CHENNAI MATHEMATICAL INSTITUTE

Ph.D. Programme in Physics

Entrance Examination, 15 May 2013

[I] Consider a simple pendulum consisting of a bob of mass m suspended from a fixed support via a massless rigid rod of length l. It is free to oscillate (not necessarily through small angles) in a plane, under the influence of the Earth's constant acceleration due to gravity g.

1. Write the Lagrangian for this pendulum and plot the potential energy $V(\theta)$. Denote the angle made by the pendulum relative to its stable equilibrium position by θ anti-clockwise. Choose the additive constant so that $V(0) = -mgl$.
2. Find Lagrange's equation for the pendulum. Obtain all static (time-independent) solutions and comment on their physical meaning and stability.
3. Find the momentum p_θ conjugate to θ. Starting from the Lagrangian, obtain the Hamiltonian expressed in terms of phase space variables via a Legendre transform.
4. What is the phase space of this system?
5. For energy $-mgl \leq E < mgl$, find the maximum angle of deflection θ_0, and show that the time period of oscillation is as below. What is the origin of the pre-factor 4?

$$T = 4 \int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2}{ml^2}[E + mgl\cos\theta]}}$$

6. Find θ_0 and T as E approaches mgl from below. What does the answer mean physically?

[II] Consider the wave function of a spin-two system given by

$$|\psi> \quad = \quad \frac{a|S=2, S_z=2> \quad + \quad b|S=2, S_z=1>}{(|a|^2 + |b|^2)^{1/2}}$$

where

$$S_z|S=2, S_z=m> \quad = \quad \hbar m|S=2, S_z=m>$$

and

$$S^2|J=2, S_z=m> \quad = \quad 6\hbar^2|S=2, S_z=m>$$

and $|S=2, S_z=m>$ is normalised to unity. Let the Hamiltonian of the system be

$$H = \lambda\vec{S}^2 + \mu S_x^2$$

where $\vec{S}$ is the spin operator and S_x is the x-component. In the above equations, a and b are complex constants and λ and μ are real constants.

1. Suppose a measurement of S_z is made on $|\psi>$. What are the possible outcomes and what is the probability of each?
2. What are the dimensions of λ and μ?
3. What are the conserved quantities of the Hamiltonian?
4. Find the expectation value of the Hamiltonian in the state $|\psi>$.

[III]

1. A is a 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

Diagonalise A and find $\exp(A)$.

2. Show that the Fourier Transform of $f(x) = \frac{1}{\mathbf{r}^2+\lambda^2}$, where $\mathbf{r} = (x, y, z)$, $\mathbf{r}^2 = x^2 + y^2 + z^2$ and $\lambda^2 > 0$ is given by $\tilde{F}(\mathbf{k}) = \frac{4\pi^2}{k} e^{-\lambda k}$, where $k = |\mathbf{k}|$.

[IV] Consider a point particle of mass m and charge q.

1. Using the Lorentz force equation, find the transformation properties of electric field $\vec{E}(\vec{r}, t)$, magnetic field $\vec{B}(\vec{r}, t)$, vector potential $\vec{A}(\vec{r}, t)$ and scalar potential $\phi(\vec{r}, t)$, under parity and (separately) time-reversal transformations.
2. Solve completely the force equation for the particle moving in a constant magnetic field in z-direction, giving the radius of the helix and the pitch angle.

[V] An ideal gas is at a temperature T_1 and volume V_1. The gas is taken through an isobaric (constant pressure) process to a state of higher temperature T_2. It is then taken via an isochoric (constant volume) process to a state of temperature T_1, and finally back to the initial state in an isothermal (constant temperature) process.

1. Calculate the amount of heat transferred (Q) to the gas in the cycle.
2. Same as above, but in the reverse cycle.
3. What would be the result if δQ were an exact differential?
4. Calculate the work done by the gas during the cycle. Is it equal to Q? Why?
5. Draw a P-V diagram to illustrate the cycle.
6. Comment on the change in entropy, if any, during the cycle.