MSC/PHD MATHEMATICS ANSWERS

Part A

(1) a, b
(2) a, b, d
(3) a
(4) c
(5) b, d
(6) c, d
(7) c, d
(8) a, c
(9) a
(10) a, d
(11) d
(12) b
(13) b, c
(14) a, c, d
(15) b, d

Part B

(1) Let H and X be the subgroups of G generated by the elements h and x, respectively. The given equation implies that H acts on X by conjugation; in other words, we have a homomorphism $\phi : H \to Aut(X)$, where $Aut(X)$ is the group of automorphisms of X. This latter group has order $p-1$, whence ϕ is the trivial homomorphism. This means that $hxh^{-1} = x$. But we are given that $hxh^{-1} = x^{10}$. It follows that x^9 is the identity element, so $p = 3$.

(2) (a) Note that $T^7 - 1 = (T-1)(T^3+T^2+1)(T^3+T+1)$. Consider the field $K = \mathbb{F}_2[T]/(T^3+T+1)$. It has a basis $1, T, T^2$ over $\mathbb{F}_2$. Multiplication by T on K is $\mathbb{F}_2$-linear, so it can be represented by a 3×3 matrix over $\mathbb{F}_2$. In the above order of the basis vectors, it is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $A^3 + A + I_3 = 0$, so $A^7 = I_3$.

(b) (i) No. For example $0x = 2$ has a solution modulo 2 but does not have a real solution.

(ii) Yes. Suppose that the system does not have real solutions. Then $\operatorname{rank}(A) < \operatorname{rank}([A|b])$, where $[A|b]$ denotes the augmented matrix. Denote these two ranks by r and s respectively. Then there is an $s \times s$ submatrix of $[A|b]$ that is invertible; let its determinant (which is an integer) be d. Then for every prime $p > d$, the corresponding $s \times s$ submatrix of $[A|b]$ has an invertible determinant modulo p, i.e., $\operatorname{rank}([A|b]) = s \mod p$ for all large enough primes p. On the other hand, rank of any integer matrix modulo a prime number can only be at most its rank considered as a real matrix, so for all primes p, $\operatorname{rank}(A) \leq r \mod p$ for all primes p. Therefore for all sufficiently large primes p, $\operatorname{rank}(A) < \operatorname{rank}([A|b])$, mod p contradicting the hypothesis that there is solution modulo every prime.

(3) Let $A \in M_n(\mathbb{C})$. We want to show that for every $\epsilon > 0$, the ϵ-ball around A contains a diagonalizable matrix. First note that this is true for A if and only if it is true PAP^{-1} for every invertible $P \in M_n(\mathbb{C})$; therefore we may assume that A is in its Jordan canonical form. For $1 \leq i \leq n$, pick $0 < \delta_i < \frac{\epsilon}{n}$, all distinct. Let B be the sum of A and the diagonal matrix with $\delta_1, \ldots, \delta_n$ on the

diagonal. We claim that B is diagonalizable. Indeed, the eigen-values of B are $a_{i,i} + \delta_i, 1 \leq i \leq n$. We may choose the δ_i to further satisfy that these are distinct, thus making B diagonalizable.

(4) The integrand extended to a function on the complex plane has simple poles at $\pm 2i$, and $-1 \pm i$. We shall compute the integral by contour integration. Let C_n be the contour given by the square with vertices $(-n, 0)$, $(n, 0)$, (n, n), $(n, -n)$, and denote by I_n the value of the contour integral about C_n (traversed counterclockwise). On one hand, for $n \gg 0$, $I_n = I_{n+1} = 2\pi i\{\text{sum of the residues in } C_n\}$. On the other hand, $\lim_{n\to\infty} I_n$ converges to the value of the required integral on the real line as the integrand goes to zero on the three sides of C_n not on the real line, as $n \to \infty$.

(5) By way of contradiction, suppose that there exists such an analytic function f. Then f has an expansion as a convergent power series around 0. It is of the form $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$. From the given information, we can, however, conclude that $f^{(n)}(0) = 0$ for all $n \geq 0$; therefore, $f \equiv 0$, contradiction.

