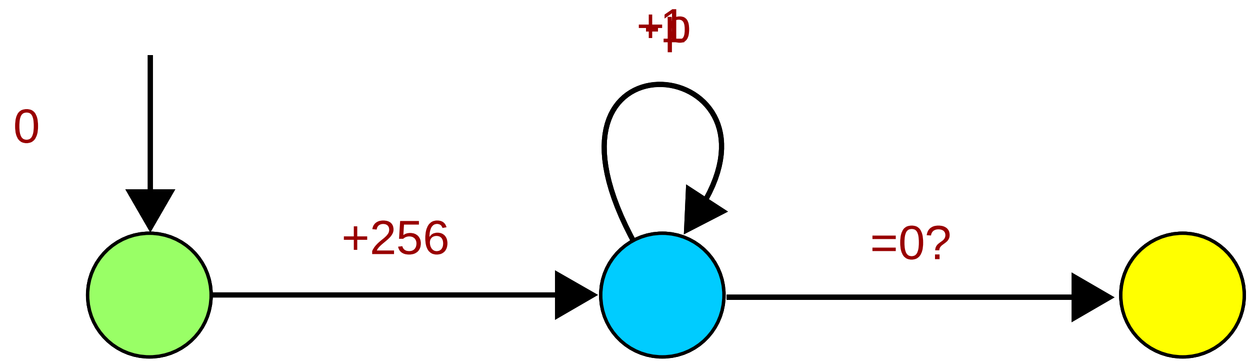
$$\alpha_0 \beta_1^* \cdots \alpha_{k-1} \beta_k^* \alpha_{k+1}$$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 1 \end{pmatrix} \cdot \bar{e} \geq \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix} \cap \begin{pmatrix} -2 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ -2 & 4 & -1 & 0 \\ -2 & 4 & -1 & 3 \end{pmatrix} \cdot \bar{e} \geq \begin{pmatrix} -2 \\ 1 \\ 5 \\ -3 \end{pmatrix}$$

small solutions for individual system

exponential blow-up in the number of columns

An Open Problem



- Decidable for one unbounded counter [H. et al., '09]
- Undecidable for three counters
- Status unknown for parametric bounded one-counter automata and 2-VASS
- Inter-reducible with reachability in parametric two-clock timed automata [Bundala, Ouaknine, '14]