2017 Entrance Examination for the BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of (4 × 10 = 40) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a number (rational/ real/ complex) or, where appropriate, one of the phrases "infinite"/"does not exist"/"not possible to decide". Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two integers.

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1. Positive integers a and b, possibly equal, are chosen randomly from among the divisors of 400. The numbers a, b are chosen independently, each divisor being equally likely to be chosen. Find the probability that $\gcd(a, b) = 1$ and $\operatorname{lcm}(a, b) = 400$.

2. Find the volume of the solid obtained when the region bounded by $y = \sqrt{x}$, $y = -x$ and the line $x = 9$ is revolved around the x-axis. (It may be useful to draw the specified region.)

3. 10 mangoes are to be placed in 5 distinct boxes labeled U, V, W, X, Y. A box may contain any number of mangoes including no mangoes or all the mangoes. What is the number of ways to distribute the mangoes so that exactly two of the boxes contain exactly two mangoes each?

4. Find all complex solutions to the equation:
$$x^4 + x^3 + 2x^2 + x + 1 = 0.$$

5. Let g be a function such that all its derivatives exist. We say g has an *inflection point* at x_0 if the second derivative g'' changes sign at x_0 i.e., if $g''(x_0 - \epsilon) \times g''(x_0 + \epsilon) < 0$ for all small enough positive ϵ.
   (a) If $g''(x_0) = 0$ then g has an inflection point at x_0. True or False?
   (b) If g has an inflection point at x_0 then $g''(x_0) = 0$. True or False?
   (c) Find all values x_0 at which $x^4(x - 10)$ has an inflection point.

6. Consider the following construction in a circle. Choose points A, B, C on the given circle such that $\angle ABC$ is $60°$. Draw another circle that is tangential to the chords AB, BC and to the original circle.
   Do the above construction in the unit circle to obtain a circle S_1. Repeat the process in S_1 to obtain another circle S_2. What is the radius of S_2?

7. Write the values of the following.

   (a) $\displaystyle\int_{-3}^{3} |3x^2 - 3| \; dx.$

   (b) $f'(1)$ where $f(t) = \displaystyle\int_{0}^{t} |3x^2 - 3| \; dx.$

8. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) that satisfies the following property: For every natural number n

$$f(n) = \text{ the smallest prime factor of } n.$$

   For example, $f(12) = 2$, $f(105) = 3$. Calculate the following.

   (a) $\lim_{x\to\infty} f(x)$.

   (b) The number of solutions to the equation $f(x) = 2016$.

9. Consider the following function:

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0, \\ a, & x = 0. \end{cases}$$

   (a) Find the value of a for which f is continuous.

   Use this value of a to calculate the following.

   (b) $f'(0)$.

   (c) $\displaystyle\lim_{x\to 0} f'(0)$.

10. For this question write your answers as a series of four letters (Y for Yes and N for No) in order. Is it possible to find a 2×2 matrix M for which the equation $M\vec{x} = \vec{p}$ has:

    (a) no solutions for some but not all $\vec{p}$; exactly one solution for all other $\vec{p}$?

    (b) exactly one solution for some but not all $\vec{p}$; more than one solution for all other $\vec{p}$?

    (c) no solutions for some but not all $\vec{p}$; more than one solution for all other $\vec{p}$?

    (d) no solutions for some $\vec{p}$, exactly one solution for some $\vec{p}$ and more than one solution for some $\vec{p}$?

Answers to part A

This is the only place that will be seen for grading part A. So carefully and clearly write the answers to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered. +

A1. ____________________

A2. ____________________

A3. ____________________

A4. ____________________

A5. ____________________

A6. ____________________

A7. ____________________

A8. ____________________

A9. ____________________

A10. ____________________

Part B. Write complete solutions for these questions from page 6 onwards.

Part B is worth a total of 85 *points (Question 1 is worth* 10 *points and the remaining questions are worth* 15 *points each). Solve these questions in the space provided for each question from page 6. You may solve only part of a question and get partial credit.* **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. Answer the following questions

   (a) Evaluate
   $$\lim_{x\to 0^+}\left(x^{x^x} - x^x\right).$$

   (b) Let $A = \frac{2\pi}{9}$, i.e., $A = 40$ degrees. Calculate the following
   $$1 + \cos A + \cos 2A + \cos 4A + \cos 5A + \cos 7A + \cos 8A.$$

   (c) Find the number of solutions to $e^x = \frac{x}{2017} + 1$.

