Chennai Mathematical Institute

MSc/PhD Entrance Examination, 2013

15th May 2013

Problems in Part A will be used for screening purposes. Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Notation: $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of integers, of the real numbers, and of the complex numbers.

Part A

This section consists of fifteen (15) multiple-choice questions, each with one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

1. Pick the correct statement(s) below.
   (a) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
   (b) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/4$.
   (c) There exists a group of order 44 with a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and a subgroup isomorphic to $\mathbb{Z}/4$.
   (d) There exists a group of order 44 without any subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or to $\mathbb{Z}/4$.

2. Let G be group. The following statements hold.
   (a) If G has nontrivial centre C, then G/C has trivial centre.
   (b) If $G \neq 1$, there exists a nontrivial homomorphism $h : \mathbb{Z} \to G$.
   (c) If $|G| = p^3$, for p a prime, then G is abelian.
   (d) If G is nonabelian, then it has a nontrivial automorphism.

3. Let $C[0,1]$ be the space of continuous real-valued functions on the interval $[0,1]$. This is a ring under point-wise addition and multiplication. The following are true.
   (a) For any $x \in [0,1]$, the ideal $M(x) = \{f \in C[0,1] \mid f(x) = 0\}$ is maximal.
   (b) $C[0,1]$ is an integral domain.
   (c) The group of units of $C[0,1]$ is cyclic.
   (d) The linear functions form a vector-space basis of $C[0,1]$ over $\mathbb{R}$.

4. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with eigenvalues $\frac{2}{3}$ and $\frac{9}{5}$. Then, there exists a non-zero vector $v \in \mathbb{R}^2$ such that
   (a) $\|Av\| > 2\|v\|$;
   (b) $\|Av\| < \frac{1}{2}\|v\|$;
   (c) $\|Av\| = \|v\|$;
   (d) $Av = 0$;

5. Let F be a field with 256 elements, and $f \in F[x]$ a polynomial with all its roots in F. Then,
   (a) $f \neq x^{15} - 1$;
   (b) $f \neq x^{63} - 1$;
   (c) $f \neq x^2 + x + 1$;
   (d) if f has no multiple roots, then f is a factor of $x^{256} - x$.

6. Let $h : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $h(0) = 0$; $h(\frac{1}{2}) = 5$, and $|h(z)| < 10$ for $|z| < 1$. Then,
   (a) the set $\{z : |h(z)| = 5\}$ is unbounded by the Maximum Principle;
   (b) the set $\{z : |h'(z)| = 5\}$ is a circle of strictly positive radius;
   (c) $h(1) = 10$;
   (d) regardless of what h' is, $h'' \equiv 0$.

7. Suppose that $f(z)$ is analytic, and satisfies the condition $|f(z)^2 - 1| = |f(z) - 1| \cdot |f(z) + 1| < 1$ on a non-empty connected open set U. Then,
   (a) f is constant.
   (b) The imaginary part of f, $Im(f)$, is positive on U.
   (c) The real part of f, $Re(f)$, is non-zero on U.
   (d) $Re(f)$ is of fixed sign on U.

8. Consider the following subsets of $\mathbb{R}^2$: $X_1 = \{(x, \sin\frac{1}{x}) | 0 < x < 1\}$, $X_2 = [0, 1] \times \{0\}$, and $X_3 = \{(0, 1)\}$. Then,
   (a) $X_1 \cup X_2 \cup X_3$ is a connected set;
   (b) $X_1 \cup X_2 \cup X_3$ is a path-connected set;
   (c) $X_1 \cup X_2 \cup X_3$ is not path-connected, but $X_1 \cup X_2$ is path-connected;
   (d) $X_1 \cup X_2$ is not path-connected, but every open neighbourhood of a point in this set contains a smaller open neighbourhood which is path-connected.

9. For a set $A \subset \mathbb{R}$, denote by $Cl(A)$ the *closure* of A, and by $Int(A)$ the *interior* of A. There is a set $A \subset \mathbb{R}$ such that
   (a) A, $Cl(A)$, and $Int(A)$ are pairwise distinct;
   (b) A, $Cl(A)$, $Int(A)$, and $Cl(Int(A))$ are pairwise distinct;
   (c) A, $Cl(A)$, $Int(A)$, and $Int(Cl(A))$ are pairwise distinct;
   (d) A, $Cl(A)$, $Int(A)$, $Int(Cl(A))$, and $Cl(Int(A))$ are pairwise distinct.

10. Let $f, g : [0,1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational;} \end{cases}$$

$$g(x) := \begin{cases} 1/q & \text{if } x = \frac{p}{q} \text{ is rational, with } gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then,

(a) g is Riemann integrable, but not f;

(b) both f and g are Riemann integrable;

(c) the Riemann integral $\int_0^1 f(x)dx = 0$;

(d) the Riemann integral $\int_0^1 g(x)dx = 0$.