(6) The function satisfies the hypotheses of Lagrange's Mean Value Theorem over [0,1] and [1,2], using which we'll prove that the difference quotient $\Delta(h) = [f(1+h) - f(1)]/h$ has a limit as $h \to 0$. Let $\epsilon > 0$ be given. Then, $\exists \delta > 0$ such that $|f'(x) - 2013| < \epsilon$ whenever $0 < 1 - x < \delta$. For $-1 < h < 0$, the MVT says that $\exists c \in (0, 1)$ such that $\Delta(h) = f'(c)$. But then $|\Delta(h) - 2013| < \epsilon$ whenever $0 < -h < \delta$. This says that $\lim_{h\to 0^-} \Delta(h) = 2013$. One similarly shows that $\lim_{h\to 0^+} \Delta(h) = 2013$.

(7) (a) By way of contradiction, suppose that f and f^{-1} are differentiable. Then $Df \circ D(f^{-1}) : \mathbb{R}^3 \to \mathbb{R}^3$ is $D(\mathrm{id}_{\mathbb{R}^3}) = I_3$ at every point in $\mathbb{R}^3$. However, $\mathrm{rank}(Df) \leq 2$ at every point in $\mathbb{R}^2$, contradiction.

(b) No. Consider $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto e^t$. Then $(Df)(x) = e^x$ for all $x \in \mathbb{R}$. Therefore $(Df)(x)$ induces an isomorphism of of tangent spaces for all x.

(8) (a) For a prime p, set $S(p) := \{a \in \mathbb{Z} : a \geq 1 \text{ and there exist } b \in \mathbb{Z}, b \geq a \text{ and } n \in \mathbb{Z} \text{ such that } p^a \mid f(p^b n)\}$. By hypothesis $S(p)$ is empty except for finitely many p, so for some p, $S(p)$ is infinite. Choose such a p. Then for all $a \in S(p)$, there exist b, n such that $p^a \mid f(p^b n)$ so $p^a \mid f(0)$ since $f(0) = f(x) - xg(x)$ for some $g(x) \in \mathbb{Z}[x]$. Therefore $f(0) = 0$, a contradiction since $q \mid f(q)$ for all primes q.

(b) Let F be a finite field; denote its group of units by $F^\times$. Let $n = |F^\times|$. Let $d > 0$ be a divisor of n. We want to show that there exists at most one subgroup of $F^\times$ of order d. By way of contradiction, suppose that there are two distinct subgroups G and H of $F^\times$ of order d. Note that for all $x \in G \cup H$, $x^d = 1$, so in F, there at least $d+1$ elements that satisfy the polynomial $T^d - 1 = 0$, a contradiction.

(9) Suppose, by way of contradiction, that K admits a disconnection, that is open subsets $U, V \subset \mathbb{R}^2$ such that $K \subset U \cup V$, $U \cap K \neq \emptyset$, $V \cap K \neq \emptyset$, but $U \cap V \cap K = \emptyset$. Let $K_i' = K_i \setminus (U \cap V)$, $i \geq 1$. If $K_i' \neq \emptyset\ \forall i$, then by the Finite Intersection Property, $\cap_{i=1}^{\infty} K_i' \neq \emptyset$, contradicting the fact that $K \subset U \cup V$. So, $\exists n$ such that $K_n \subset U \cup V$. Similarly, working with $K_i'' = K_i \setminus (U \cup V)$ we can show that $\exists m$ such that $U \cap V \cap K_m = \emptyset$. If we set $N = \max\{m, n\}$, then U, V form a disconnection of K_N, contradicting the hypothesis that the K_i are connected.

(10) It suffices to show that A is bounded and closed. The projection maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are continuous. Therefore A is bounded. Since every continuous $\mathbb{R}$-valued function on A attains its maximum, it also attains its minimum. Let p be any limit point of A. The continuous function $A \to \mathbb{R}$, $q \mapsto d(q, p)$, where d is the usual metric on $\mathbb{R}^2$ attains its minimum, so $p \in A$. Hence A is closed.