2. Let L be the line of intersection of the planes $x + y = 0$ and $y + z = 0$.

   (a) Write the vector equation of L, i.e., find (a, b, c) and (p, q, r) such that
   $$L = \{(a, b, c) + \lambda(p, q, r) \mid \lambda \text{ is a real number.}\}$$

   (b) Find the equation of a plane obtained by rotating $x + y = 0$ about L by $45°$.

3. Let $p(x)$ be a polynomial of degree strictly less than 100 and such that it does not have $x^3 - x$ as a factor. If
   $$\frac{d^{100}}{dx^{100}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$
   for some polynomials $f(x)$ and $g(x)$ then find the smallest possible degree of $f(x)$. Here $\frac{d^{100}}{dx^{100}}$ means taking the 100th derivative.

4. The domain of a function f is the set of natural numbers. The function is defined as follows:
   $$f(n) = n + \left\lfloor \sqrt{n} \right\rfloor$$
   where $\lfloor k \rfloor$ denotes the nearest integer smaller than or equal to k. For example, $\lfloor \pi \rfloor = 3, \lfloor 4 \rfloor = 4$. Prove that for every natural number m the following sequence contains at least one perfect square
   $$m, f(m), f^2(m), f^3(m), \ldots$$
   The notation f^k denotes the function obtained by composing f with itself k times, e.g., $f^2 = f \circ f$.

5. Each integer is colored with exactly one of three possible colors - black, red or white - satisfying the following two rules: the negative of a black number must be colored white, and the sum of two white numbers (not necessarily distinct) must be colored black.

   (a) Show that the negative of a white number must be colored black and the sum of two black numbers must be colored white.

   (b) Determine all possible colorings of the integers that satisfy these rules.

6. You are given a regular hexagon. We say that a square is inscribed in the hexagon if it can be drawn in the interior such that all the four vertices lie on the perimeter of the hexagon.

   (a) A line segment has its endpoints on opposite edges of the hexagon. Show that it passes through the center of the hexagon if and only if it divides the two edges in the same ratio.

   (b) Suppose a square $ABCD$ is inscribed in the hexagon such that A and C are on the opposite sides of the hexagon. Prove that center of the square is same as that of the hexagon.

   (c) Suppose the side of the hexagon is of length 1. Then find the length of the side of the inscribed square whose one pair of opposite sides is parallel to a pair of opposite sides of the hexagon.

   (d) Show that, up to rotation, there is a unique way of inscribing a square in a regular hexagon.

Write answers to part B from the next page.

Answers to part B

If you need extra space for any problem,
continue on one of the colored blank pages at the end and write a note to that effect.

1. Answer the following questions

   (a) Evaluate

$$\lim_{x \to 0^+} (x^{x^x} - x^x).$$

(b) Let $A = \frac{2\pi}{9}$, i.e., $A = 40$ degrees. Calculate the following

$$1 + \cos A + \cos 2A + \cos 4A + \cos 5A + \cos 7A + \cos 8A.$$

(c) Find the number of solutions to $e^x = \frac{x}{2017} + 1$.

2. Let L be the line of intersection of the planes $x + y = 0$ and $y + z = 0$.

   (a) Write the vector equation of L, i.e., find (a, b, c) and (p, q, r) such that

   $$L = \{(a, b, c) + \lambda(p, q, r) \mid \lambda \text{ is a real number.}\}$$

   (b) Find the equation of a plane obtained by rotating $x + y = 0$ about L by $45°$.

3. Let $p(x)$ be a polynomial of degree strictly less than 100 and such that it does not have $x^3 - x$ as a factor. If

$$\frac{d^{100}}{dx^{100}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for some polynomials $f(x)$ and $g(x)$ then find the smallest possible degree of $f(x)$. Here $\frac{d^{100}}{dx^{100}}$ means taking the 100th derivative.

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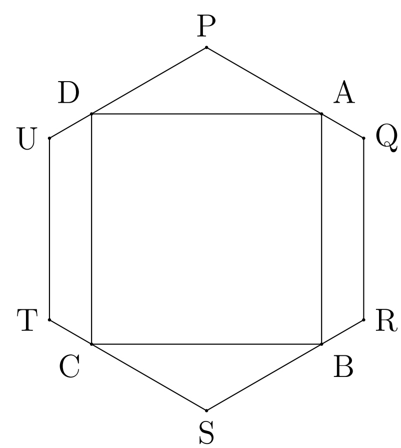

   (d) Show that, up to rotation, there is a unique way of inscribing a square in a regular hexagon.