11. Let C be the ellipse $24x^2 + xy + 5y^2 + 3x + 2y + 1 = 0$. Then, the line integral $\oint(x^2ydy + xy^2dx)$

(a) lies in $(0,1)$;

(b) is 1;

(c) is either 1 or -1 depending on whether C is traversed clockwise or counterclockwise;

(d) is 0.

12. The series $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^{n+1}n^4e^{-n^2}$

(a) has unbounded partial sums;

(b) is absolutely convergent;

(c) is convergent but not absolutely convergent;

(d) is not convergent, but partial sums oscillate between -1 and $+1$.

13. Let f be continuously differentiable on $\mathbb{R}$. Let $f_n(x) = n\left(f(x + \frac{1}{n}) - f(x)\right)$. Then,

(a) f_n converges uniformly on $\mathbb{R}$;

(b) f_n converges on $\mathbb{R}$, but not necessarily uniformly;

(c) f_n converges to the derivative of f uniformly on $[0,1]$;

(d) there is no guarantee that f_n converges on any open interval.

14. Let $f : X \to Y$ be a nonconstant continuous map of topological spaces. Which of the following statements are true?

(a) If $Y = \mathbb{R}$ and X is connected then X is uncountable.

(b) If X is Hausdorff then $f(X)$ is Hausdorff.

(c) If X is compact then $f(X)$ is compact.

(d) If X is connected then $f(X)$ is connected.

15. Let X be a set with the property that for any two metrics d_1, and d_2 on X, the identity map

$$id : (X, d_1) \to (X, d_2)$$

is continuous. Which of the following are true?

(a) X must be a singleton.

(b) X can be any finite set.

(c) X cannot be infinite.

(d) X may be infinite but not uncountable.

Part B

Solve six (6) problems from below, **clearly indicating** which problems you would like us to mark. Every problem is worth ten (10) marks. Justify all your arguments to receive credit.

1. Let G be a finite group, p the smallest prime divisor of $|G|$, and $x \in G$ an element of order p. Suppose $h \in G$ is such that $hxh^{-1} = x^{10}$. Show that $p = 3$.
2. (a) Show that there exists a 3×3 invertible matrix $M \neq I_3$ with entries in the field $\mathbb{F}_2$ such that $M^7 = I_3$.

   (b) Let A be an $m \times n$ matrix, and $\mathbf{b}$ an $m \times 1$ vector, both with integer entries.

      1. Suppose that there exists a prime number p such that the equation $A\mathbf{x} = \mathbf{b}$ seen as an equation over the finite field $\mathbb{F}_p$ has a solution. Then does there exist a solution to $A\mathbf{x} = \mathbf{b}$ over the real numbers?
      2. If $A\mathbf{x} = \mathbf{b}$ has a solution over $\mathbb{F}_p$ for every prime p, is a real solution guaranteed?
3. Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices over $\mathbb{C}$. Think of $M_n(\mathbb{C})$ as the $2n^2$-dimensional Euclidean space $\mathbb{R}^{2n^2}$. Show that the set of all diagonalizable $n \times n$ matrices is dense in $M_n(\mathbb{C})$.
4. Compute the integral
$$\int_{-\infty}^{\infty} \frac{x}{(x^2 + 2x + 2)(x^2 + 4)} dx.$$
5. Show that there does not exists an analytic function f defined in open unit disk for which $f(\frac{1}{n})$ is 2^{-n}.
6. Let f be a real valued continuous function on $[0, 2]$ which is differentiable at every point except possibly at $x = 1$. Suppose that $\lim_{x \to 1} f'(x) = 2013$. Show that f is differentiable at x.
7. (a) Show that there exists no bijective map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that f and f^{-1} are differentiable.

   (b) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map such that the derivative $Df(x)$ is surjective for all x. Is f surjective?
8. (a) Let $f \in \mathbb{Z}[x]$ be a non-constant polynomial with integer coefficients. Show that as a varies over the integers, the set of divisors of $f(a)$ includes infinitely many different primes.

   (b) Assume known the following result: If G is a finite group of order n such that for integer $d > 0$, $d|n$, there is no more than one subgroup of G of order d, then G is cyclic. Using this (or otherwise) prove that the multiplicative group of units in any finite field is cyclic.
9. Let $K_1 \supset K_2 \supset \dots$ be a sequence of connected compact subsets of $\mathbb{R}^2$. Is it true that their intersection $K = \cap_{i=1}^{\infty} K_i$ is connected also? Provide either a proof or a counterexample.
10. Let A be a subset of $\mathbb{R}^2$ with the property that every continuous function $f : A \to \mathbb{R}$ has a maximum in A. Prove that A is compact.