

CHENNAI MATHEMATICAL INSTITUTE
MSc Applications of Mathematics Entrance Examination
15 May 2014

Instructions:

• Enter your *Registration Number* here **CMI PG**—

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OR here **PG**—

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• Enter the name of the city where you write this test:

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- The allowed time is 3 hours.
- This examination has two parts. Part A has multiple-choice questions, while questions in Part B require detailed answers.
- **The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.**
- **Answers to questions in Part A must be recorded on the sheet provided for the purpose.**
- You may use the blank pages at the end for your rough-work.

For office use only

Part B

Qno	1	2	3	4	5	6	7	8	9	10	11	12
Marks												

.	Part A	Part B	Total
Score			

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You must record your answers to Part A here by filling in the appropriate circles:
For example, if your answer to question number 17 is (A) and (D), record it as follows:

17. B C

Part A

1. A B C D
2. A B C D
3. A B C D
4. A B C D
5. A B C D
6. A B C D
7. A B C D
8. A B C D
9. A B C D
10. A B C D

Important

The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Part A

This section consists of Ten (10) multiple-choice questions, each with one or more correct answers. Record your answers on the attached sheet by filling in the appropriate circles.

Every question is worth five (5) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

- Suppose $\{a_n : n \geq 1\}$ is a sequence of positive numbers with $a_n \geq a_{n+1}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then we can conclude that
 - $\lim_{n \rightarrow \infty} a_n = 0$.
 - $\lim_{n \rightarrow \infty} na_n = 0$.
 - $\lim_{n \rightarrow \infty} n^2 a_n = 0$.
 - $\lim_{n \rightarrow \infty} (\sum_{m=n}^{\infty} a_m) = 0$.
- Which of the following functions is/are uniformly continuous on $(0, 1)$?
 - $f(x) = x(1 - x)$.
 - $g(x) = \frac{x}{(1-x)}$.
 - $u(x) = \frac{(1-x)}{x}$.
 - $v(x) = \frac{x}{(2-x)}$.
- Let $f : (0, 1) \mapsto (0, 1)$ be a continuously differentiable function. Then we can conclude that
 - $g = \frac{1}{f}$ is a continuous function on $(0, 1)$.
 - $g = \frac{1}{f}$ is a continuously differentiable function on $(0, 1)$.
 - $g = \frac{1}{f}$ is a uniformly continuous function on $(0, 1)$.
 - h defined by $h(x) = x(1 - x)f(x)$ for $x \in (0, 1)$ is uniformly continuous.
- Let A be a $n \times n$ non-singular symmetric matrix with entries in $(0, \infty)$. Then we can conclude that
 - $|A| > 0$ ($|A|$ denotes the determinant of A).
 - A is a positive definite matrix.
 - $B = A^2$ is a positive definite matrix.
 - $C = A^{-1}$ is a matrix with entries in $(0, \infty)$.
- Let A be an $n \times n$ matrix with real entries and let $B = A^t A$ (where A^t is the transpose of A). Which of the following statements is/are always true?
 - If B is invertible then A must be invertible.
 - If λ is an eigenvalue of B then $\lambda \geq 0$.
 - If λ is an eigenvalue of A then λ^2 is an eigenvalue of B .
 - Let $C = I + B$. Then C is invertible.

6. Let A be a $m \times n$ matrix. Consider the equation

$$Ax = b \tag{1}$$

Which of the following is/are correct

- (a) If the equation (1) admits a solution for all $b \in \mathbb{R}^m$ then n must be greater than or equal to m .
 - (b) If the equation (1) admits a unique solution for some $b \in \mathbb{R}^m$, then n must be greater than or equal to m .
 - (c) If the equation (1) admits two distinct solutions for some $b \in \mathbb{R}^m$, then m must be greater than or equal to n .
 - (d) If for all $b \in \mathbb{R}^m$, the equation (1) admits a solution that is unique then n must be equal to m .
7. Let f, g be real valued continuous functions on $[0, 1]$. Which of the following statements is/are always true?
- (a) f is uniformly continuous.
 - (b) g is bounded.
 - (c) If $f(r) = 1 - g(r)$ for all rational numbers $r \in [0, 1]$ then $f(x) = 1 - g(x)$ for all $x \in [0, 1]$.
 - (d) The function v defined by $v(x) = f^2(x) + g^2(x)$, $0 \leq x \leq 1$ is continuous.
8. Let A be an $n \times m$ matrix and B be an $m \times n$ matrix. For a square matrix D , let $Tr(D)$ denote trace of D , $|D|$ denote the determinant of D . Suppose that AB is invertible. Then which of the following is/are always true?
- (a) $Tr(AB) = Tr(BA)$.
 - (b) $|AB| = |BA|$.
 - (c) $m \geq n$.
 - (d) BA must be invertible.
9. Let f be a continuously differentiable function on \mathbb{R} (and let f' denote the derivative of f). Which of the following statements is/are always true?
- (a) f is uniformly continuous on $(-1, 1)$.
 - (b) If $f'(x) \neq 0$ for all $x \in (a, b)$ then

$$\max(f(a), f(b)) > f(x) \quad \forall x \in (a, b).$$

- (c) If f' is a bounded function, then f is a uniformly continuous function on \mathbb{R} .
- (d) If f is a bounded function, then f' is a uniformly continuous function on \mathbb{R} .

10. Let $\{a_n : n \geq 1\}$ be a sequence of nonnegative numbers such that

$$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = 1.$$

Which of the following statements is/are always true?

- (a) The series $\sum_n a_n$ converges.
- (b) The series $\sum_n a_n x^n$ converges uniformly for $x \in [-0.5, 0.5]$.
- (c) The series $\sum_n a_n x^n$ converges uniformly for $x \in [-1, 1]$.
- (d) $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$.

Part B

Answer any six questions. Each question carries 10 marks.

1. In how many ways can 7 girls and 8 boys be seated in a line so that no two girls are together.
2. Is the series given below convergent? Justify.

$$\sum_2^{\infty} \frac{\log(n+1) - \log n}{\sqrt{n}}.$$

3. Let $t > 0$ be fixed. Show that

$$\frac{1}{n} \left\{ \sin \frac{t}{n} + \sin \frac{2t}{n} + \sin \frac{3t}{n} + \dots + \sin \frac{(n-3)t}{n} \right\} \rightarrow \frac{1 - \cos t}{t}.$$

4. Show that

$$\int_0^{2\pi n} \frac{\sin(n^3 x)}{n^2 + x^2} dx \rightarrow 0.$$

5. Does the series $\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n^2}$ converge uniformly for $x \in \mathbb{R}$.
6. Let f be a continuous function on an open interval (a, b) and let $x_1, x_2, \dots, x_n \in (a, b)$. Show that there exists a number $u \in (a, b)$ such that

$$f(u) = \frac{1}{n} \sum_{i=1}^n f(x_i).$$

7. Show that there is no continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that takes every value exactly twice *i.e.* (here $\#(A)$ denotes number of elements in the set A).

$$\#\{x \in \mathbb{R} : f(x) = y\} = 2 \quad \forall y \in \mathbb{R}.$$

8. Let

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is continuous function on \mathbb{R}^2 .

9. The numbers 2475; 1287; 4598; 5742 are all divisible by 11. Show that the following determinant is also divisible by 11.

$$\begin{vmatrix} 2 & 1 & 4 & 5 \\ 4 & 2 & 5 & 7 \\ 7 & 8 & 9 & 4 \\ 5 & 7 & 8 & 2 \end{vmatrix}.$$

10. Let T be the linear transform of \mathbb{R}^3 given by

$$T(1, 0, 0) = (1, 0, 0); \quad T(1, 1, 0) = (1, 1, 1); \quad T(1, 1, 1) = (1, 1, 0).$$

find formula for $T(x, y, z)$. Find $\text{Kernel}(T)$. Find $\text{range}(T)$. Show $T^3 = T$.

11. Let V be the linear space of functions on \mathbb{R} spanned by the functions $\{1, x, e^x, xe^x\}$. Define linear map $T : V \rightarrow V$ by $T(f) = f'$, the derivative of f . Calculate the matrix of T w.r.t. the above basis.
12. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T \neq 0$ but $T^2 \equiv 0$. Show that

$$\dim(\text{Range}(T)) = 1.$$